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In the Name of Allah
Most Compassionate, Mort Merciful

Edition Word

O Allah, my Lord

Cast felicity in me , facilitate my cause and unknot my tongue to perceive my speech , thanks be upon Him the Evolver of the universe and peace be upon Mohammad and his immaculate and benevolent progeny .

A fledged edition of Al-Bahr , peer reviewed scientific journal, embraces a constellation of research studies pertinent to engineering and natural sciences we do hope to overlap a scientific gap the specialists observe as an academic phenomenon worth being under the lenses of the researchers, that is why there is diversity in the studies to meet the requirements of the journal readership . For the journal, now, comes to the fore , at the efforts of the editorial and advisory boards and the researchers who strain every sinew to publish in Al-Bahr, to be global as to be published in an international publishing house in line with the global scientific journals.

On such an occasion we do pledge the promise of fealty and loyalty to those who observe our issues with love and heed in the International Al-`Ameed for Research and Studies , Department of Cultural and Intellectual Affairs in the Holy Al-`Abbas Shrine and the strenuous endeavour to cull whatever invigorates the scientific interaction and academic research in Iraq and worldwide to create a new generation keeping pace with the development of the current scientific phase and to lay the hands of the researchers, nationwide and worldwide, upon the desired missions.

Thanks be upon Him ,the Evolver ad infinitum .

■ Consultation and Editaril

Minimax and Semi-Minimax Estimators for the Parameter of the Inverted Exponential Distribution under Quadratic and Precautionary Loss Functions

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الخلاصة

ركز هذا البحث على مسألة ايجاد مقدرات صغرى الكبريات وشبه صغرى الكبريات لمعلمة القياس للتوزيع الاسي المعكوس (IED) من خلال تطبيق نظرية ليمان بالتوافق مع التوزيعات الاولية (المسبقة) المعلوماتية وغير المعلوماتية بدالتي الخسارة التربيعية المتماثلة والوقائية غير المتماثلة. وقد تم مقارنة اداء المقدرات تجريبياً من خلال دراسة محاكاة استنادا الى متوسط مربعات الخطأ ومتوسط الخطأ النسبي المطلق.

الكلمات المفتاحية

التوزيع الآسي المعكوس، نظرية ليمان، مقدرات صغرى الكبريات، مقدرات شبه صغرى الكبريات، متوسط الخطأ النسبي المطلق.

Abstract

This paper is concerned with the problem of finding the mini max furthermore semi-mini max estimators for the scale parameter of the inverted exponential distribution(IED) in the direction of applying the theorem of Lehmann corresponding to non-informative and informative prior distributions under symmetric «quadratic» and asymmetric «precautionary» loss functions. The performance of the obtained estimators have been compared empirically through simulation experiment with respect to their mean square errors and mean absolute percentage errors.

Keywords

Inverted Exponential Distribution, Theorem of Lehmann, Mini max Estimator, Semi-Mini max Estimator, Mean Absolute Percentage Errors.

