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Mobile: +964 760 235 5555
+964 771 948 7257
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Most Compassionate, Mort Merciful

Edition Word

O Allah, my Lord

Cast felicity in me , facilitate my cause and unknot my tongue to perceive my speech , thanks be upon Him the Evolver of the universe and peace be upon Mohammad and his immaculate and benevolent progeny .

A fledged edition of Al-Bahr , peer reviewed scientific journal, embraces a constellation of research studies pertinent to engineering and natural sciences we do hope to overlap a scientific gap the specialists observe as an academic phenomenon worth being under the lenses of the researchers, that is why there is diversity in the studies to meet the requirements of the journal readership . For the journal, now, comes to the fore , at the efforts of the editorial and advisory boards and the researchers who strain every sinew to publish in Al-Bahr, to be global as to be published in an international publishing house in line with the global scientific journals.

On such an occasion we do pledge the promise of fealty and loyalty to those who observe our issues with love and heed in the International Al-`Ameed for Research and Studies , Department of Cultural and Intellectual Affairs in the Holy Al-`Abbas Shrine and the strenuous endeavour to cull whatever invigorates the scientific interaction and academic research in Iraq and worldwide to create a new generation keeping pace with the development of the current scientific phase and to lay the hands of the researchers, nationwide and worldwide, upon the desired missions.

Thanks be upon Him ,the Evolver ad infinitum .

Consultation and Editaril

Bivariate Generalized Double Weighted Exponential Distribution

Kareema Abed Al-Kadim and Mohannad Mohammad Fadhil
Department of Mathematics , College of Education of pure Sciences, University of Babylon , Hilla, Iraq

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الخلاصة

في هذا البحث نقترح التوزيع الاسي الموزون المضاعف المعمم للمتغيرين مع مناقشة بعض خواصة، مثل دالة الكثافة الاحتمالية المشتركة والهامشية، دالة الموثوقية المشتركة، التوقع الرياضي، الدالة المولدة للعزوم الهامشية وفي النهاية ، نستخدم طريقة الامكان الاعظم لتقدير معلماته.

الكلمات المفتاحية

التوزيع الاسي الموزون المضاعف المعمم، دالة الكثافة الاحتمالية الشرطية، دالة الموثوقية المشتركة، مقدرات الامكان الاعظم.

Abstract

In this article we suggest abivariate generalized double weighted exponential distribution with discussion some of its properties , suchasjoint probability density function and its marginal,joint reliabilityfunction,the mathematical expectation , the marginal moment generating functionand,we usethe maximum likelihood method to estimate its parameters.

Keywords

Generalized double weighted exponential distribution, Conditional probability density function, Joint reliability function.

1. Introduction

Abed Al-Kadim and Hantoosh [1] introduced the double weighted distribution and double weighted exponential (DWE)distribution.

So that our object of this article is to display abivariate generalized double weighted exponential (BGDWE) distribution , which is a special case of the multivariate distributions. Its marginal's are generalized double weighted exponential (GDWE) distribution by using the method similar to those used by Marshall and Olkin [2], Sarhan and Balakrishnan [3] defined a new bivariate distribution using generalized distribution and exponential distribution and derived some properties of this new distribution, Al-Khedhairi and El-Gohary [4] presenteda class of bivariate Gompertz distributions, Kundu and Gupta [5] proposed the bivariate generalized exponential distribution ,El-Sherpienyetal. [6] presenteda new bivariate distribution with generalized gompertzmarginals and Davarzanietal. [7] studied the bivariate life time geometric distribution in presence of cure fractions.

Plan of the Article:

In this article, we define the BVGDWE distribution and discuss its different properties in Section 2. Section 3 present the reliability analysis. In Section 4weintroducethemathematical expectation. In Section 5 we derive the marginal moment generating function. Section 6 obtains the parameter estimation using MLE. Finally, a conclusion for the results is given in Section 7.