1. Introduction

The inverted exponential distribution (IED) is one of the continuous probability distributions. It had been introduced by Keller and Kamathin (1982) [1]. Recently, IED has been received attention from many researchers. Lin et al. (1989) [2] obtained maximum likelihood estimates, confidence limits and uniformly minimum variance unbiased estimators for the parameter and reliability function with complete samples. Stefanski (1996) [3] discussed some basic properties of the IED. Nadarajah and Kotz (2003) [4] discussed some properties of generalized IED. Dey (2007) [5] considered the IED as a life distribution and studied Bayes estimation of the parameter under LINEX loss function. Abouammoh and Alshingiti (2009) [6] introduced a shape parameter in the IED to obtain the generalized IEDas well as they discussed the statistical and reliability properties. Prakash (2009) [7] discussed the properties of Bayes estimator, Shrinkage estimator and minimax estimator for the parameter of an IE model under the squared error and general entropy loss functions. Khan (2011) [8] pointed that the inverse generalized exponential distribution approaches to the IED when its shape parameter becomes one and its location parameter becomes zero. Majeed and Aslam (2012) [9] studied the IED as a prospective life distribution. Prakash (2012) [10] examined the properties of Bayes estimators of the parameter, reliability function and hazard rate under squared error and LINEX loss functions. Singh et al. (2012) [11] obtained maximum likelihood estimators of the parameter and reliability functionas well as Bayes estimators under the general entropy loss function for complete, type I and type II censored. Zhou (2012) [12] obtained Bayes estimators of the parameter of the IED for the well-known weighted square error loss, square log error loss and Modified linear Exponential (MLINEX) loss functions. Further minimax estimators are derived by using Lehmann’s Theorem. Hussian (2013) [13] introduced a generalized version of the IED called the weighted IED. The weighted IED is reduced to the IED when its shape parameter approaches to zero.Vishwakarma et al. (2013) [14] obtained Bayes estimators of model parameter, reliability and hazard functions based on upper and lower record values. Oguntunde et al. (2014) [15] combined the IED with Kumaraswamy distribution to introduced a three parameter Kumaraswamy-inverse exponential distribution and investigate some of its statistical properties. Pundir et al. (2014) [16] deal with the estimation procedure of the parameter of IED based on hybrid censored data. Al-Noor and Bawi (2015) [17] comparedthe maximum likelihood estimator for the unknown scale parameter of the IEDas well as Bayes estimators,under symmetric “squared error” and asymmetric “precautionary” loss functions,in terms of two statistical criteria which are mean square errorsand mean absolute percentage errors.

The probability density function and distribution function of IED are defined as [11]:

$$f(t;\theta) = \frac{1}{\theta t^2} e^{-1/\theta t} \; ; t > 0, \theta > 0 \quad (1)$$
$$F(t;\theta) = e^{-1/\theta t} \quad ; \theta > 0 \quad (2)$$

The IED has no finite moments where the r^{th} moment is given by [18]:

$$E(T^r) = \frac{1}{\theta^r}\Gamma(1-r) \; ; r < 1 \quad (3)$$

Thus the expectation and the variance of the IED do not exist.

2. Mini max and Semi-Mini max Estimation

The mini max estimation was introduced by Wald (1950) from the concept of the game theory. According to Wald, "mini max approach tries to guard against the worst by requiring that the chosen decision rule should provide maximum protection against the highest possible risk". An estimator having this property is called a mini max estimator [19]. The derivation of mini max estimators depends basically on a theorem due to Hodge and Lehmann (1950) which can be stated as follows:

Lehmann's Theorem [7]:

Let $\tau=\{F_\theta ; \theta \in \Theta\}$ be a family of distribution functions and *D* be a class of estimators of the parameter Θ. Suppose that $d^* \in D$ is a Bayes estimator against a prior distribution $\pi(\theta)$ on the parameter space Θ. Then Bayes estimator d^* is said to be mini max estimator if the risk function of d^* is independent on Θ.

Mathematically, this theorem can be proved through applying two steps: first step is finding Bayes estimator $\hat{\theta}$ of θ while the second step is showing that the risk function of $\hat{\theta}$, $R(\hat{\theta},\theta)$, is a constant or not.

In order to find Bayes estimators, we consider two informative priors «inverted gamma and Gumbel type II» as well as two non-informative priors «Jeffreys and extension of Jeffreys».

1. Inverted Gamma Prior:

The probability density function of inverted gamma prior is defined as [20]:

$$\pi_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)\theta^{\alpha+1}} e^{-\beta/\theta} \quad ; \theta > 0, \alpha > 0, \beta > 0 \quad (4)$$

2. Gumbel Type II Prior:

The probability density function of Gumbel type II prior defined as [20]:

$$\pi_2(\theta) = b\left(\frac{1}{\theta}\right)^2 e^{-b/\theta} \quad ; \theta > 0, b > 0 \quad (5)$$

3. Jeffreys' Prior:

Jeffreys' prior is proposed by Harold Jeffreys in (1946). It is based on Fisher information [21], such that: $\pi_3(\theta) \propto \sqrt{I(\theta)}$ where $I(\theta)=-nE[\partial^2 \ln f(t,\theta)/\partial\theta^2]$ is Fisher's information matrix. For the model (1),

$$\pi_3(\theta) = \frac{w\sqrt{n}}{\theta} \; ; \theta > 0 \quad (6)$$

4. Extension of Jeffreys' Prior:

The extension of Jeffreys' prior is considered as [22]:

$\pi_4(\theta) \propto [I(\theta)]^k$; $k \in R^+$ Where $I(\theta)$ is Fisher's information matrix. For the model (1),

$$\pi_4(\theta) = \frac{wn^k}{\theta^{2k}} \quad ; \quad \theta > 0 \quad (7)$$

The posterior density of (θ) corresponding to the j^{th} prior, $\pi_j(\theta)$, j=1,2,3,4 , is obtained as:

$$H_j(\theta|\underline{t}) = \frac{\prod_{i=1}^{n} f(t_i,\theta)\pi_j(\theta)}{\int_\theta \prod_{i=1}^{n} f(t_i,\theta)\pi_j(\theta)d\theta} \quad (8)$$

Let $T=(t_1, t_2, \ldots, t_n)$ be independent identically distributed observations drawn from the IED defined by (1). By setting $S=\sum_{i=1}^{n} 1/t_i$, the posterior density of (θ) corresponding to the four given priors are given respectively by:

$$H_1(\theta|\underline{t}) = \frac{(S+\beta)^{n+\alpha}}{\Gamma(n+\alpha)\theta^{n+\alpha+1}} e^{-(S+\beta)/\theta} \quad (9)$$

which implies that $(\theta|\underline{t})_{H_1} \backsim$ Inverted Gamma(n+α ,S+β).

$$H_2(\theta|\underline{t}) = \frac{(S+b)^{n+1}}{\Gamma(n+1)\theta^{n+2}} e^{-(S+b)/\theta} \quad (10)$$

which implies that $(\theta|\underline{t})_{H_2} \backsim$ Inverted Gamma(n+1 ,S+b).

$$H_3(\theta|\underline{t}) = \frac{S^n}{\Gamma(n)\theta^{n+1}} e^{-S/\theta} \quad (11)$$

which implies that $(\theta|\underline{t})_{H_3} \backsim$ Inverted Gamma (n ,S).

$$H_4(\theta|\underline{t}) = \frac{S^{n+2k-1}}{\Gamma(n+2k-1)\theta^{n+2k}} e^{-S/\theta} \quad (12)$$

which implies that $(\theta|\underline{t})_{H_4} \backsim$ Inverted Gamma(n+2k-1,S).

2.1. Estimators under Quadratic Loss Function

Quadratic loss function(in some paper named as modified square error loss function) is a non-negative symmetric and continuous loss function of θ and $\hat{\theta}_Q$. The formula of quadratic loss function is [20]:

$$L(\hat{\theta}_Q, \theta) = \left(\frac{\hat{\theta}_Q - \theta}{\theta}\right)^2 \quad (13)$$

whereBayes estimator of θ under quadratic loss function is obtained as:

$$\hat{\theta}_Q = \frac{E_H\left(\frac{1}{\theta}|\underline{t}\right)}{E_H\left(\frac{1}{\theta^2}|\underline{t}\right)} \quad (14)$$