2. Bivariate Generalized Double Weighted Exponential Distribution

Suppose Y Y is anon-negative random variable with probability density function (PDF),, then the double weighted exponential distribution by using probability density function is:

$$f_{DWE}(y) = \frac{[w(y)\,f(y)]\,f(y)}{\mu_w} = \frac{w(y)[f(y)]^2}{\mu_w},\ y > 0\ \ and\ \ \mu_w = E[w(y)\,f(y)] < \infty$$

The first weight is $w(y) = y$and the second is$f(y)$, where$f(y)$ is probability density function of exponential distribution. Then

$$f_{DWE}(y;\lambda) = 4\lambda^2 y e^{-2\lambda y}\ \ ,\ y > 0\ ,\lambda > 0 \quad (1)$$

also the cumulative distribution function is:

$$F_{DWE}(y;\lambda) = 1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}$$

(2)The univariate GDWE distribution has the following PDF and CDF respectively for

$$y > 0;\ f_{GDWE}(y;\alpha,\lambda) = 4\alpha\lambda^2 y e^{-2\lambda y}\left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha-1} \quad (3)$$

$$F_{GDWE}(y;\alpha,\lambda) = \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha} \quad (4)$$

where $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively.. Suppose that $D_1 \sim \text{GDWE}(\alpha_1,\lambda)$, $D_2 \sim \text{GDWE}(\alpha_2,\lambda)$ and$D_3 \sim \text{GDWE}(\alpha_3,\lambda)$and they are mutually in dependent. Here$' \sim '$ means is distributed GDWE .Define $Y_1 = \max(D_1,D_3)$and$Y_2 = \max(D_2,D_3)$. Then we say that the bivariate vector (Y_1,Y_2) has a bivariate generalized double weighted exponential distribution with the shape parametersα_1,α_2 and$\alpha_3$and the scale parameterλ .We will denoteit by BGDWE $(\alpha_1,\alpha_2,\alpha_3,\lambda)$.

2.1. The Joint Cumulative Distribution Function

We now introduce the joint distribution

of random variables Y_1 and Y_2 considered the following theorem of the joint CDF of the BGDWE$(\alpha_1, \alpha_2, \alpha_3, \lambda)$.

2.1.1. Theorem [8].

If $(Y_1, Y_2) \sim$ BGDWE$(\alpha_1, \alpha_2, \alpha_3, \lambda)$, then the joint CDF of (Y_1, Y_2) for $y_1 > 0$, $y_2 > 0$, is:

$$F_{BGDWE}(y_1, y_2) = \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2} \times \left(1 - 2\lambda t e^{-2\lambda t} - e^{-2\lambda t}\right)^{\alpha_3} \quad (5)$$

where $t = \min(y_1, y_2)$

Proof.

Since $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$ we get

$$F(y_1, y_2) = P(\max(D_1, D_3) \le y_1, \max(D_2, D_3) \le y_2) = P(D_1 \le y_1, D_2 \le y_2, D_3 \le \min(y_1, y_2))$$

where $D_j\ (j = 1,2,3)$ are mutually independent, we readily obtain

$$F_{BGDWE}(y_1, y_2) = P(D_1 \le y_1)P(D_2 \le y_2)P(D_3 \le \min(y_1, y_2)) = F_{GDWE}(y_1; \alpha_1, \lambda)F_{GDWE}(y_2; \alpha_2, \lambda)F_{GDWE}(t; \alpha_3, \lambda) \quad (6)$$

Substituting (4) into (6) we obtain (5) which completes the proof of the theorem 2.1.

2.2. The Joint Probability Density Function

2.2.1. Lemma

If $(Y_1, Y_2) \sim BGDWE(\alpha_1, \alpha_2, \alpha_3, \lambda)$, then the joint PDF of (Y_1, Y_2) for $y_1 > 0, y_2 > 0$, is:

$$f_{BGDWE}(y_1, y_2) = \begin{cases} f_1(y_1, y_2) & if\ 0 < y_1 < y_2 < \infty \\ f_2(y_1, y_2) & if\ 0 < y_2 < y_1 < \infty \\ f_3(y, y) & if\ 0 < y_1 = y_2 = y < \infty \end{cases} \quad (7)$$

where

$$f_1(y_1, y_2) = f_{GDWE}(y_1; \alpha_1 + \alpha_3, \lambda)f_{GDWE}(y_2; \alpha_2, \lambda) = (\alpha_1 + \alpha_3)16\lambda^4 y_1 e^{-2\lambda y_1}\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3 - 1} \times \alpha_2 y_2 e^{-2\lambda y_2}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2 - 1} \quad (8)$$