Now, Bayes estimators for the parameter θ underquadratic loss function corresponding to four posterior distributions are given respectively as:

$$\hat{\theta}_{Q_1} = \frac{S+\beta}{n+\alpha+1} \quad (15)$$

$$\hat{\theta}_{Q_2} = \frac{S+b}{n+2} \quad (16)$$

$$\hat{\theta}_{Q_3} = \frac{S}{n+1} \quad (17)$$

$$\hat{\theta}_{Q_4} = \frac{S}{n+2k} \quad (18)$$

The risk function $R(\hat{\theta},\theta)$ under quadratic loss function (13) is:
$R(\hat{\theta}_Q,\theta)=E[L(\hat{\theta}_Q,\theta)]=E((\hat{\theta}_Q-\theta)/\theta)^2=E((\hat{\theta}_Q^2-2\hat{\theta}_Q\theta+\theta^2)/\theta^2)$

$$\Rightarrow R(\hat{\theta}_Q, \theta) = \frac{1}{\theta^2}E(\hat{\theta}_Q^2) - \frac{2}{\theta}E(\hat{\theta}_Q) + 1 \quad (19)$$

■ For $\hat{\theta}_{Q_1}$(15) the risk function (19) will be:

$$R(\hat{\theta}_{Q_1}, \theta) = \frac{1}{\theta^2(n+\alpha+1)^2}E(S+\beta)^2 - \frac{2}{\theta\,(n+\alpha+1)}E(S+\beta) + 1 \quad (20)$$

We have to find $E(S+\beta)^2$ and E(S+β). Since the random variable *T* distributed as IED, then *Y*=1/T distributed as exponential distribution and $S=\sum_{i=1}^{n} 1/t_i$ distributed as gamma with probability density function;

$$f(s) = \frac{s^{n-1}}{\Gamma(n)\theta^n} e^{-s/\theta} \quad (21)$$

with the expected value and variance as:

$$E\left(\sum_{i=1}^{n} \frac{1}{t_i}\right) = E(S) = n\theta \quad (22)$$

$$var\left(\sum_{i=1}^{n} \frac{1}{t_i}\right) = var(S) = E(S^2) - \left(E(S)\right)^2 = n\theta^2 \quad (23)$$

Now, depending on (22) and (23), we can get:

$$E(S+\beta)^2 = E(S^2) + 2\beta E(S) + \beta^2 \Rightarrow$$

$$E(S+\beta)^2 = n\theta^2 + n^2\theta^2 + 2\beta n\theta + \beta^2 \quad (24)$$

And $E(S+\beta) = E(S) + \beta \Rightarrow E(S+\beta) = n\theta + \beta$ (25)

Substituting (24) and (25) in (2.80), we get:

$R(\hat{\theta}_{Q_1}),\theta)=(1/(\theta^2(n+\alpha+1)^{2)}\ (n\theta^2+n^2\theta^2+2\beta n\theta+\beta^2)-2/\theta\ (n+\alpha+1)\ (n\theta+\beta)+1$

Let $\lambda_1=1/(n+\alpha+1)$, then:

$$R(\hat{\theta}_{Q_1},\theta) = \lambda_1^2\left(n(n+1) + \frac{2n\beta}{\theta} + \frac{\beta^2}{\theta^2}\right) - 2\lambda_1\left(n + \frac{\beta}{\theta}\right) + 1 \quad (26)$$

From (26), it›s clear that $R(\hat{\theta}_{Q_1},\theta)$ is not constant. That is, $\hat{\theta}_{Q_1}$ is not minimax estimator.

Now, when $\beta \longrightarrow 0$, $R(\hat{\theta}_{Q_1},\theta)$ becomes:

$$R(\hat{\theta}_{Q_1},\theta) = n(n+1)\lambda_1^2 - 2n\lambda_1 + 1 \quad (27)$$

From (27), the $R(\hat{\theta}_{Q_1},\theta)$ is constant. That is, $\hat{\theta}_{Q_1}$ is semi - minimax estimator when $\beta \longrightarrow 0$.