$$f_2(y_1, y_2) = f_{GDWE}(y_1; \alpha_1, \lambda)f_{GDWE}(y_2; \alpha_2 + \alpha_3, \lambda) = \alpha_1(\alpha_2 + \alpha_3)16\lambda^4 y_1 e^{-2\lambda y_1}\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 - 1} \times y_2 e^{-2\lambda y_2}(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2})^{\alpha_2 + \alpha_3 - 1} \quad (9)$$

$$f_3(y, y) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{GDWE}(y; \alpha_1 + \alpha_2 + \alpha_3, \lambda) = (\alpha_3)4\lambda^2 y e^{-2\lambda y}(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y})^{\alpha_1 \alpha_2 + \alpha_3 - 1} \quad (10)$$

Proof.

Let us first suppose that $y_1 < y_2$. Then, $F_{BGDWE}(y_1, y_2)$ in (5) will be denoted by $F_1(y_1, y_2)$ and becomes

$$F_1(y_1, y_2) = \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2}$$

By taking $\frac{\partial^2 F_1(y_1, y_2)}{\partial y_1 \partial y_2} = f_1(y_1, y_2)$, we get equation (8). By the same way we find $f_2(y_1, y_2)$ when $y_2 < y_1$. But $f_3(y, y)$ cannot be derived in a similar way. Using the facts that:

$$\int_0^\infty \int_0^{y_2} f_1(y_1, y_2)\,dy_1 dy_2 + \int_0^\infty \int_0^{y_1} f_2(y_1, y_2)\,dy_2 dy_1 + \int_0^\infty f_3(y, y)dy = 1 \quad (11)$$

Let

$$T_1 = \int_o^\infty \int_0^{y_2} f_1(y_1, y_2)\,dy_1 dy_2 \quad and \quad T_2 = \int_0^\infty \int_0^{y_1} f_2(y_1, y_2)\,dy_2 dy_1$$

Then

$$T_1 = \int_o^\infty \int_0^{y_2} (\alpha_1 + \alpha_3)16\lambda^4 y_1 e^{-2\lambda y_1}\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3 - 1} \times \alpha_2 y_2 e^{-2\lambda y_2}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2 - 1} dy_1\, dy_2 = \int_o^\infty \alpha_2 4\lambda^2 y_2 e^{-2\lambda y_2}(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2})^{\alpha_1 + \alpha_2 + \alpha_3 - 1}\, dy_2 \quad (12)$$

Similarly

$$T_2 = \int_o^\infty \alpha_1 4\lambda^2 y_1 e^{-2\lambda y_1}\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy_1 \quad (13)$$

By substituting (12) and (13) in equation(11) , we get

$$-\int_o^\infty \alpha_2 4\lambda^2 y e^{-2\lambda y}\left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy$$

$$-\int_o^\infty \alpha_1 4\lambda^2 y e^{-2\lambda y}\left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy$$

This is

$$f_3(y, y) = \alpha_3 4\lambda^2 y e^{-2\lambda y}\left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} \ \blacksquare \quad (14)$$

2.3. Marginal Probability Density Function

The following theorem gives the marginal density function of Y_1 and Y_2.

2. 3.1. Theorem

The marginal probability density functions of $Y_i (i = 1,2)$ is given by

$$f_{Y_i}(y_i) = (\alpha_i + \alpha_3) 4\lambda^2 y_i e^{-2\lambda y_i} \left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} \quad (15)$$

$$= f_{GDWE}(y_i; \alpha_i + \alpha_3, \lambda), \qquad y_i > 0, (i = 1,2)$$

Proof.

The marginal cumulative distribution function of Y_i, say $F_{Y_i}(y_i)$, written as:

$$F_{Y_i}(y_i) = P(Y_i \le y_i) \quad = P(\max(D_i, D_3) \le y_i) \qquad = P(D_i \le y_i, D_3 \le y_i)$$

and since D_i is independent of D_3, we simply have

$$F_{Y_i}(y_i) = \left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_i} \left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_3} = \left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3} \quad (16)$$

$$= F_{GDWE}(y_i; \alpha_i + \alpha_3, \lambda) y_i > 0, i = 1,2$$

By differentiating w.r.t. y_i, we get (15) .