■ For $\hat{\theta}_{Q_2}$ (16) the risk function (19) will be:

$$R(\hat{\theta}_{Q_2},\theta) = \frac{1}{\theta^2(n+2)^2}E(S+b)^2 - \frac{2}{\theta(n+2)}E(S+b) + 1 \quad (28)$$

Depending on (22) and (23), we get:

$$E(S+b)^2 = n\theta^2 + n^2\theta^2 + 2bn\theta + b^2 \quad (29)$$

And

$$E(S+b)=n\theta+b \quad (30)$$

Substituting (29) and (30) in (28), we get:

$$R(\hat{\theta}_{Q_2},\theta) = \frac{1}{\theta^2(n+2)^2}(n\theta^2 + n^2\theta^2 + 2bn\theta + b^2) - \frac{2}{\theta(n+2)}(n\theta + b) + 1$$

Let $\lambda_2=1/(n+2)$, then:

$$R(\hat{\theta}_{Q_2},\theta) = \lambda_2^2\left(n(n+1) + \frac{2nb}{\theta} + \frac{b^2}{\theta^2}\right) - 2\lambda_2\left(n + \frac{b}{\theta}\right) + 1 \quad (31)$$

From(31), it's clear that $R(\hat{\theta}_{Q_2},\theta)$ is not constant. That is, $\hat{\theta}_{Q_2}$ is not minimax estimator.

Now, when $b \longrightarrow 0$, $R(\hat{\theta}_{Q_2},\theta)$ becomes:

$$R(\hat{\theta}_{Q_2},\theta)=n(n+1)\ \lambda_2^2-2n\lambda_2+1 \quad (32)$$

From (32), the $R(\hat{\theta}_{Q_2},\theta)$ is constant. That is, $\hat{\theta}_{Q_2}$ is semi-minimax estimator when $b \longrightarrow 0$.

■ For $\hat{\theta}_{Q_3}$(17) the risk function (19) will be:

$$R(\hat{\theta}_{Q_3},\theta) = \frac{1}{\theta^2(n+1)^2}E(S^2) - \frac{2}{\theta(n+1)}E(S) + 1 \quad (33)$$

Substituting (22) and (23) in (33), we get:

$$R(\hat{\theta}_{Q_3},\theta) = \frac{1}{\theta^2(n+1)^2}(n\theta^2 + n^2\theta^2) - \frac{2}{\theta(n+1)}(n\theta) + 1$$

Let $\lambda_3=1/(n+1)$, then:

$$R(\hat{\theta}_{Q_3},\theta) = n(n+1)\lambda_3^2 - 2n\lambda_3 + 1$$

From (34), it's clear that $R(\hat{\theta}_{Q_3},\theta)$ is constant. That is, $\hat{\theta}_{Q_3}$ is minimax estimator.

▪ For $\hat{\theta}_{Q_3}$(18) the risk function (19) will be:

$$R(\hat{\theta}_{Q_4},\theta) = \frac{1}{\theta^2(n+2k)^2}E(S^2) - \frac{2}{\theta(n+2k)}E(S) + 1 \quad (35)$$

Substituting (22) and (23) in (35), we get:

$$R(\hat{\theta}_{Q_4},\theta) = \frac{1}{\theta^2(n+2k)^2}(n\theta^2 + n^2\theta^2) - \frac{2}{\theta(n+2k)}(n\theta) + 1$$

Let $\lambda_4=1/(n+2k)$, then:

$$R(\hat{\theta}_{Q_4},\theta) = n(n+1)\lambda_4^2 - 2n\lambda_4 + 1 \quad (36)$$

From (36), it's clear that $R(\hat{\theta}_{Q_4},\theta)$ is constant. That is, $\hat{\theta}_{Q_4}$ is minimax estimator.

2.2. Estimators under Precautionary Loss Function

A very useful and simple an asymmetric precautionary loss function is as in [23]:

$$L(\hat{\theta}_P,\theta) = \frac{(\hat{\theta}_P - \theta)^2}{\hat{\theta}_P\theta} \quad (37)$$

Then, Bayes estimator of θ under precautionary loss function is obtained as:

$$\hat{\theta}_P = \sqrt{\frac{E_H(\theta|\underline{t})}{E_H\left(\frac{1}{\theta}|\underline{t}\right)}} \quad (38)$$

Now, Bayes estimators for the parameter θ under precautionary loss function corresponding to four posterior distributions are given respectively by:

$$\hat{\theta}_{P_1} = \frac{S+\beta}{\sqrt{(n+\alpha)(n+\alpha-1)}}$$

$$\hat{\theta}_{P_2} = \frac{S+b}{\sqrt{n(n+1)}}$$

$$\theta_{P_3} = \frac{S}{\sqrt{n(n-1)}}$$

$$\hat{\theta}_{P_4} = \frac{S}{\sqrt{(n+2k-1)(n+2k-2)}}$$

The risk function $R(\hat{\theta},\theta)$ under precautionary loss function (37) is:

$$R(\hat{\theta}_P,\theta)=E[L(\hat{\theta}_P,\theta)]=E\left(\frac{(\hat{\theta}_P-\theta)^2}{\hat{\theta}_P\theta}\right)=E\left(\frac{\hat{\theta}_P^2-2\hat{\theta}_P\theta+\theta^2}{\hat{\theta}_P\theta}\right)$$

$$\Rightarrow R(\hat{\theta}_P,\theta)=\frac{1}{\theta}E(\hat{\theta}_P)+\theta E\left(\frac{1}{\hat{\theta}_P}\right)-2 \quad (43)$$

- For $\hat{\theta}_{p_1}$ (39) the risk function (43) will be:

$$R(\hat{\theta}_{P_1},\theta)=\frac{E(S+\beta)}{\theta\sqrt{(n+\alpha)(n+\alpha-1)}}+\theta\sqrt{(n+\alpha)(n+\alpha-1)}E\left(\frac{1}{S+\beta}\right)-2 \quad (44)$$

From (25), $E(S+\beta)=n\theta+\beta$, and we have to find $E(1/(S+\beta))$;

$$E\left(\frac{1}{S+\beta}\right)=\int_0^\infty\frac{1}{S+\beta}f(s)\,dS=\int_0^\infty\frac{1}{S+\beta}\frac{S^{n-1}}{\Gamma(n)\,\theta^n}e^{-S/\theta}dS=\frac{1}{\beta}\int_0^\infty\frac{1}{\left(1+\frac{S}{\beta}\right)}\frac{S^{n-1}}{\Gamma(n)\,\theta^n}e^{-S/\theta}dS$$

$$Recall\ that:\frac{1}{1+x}=\sum_{m=0}^{\infty}(-1)^m x^m \Rightarrow \frac{1}{1+\frac{S}{\beta}}=\sum_{m=0}^{\infty}(-1)^m\left(\frac{S}{\beta}\right)^m$$

$$E\left(\frac{1}{S+\beta}\right)=\sum_{m=0}^{\infty}(-1)^m\frac{1}{\Gamma(n)\,\theta^n\beta^{m+1}}\int_0^\infty S^{n+m-1}e^{-S/\theta}dS$$

By using the transformation, $y=S/\theta$, we get:

$$E\left(\frac{1}{S+\beta}\right)=\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\frac{\theta^m}{\beta^{m+1}} \quad (45)$$

Substituting (25) and (45) in (44), we get:

$$R(\hat{\theta}_{P_1},\theta)=\frac{n\theta+\beta}{\theta\sqrt{(n+\alpha)(n+\alpha-1)}}+\theta\sqrt{(n+\alpha)(n+\alpha-1)}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\frac{\theta^m}{\beta^{m+1}}-2$$

Let $\lambda_5=\frac{1}{\sqrt{(n+\alpha)(n+\alpha-1)}}$, then:

$$R(\hat{\theta}_{P_1},\theta)=\lambda_5\left(n+\frac{\beta}{\theta}\right)+\frac{1}{\lambda_5}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\left(\frac{\theta}{\beta}\right)^{m+1}-2 \quad (46)$$

From (46), it's clear that $\hat{\theta}_{p_1}$ is not constant. That is, $\hat{\theta}_{p_1}$ is not minimax estimator.