2.4. Conditional Probability Density Functions

We present the conditional probability density functions of Y_1 and Y_2 by using the marginal probability density functions in the following theorem.

2.4.1. Theorem

The conditional probability density functions of Y_i, given $Y_j = y_j$ denoted by $f_{Y_i/Y_j}(y_i/y_j)$, $i, j = 1,2; i \ne j$, is:

$$f_{Y_i/Y_j}(y_i/y_j) = \begin{cases} f^{(1)}_{Y_i/Y_j}(y_i/y_j) & \text{if } y_i < y_j \\ f^{(2)}_{Y_i/Y_j}(y_i/y_j) & \text{if } y_j < y_i \\ f^{(3)}_{Y_i/Y_j}(y_i/y_j) & \text{if } y_i = y_j = y \end{cases} \quad (17)$$

where

$$f^{(1)}_{Y_i/Y_j}(y_i/y_j) = \frac{(\alpha_i+\alpha_3)\alpha_j 4\lambda^2 y_i e^{-2\lambda y_i}\left(1-2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1}}{(\alpha_j+\alpha_3)\left(1-2\lambda y_j e^{-2\lambda y_j} - e^{-2\lambda y_j}\right)^{\alpha_3}} \quad (18)$$

$$f^{(2)}_{Y_i/Y_j}(y_i/y_j) = \alpha_i 4\lambda^2 y_i e^{-2\lambda y_i}\left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{\alpha_i-1} \quad (19)$$

and

$$f^{(3)}_{Y_i/Y_j}(y_i/y_j) = \frac{\alpha_3\left(1-2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1}}{(\alpha_2+\alpha_3)} \quad (20)$$

Proof.

We get (18),(19)and (20), using the joint PDF of (Y_1, Y_2) given in (7) and $f_{Y_i}(y_i)$ in (15) in the following formula:

$$f_{Y_i/Y_j}(y_i/y_j) = \frac{f_{Y_i/Y_j}(y_i/y_j)}{f_{Y_j}(y_j)} \quad , i \ne j = 1,2 \quad (21)$$

3. Reliability Analysis [9]

We discuss some reliability measures , the joint reliability function , joint hazard function and joint reversed hazard function .

3.1. The Joint Reliability Function

In the following Proposition, we find the joint reliability function of Y_1 and Y_2 .

3.1.1. Proposition

The joint reliability function of Y_1 and Y_2 is given by:

$$R_{BGDWE}(y_1, y_2) = \begin{cases} R_1(y_1, y_2) & \text{if } y_1 < y_2 \\ R_2(y_1, y_2) & \text{if } y_2 < y_1 \\ R_3(y, y) & \text{if } y_1 = y_2 = y \end{cases} \quad (22)$$

then

$$R_1(y_1, y_2) = 1 - \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} - \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} + \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2} \quad (23)$$

$$R_2(y_1, y_2) = 1 - \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} - \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} + \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} \quad (24)$$

and

$$R_3(y, y) = 1 - \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1+\alpha_3} - \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_2+\alpha_3} + \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1+\alpha_2+\alpha_3} \quad (25)$$

Proof.

The joint reliability function of Y_1 and Y_2 is:

$$R_{BGDWE}(y_1, y_2) = 1 - [F_{Y_1}(y_1) + F_{Y_2}(y_2) - F_{BGDWE}(y_1, y_2)] \quad (26)$$

substituting from equation (16) and (5)in equation (26) ,we get

$$R_{BGDWE}(y_1,y_2) = 1 - \left(1-2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3}$$
$$-\left(1-2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3}$$
$$+\left(1-2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1}\left(1-2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2}$$
$$\times\left(1-2\lambda t e^{-2\lambda t} - e^{-2\lambda t}\right)^{\alpha_3}$$

where $t = min(y_1, y_2)$,

if $y_1 < y_2$, we have obtain the expression of given in (23),

if $y_2 < y_1$, we have obtain the expression of given in (24) and

if $y_1 = y_2 = y$we have obtain the expression of given in (25).