Now, return to (44) and let $\beta\to 0$, we get:

$$R(\hat{\theta}_{P_1},\theta)=\frac{E(S)}{\theta\sqrt{(n+\alpha)(n+\alpha-1)}}+\theta\sqrt{(n+\alpha)(n+\alpha-1)}\,E\left(\frac{1}{S}\right)-2 \quad (47)$$

Since, $S=\sum_{i=1}^{n}\frac{1}{t_i}\sim Gamma(n,\theta)$, then $\frac{1}{S}\sim Inverted\ Gamma(n,\theta)$, with expected value equal to:

$$E\left(\frac{1}{S}\right)=\frac{1}{\theta(n-1)} \quad (48)$$

Substituting (22) and (48) in (47), we get:

$$R(\hat{\theta}_{P_1},\theta)=n\lambda_5+\frac{1}{(n-1)\lambda_5}-2 \quad (49)$$

From (49), $R(\hat{\theta}_{P_1},\theta)$ becomes constant. So, $\hat{\theta}_{p_1}$ is semi-minimax estimator when $\beta\to 0$.

- For the risk function will be:

$$R(\hat{\theta}_{P_2},\theta)=\frac{1}{\theta\sqrt{n(n+1)}}E(S+b)+\theta\sqrt{n(n+1)}E\left(\frac{1}{S+b}\right)-2 \quad (50)$$

From (30), $E(S+b)=n\theta+b$

Similar to (45), we can get:

$$E\left(\frac{1}{S+b}\right)=\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\frac{\theta^m}{b^{m+1}} \quad (51)$$

Substituting (30) and (51) in (50), we get:

$$R(\hat{\theta}_{P_2},\theta)=\frac{1}{\theta\sqrt{n(n+1)}}(n\theta+b)+\theta\sqrt{n(n+1)}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\frac{\theta^m}{b^{m+1}}-2$$

Let $\lambda_6=\frac{1}{\sqrt{n(n+1)}}$, then:

$$R(\hat{\theta}_{P_2},\theta)=\lambda_6\left(n+\frac{b}{\theta}\right)+\frac{1}{\lambda_6}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n+m)}{\Gamma(n)}\left(\frac{\theta}{b}\right)^{m+1}-2 \quad (52)$$

From (52), it's clear that $\hat{\theta}_{p_1}$ is not minimax estimator .

Now, return to (50), and let $b\to 0$, we get:

$$R(\hat{\theta}_{P_2},\theta)=\frac{1}{\theta\sqrt{n(n+1)}}E(S)+\theta\sqrt{n(n+1)}E\left(\frac{1}{S}\right)-2 \quad (53)$$

Substituting (22) and (48) in (53), we get:

$$R(\hat{\theta}_{P_2},\theta)=n\lambda_7+\frac{1}{(n-1)\lambda_7}-2 \quad (54)$$

From (54), $R(\hat{\theta}_{p_2},\theta)$ becomes constant. So, $\hat{\theta}_{p_2}$ is semi-minimax estimator when $b\to 0$.

- For $\hat{\theta}_{p_3}$ (41) the risk function (43) will be:

$$R(\hat{\theta}_{P_3},\theta)=\frac{1}{\theta\sqrt{n(n-1)}}E(S)+\theta\sqrt{n(n-1)}\,E\left(\frac{1}{S}\right)-2 \quad (55)$$

Substituting (22) and (48) in (55), we get:

$$R(\hat{\theta}_{P_3},\theta)=\frac{1}{\theta\sqrt{n(n-1)}}(n\theta)+\theta\sqrt{n(n-1)}\frac{1}{\theta(n-1)}-2$$

Let $\lambda_8=\frac{1}{\sqrt{n(n-1)}}$, then:

$$R(\hat{\theta}_{P_3},\theta)=n\lambda_8+\frac{1}{(n-1)\lambda_8}-2 \quad (56)$$

From (56), it's clear that $R(\hat{\theta}_{p_3},\theta)$ is constant. So, $\hat{\theta}_{p_3}$ is minimax estimator.