3.2. Joint Hazard Function

Let(Y_1, Y_2)be two random variables with probability density function $f_{BGDWE}(y_1, y_2)$. defined joint hazard function as:

$$h_{BGDWE}(y_1, y_2) = \frac{f_{BGDWE}(y_1, y_2)}{R_{BGDWE}(y_1, y_2)} \quad (27)$$

Then, the joint hazard function is:

$$h_{BGDWE}(y_1,y_2) = \begin{cases} h_1(y_1,y_2) & if\ y_1 < y_2 \\ h_2(y_1,y_2) & if\ y_2 < y_1 \\ h_3(y,y) & if\ y_1 = y_2 = y \end{cases} \quad (28)$$

$if\ y_1 < y_2$, then

$$h_1(y_1,y_2) = \frac{f_1(y_1,y_2)}{R_1(y_1,y_2)} \quad (29)$$

where $f_1(y_1,y_2)$ from equation (8) and $R_1(y_1,y_2)$ from equation(23),

$if\ y_2 < y_1$, then$h_2(y_1,y_2) = \frac{f_2(y_1,y_2)}{R_2(y_1,y_2)}$ (30)

where$f_2(y_1,y_2)$from equation(9)

and $R_2(y_1,y_2)$from equation(24),

$if\ y_1 = y_2 = y$,then$h_3(y,y) = \frac{f_3(y,y)}{R_3(y,y)}$(31)

where $f_3(y,y)$ from equation(10)and $R_3(y,y)$ from equation(25).

3.3. Joint Reversed Hazard Function

The joint reversed hazard function is defined as the ratio of the PDF and the corresponding CDF .

1.The joint reversed hazard function of (Y_1, Y_2) is defined as:

$$r_{BGDWE}(y_1,y_2) = \frac{f_{BGDWE}(y_1,y_2)}{F_{BGDWE}(y_1,y_2)} \quad (32)$$

so that

$$r_{BGDWE}(y_1,y_2) = \begin{cases} r_1(y_1,y_2) & if\ y_1 < y_2 \\ r_2(y_1,y_2) & if\ y_2 < y_1 \\ r_3(y,y) & if\ y_1 = y_2 = y \end{cases} \quad (33)$$

then

$$r_1(y_1,y_2) = (\alpha_1+\alpha_3)\alpha_2 16\lambda^4 y_1 y_2\, e^{-2\lambda(y_1+y_2)}$$
$$\times\left[\left(1-2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)\left(1-2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)\right]^{-1} \quad (34)$$

$$r_2(y_1,y_2) = \alpha_1(\alpha_2+\alpha_3)16\lambda^4 y_1 y_2\, e^{-2\lambda(y_1+y_2)}$$
$$\times\left[\left(1-2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)\left(1-2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)\right]^{-1} \quad (35)$$

and

$$r_3(y,y) = \alpha_3 4\lambda^2 y\, e^{-2\lambda y}\left(1-2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{-1} \quad (36)$$

2.The gradient vector of the joint reversed hazard function is given by:

$r(y_1,y_2) = \left(r_{Y_1}(y_1), r_{Y_2}(y_2)\right)$, where

$$r_{Y_i}(y_i) = \frac{f_{Y_i}(y_i)}{F_{Y_i}(y_i)} = \frac{\partial}{\partial y_i}\ln F_{Y_i}(y_i) \quad ,i = 1,2 \text{ ,then} \quad (37)$$

$$r_{Y_i}(y_i) = (\alpha_i+\alpha_3)4\lambda^2 y_i\, e^{-2\lambda y_i}\left(1-2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{-1} ,i = 1,2 \quad (38)$$

4. The Mathematical Expectation

In the following Proposition ,we can derive the mathematical expectation of Y_i, $(i = 1,2)$.

4.1. Proposition

If $Y_i \sim$ GDWE$(\alpha_i + \alpha_3, \lambda)$,then the

r^{th}moment of Y_i as following:

$$E(Y_i^r) = (\alpha_i + \alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}, i = 1,2 \quad (39)$$

Proof.

$$E(Y_i^r) = \int_0^{\infty} y_i^r f_{Y_i}(y_i)\, dy_i$$

$= \int_0^{\infty}(\alpha_i+\alpha_3)\,4\lambda^2 y_i^{r+1} e^{-2\lambda y_i}\left((1-2\lambda y_i e^{-2\lambda y_i})-e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} dy_i$ Since $0 < \left((1-2\lambda y_i e^{-2\lambda y_i})-e^{-2\lambda y_i}\right) < 1$ for $y_i > 0$, then by using the binomial series expansion we have

$$\left((1-2\lambda y_i e^{-2\lambda y_i})-e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} = \sum_{j=0}^{\infty}(1-2\lambda y_i e^{-2\lambda y_i})^j \times (-1)^{\alpha_i+\alpha_3-1-j}\binom{\alpha_i+\alpha_3-1}{j}\left(e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1-j} \quad (40)$$

also

$$(1-2\lambda y_i e^{-2\lambda y_i})^j = \sum_{k=0}^{\infty}(-1)^k\binom{j}{k}(2\lambda y_i e^{-2\lambda y_i})^k \quad (41)$$

then

$$E(Y_i^r) = (\alpha_i+\alpha_3)\,4\lambda^2\int_0^{\infty} y_i^{r+1} e^{-2\lambda y_i}\sum_{j=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j}\binom{\alpha_i+\alpha_3-1}{j} \times \left(e^{-2\lambda y}\right)^{\alpha_i+\alpha_3-1-j}\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}(2\lambda y_i e^{-2\lambda y_i})^k\, dy_i$$

$$= (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \int_0^{\infty} y_i^{r+1+k} e^{-(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)y_i}\, dy_i$$

$$E(Y_i^r) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}$$

Then the r^{th}moment of Y_i is:

$$E(Y_i^r) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}, i = 1,2 \blacksquare$$

5. The Marginal Moment Generating Function

We find the marginal moment generating function of Y_i , $(i = 1,2)$ in the following lemma

5.1. Lemma

If $Y_i \sim \text{GDWE}(\alpha_i + \alpha_3, \lambda)$, then the marginal moment generating function of Y_i , $(i = 1,2)$ as following:

$$M_{Y_i}(t_i) = (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \frac{t_i^r\Gamma(r+k+2)}{r!(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}, i = 1,2 \quad (42)$$

Proof.

$$M_{Y_i}(t_i) = E(e^{t_i y_i})$$

$$= \int_0^{\infty} e^{t_i y_i} f_{Y_i}(y_i)\, dy_i$$

$$= \sum_{r=0}^{\infty}\frac{t_i^r}{r!}\int_0^{\infty} y_i^r f_{Y_i}(y_i)\, dy_i$$

$$= \sum_{r=0}^{\infty}\frac{t_i^r}{r!}E(Y_i^r)$$

$$= (\alpha_i+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{\alpha_i+\alpha_3-1-j+k}\binom{\alpha_i+\alpha_3-1}{j}\binom{j}{k}(2\lambda)^{2+k} \times \frac{t_i^r\Gamma(r+k+2)}{r!(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}, i = 1,2 \blacksquare$$

6. Maximum Likelihood Estimation

To estimate the unknown parameters of the BGDWE distribution , we use the method of maximum likelihood estimators (MLEs).

Let$\left((Y_{11},Y_{21}),(Y_{12},Y_{22}),\dots,(Y_{1n},Y_{2n})\right)$ is a random sample from BGDWE$(\alpha_1,\alpha_2,\alpha_3,\lambda)$,where

$$n_1 = (i; Y_{1i} < Y_{2i}), n_2 = (i; Y_{1i} > Y_{2i}), n_3 = (i; Y_{1i} = Y_{2i} = Y_i), n = \sum_{k=1}^{3} n_k \quad (43)$$

By using the equations (8) , (9) ,(10) and (43), we find that the likelihood of the sample as following:

$$l(\alpha_1,\alpha_2,\alpha_3,\lambda) = \prod_{i=1}^{n_1} f_1(y_{1i},y_{2i})\prod_{i=1}^{n_2} f_2(y_{1i},y_{2i})\prod_{i=1}^{n_3} f_3(y_i,y_i)$$

The log-likelihood function becomes:

$$L(\alpha_1,\alpha_2,\alpha_3,\lambda) = n_1 ln(\alpha_1+\alpha_3) + n_1\ln(4) + 2n_1 ln(\lambda) + \sum_{i=1}^{n_1} ln(y_{1i}) - 2\lambda\sum_{i=1}^{n_1}(y_{1i}) + (\alpha_1+\alpha_3-1)\times\sum_{i=1}^{n_1} ln\left(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right) + n_1\ln(\alpha_2) + n_1\ln(4) + 2n_1 ln(\lambda) + \sum_{i=1}^{n_1} ln(y_{2i}) - 2\lambda\sum_{i=1}^{n_1}(y_{2i}) + (\alpha_2-1)\times\sum_{i=1}^{n_1} ln\left(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right) + n_2\ln(\alpha_1) + n_2\ln(4) + 2n_2 ln(\lambda) + \sum_{i=1}^{n_2} ln(y_{1i}) - 2\lambda\sum_{i=1}^{n_2}(y_{1i}) + (\alpha_1-1)\times\sum_{i=1}^{n_2} ln\left(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right) + n_2\ln(\alpha_2+\alpha_3) + n_2\ln(4) + 2n_2 ln(\lambda) + \sum_{i=1}^{n_2} ln(y_{2i}) - 2\lambda\sum_{i=1}^{n_2}(y_{2i}) + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} ln\left(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right) + n_3 ln(\alpha_3) + n_3 ln(4) + 2n_3 ln(\lambda) + \sum_{i-1}^{n_3} ln(y_i) - 2\lambda\sum_{i-1}^{n_3}(y_i) + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i-1}^{n_3} ln\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \qquad (44)$$

Taking the first partial derivatives of (44) with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ, and setting the results equal zero:

$$\frac{\partial L}{\partial\alpha_1} = \frac{n_1}{(\alpha_1+\alpha_3)} + \sum_{i=1}^{n_1} ln\left(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right) + \frac{n_2}{\alpha_1} + \sum_{i=1}^{n_2} ln\left(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right) + \sum_{i-1}^{n_3} ln\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \qquad (45)$$

$$\frac{\partial L}{\partial\alpha_2} = \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_1} ln\left(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right) + \frac{n_2}{(\alpha_2+\alpha_3)} + \sum_{i=1}^{n_2} ln\left(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right) + \sum_{i-1}^{n_3} ln\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \qquad (46)$$

$$\frac{\partial L}{\partial\alpha_3} = \frac{n_1}{(\alpha_1+\alpha_3)} + \sum_{i=1}^{n_1} ln\left(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right) + \frac{n_2}{(\alpha_2+\alpha_3)} + \sum_{i=1}^{n_2} ln\left(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right) + \frac{n_3}{\alpha_3} + \sum_{i-1}^{n_3} ln\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \qquad (47)$$

$$\begin{aligned}\frac{\partial L}{\partial\lambda} = \frac{4n_1}{\lambda} &- 2\sum_{i=1}^{n_1}(y_{1i}) + (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\frac{4\lambda(y_{1i})^2e^{-2\lambda y_{1i}}}{(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}})}\\ &- 2\sum_{i=1}^{n_1}(y_{2i}) + (\alpha_2-1)\sum_{i=1}^{n_1}\frac{4\lambda(y_{2i})^2e^{-2\lambda y_{2i}}}{(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}})} + \frac{4n_2}{\lambda}\\ &- 2\sum_{i=1}^{n_2}(y_{1i}) + (\alpha_1-1)\sum_{i=1}^{n_2}\frac{4\lambda(y_{1i})^2e^{-2\lambda y_{1i}}}{(1-2\lambda y_{1i}e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}})}\\ &- 2\sum_{i=1}^{n_2}(y_{2i}) + (\alpha_2+\alpha_3 1)\end{aligned}$$

$$\times\sum_{i=1}^{n_2}\frac{4\lambda(y_{2i})^2e^{-2\lambda y_{2i}}}{(1-2\lambda y_{2i}e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}})} + \frac{2n_3}{\lambda} - 2\sum_{i-1}^{n_3}(y_i)$$

$$+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i-1}^{n_3}\frac{4\lambda(y_i)^2e^{-2\lambda y_i}}{(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i})} \qquad (48)$$

These equations cannot easy to solve , but numerically by using the statistical software, to get the MLEs of the unknown parameters.

7. Conclusion

This article introduced the bivariate generalized double weighted exponential distribution whose marginals are generalized double weighted exponential distribution. Some statistical properties of this distribution. It is observed that the MLEs of the unknown parameters can be obtained by solving four non-linear equations using numerical technique.

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