- For $\hat{\theta}_{p_4}$ (42) the risk function (43) will be:

$$R(\hat{\theta}_{P_4},\theta)=\frac{1}{\theta\sqrt{(n+2k-1)(n+2k-2)}}E(S)+\theta\sqrt{(n+2k-1)(n+2k-2)}\,E\left(\frac{1}{S}\right)-2 \quad (57)$$

Substituting (22) and (48) in (57), we get:

$$R(\hat{\theta}_{P_4},\theta)=\frac{n\theta}{\theta\sqrt{(n+2k-1)(n+2k-2)}}+\theta\sqrt{(n+2k-1)(n+2k-2)}\frac{1}{\theta(n-1)}-2$$

Let $\lambda_9=\frac{1}{\sqrt{(n+2k-1)(n+2k-2)}}$, then:

$$R(\hat{\theta}_{P_4},\theta)=n\lambda_9+\frac{1}{(n-1)\lambda_9}-2 \quad (58)$$

From (58), it's clear that $R(\hat{\theta}_{p_4},\theta)$ is constant. So, $\hat{\theta}_{p_4}$ is minimax estimator.

3. Simulation Experiment

Mean Square Error (MSE) and Mean

Absolute Percentage Error (MAPE) of the estimator have been considered to compare the performance of the obtained estimators $\hat{\theta}_{Q_1}\hat{\theta}_{Q_2}\hat{\theta}_{Q_3}\hat{\theta}_{Q_4}\hat{\theta}_{p_1}\hat{\theta}_{p_2}\hat{\theta}_{p_3}$ and $\hat{\theta}_{p_4}$ which appear in equations (15, 16, 17, 18, 39, 40, 41 and 42 respectively) . The MSE and MAPE of an estimator are defined as:

$$MSE(\hat{\theta}) = \frac{\sum_{j=1}^{L}(\hat{\theta}_j - \theta)^2}{L} \quad (59)$$

$$MAPE(\hat{\theta}) = \frac{\sum_{j=1}^{L}\frac{|\hat{\theta}_j - \theta|}{\theta}}{L} \quad (60)$$

where $\hat{\theta}_j$ is the estimate of at the j^{th} replicate (run).

In this simulation study, we simulate data bygenerating observations from IE distribution with θ=1,1.5 and 3. The values of hyper-parameters considered are (α,β)=(4,4),(3,2),(6,10) and b=3,5. The values of constant of the extension of Jeffreys' prior considered are k=1 and 3. The sample sizes considered are *n*=10, 15, 25, 30, 50and 100 with numberof repetitions as L=3000. The simulation program has been written by using MATLAB (R2011b) program and the results have been summarized in the Tables (1)…(4).

4. Conclusion and Recommendation

The most important conclusions of simulation experiment are:

1. From Table (1)…

- When (θ=1), the performance of semi-mini max estimates corresponding to inverted gamma prior with hyper-parameters (α=β=4) and (α=6,β=10) under quadratic loss function is better than that under precautionary loss function for all sample sizes while the reverse is true with hyper-parameters (α=3,β=2) .
- When (θ=1.5), the performance of semi-mini max estimates corresponding to inverted gamma prior only with hyper-parameters (α=6,β=10) under quadratic loss function is better than that under precautionary loss function for all sample sizes while the reverse is true with hyper-parameters (α=β=4) and (α=3,β=2).
- When (θ=3), the performance of semi-mini max estimates corresponding to inverted gamma prior with all different values of hyper-parameters (α,β) under precautionary loss function is better than that under quadratic loss function for all sample sizes.

2. From Table (2)… In general the performance of semi-mini max estimates corresponding to Gumbel type II prior under quadratic loss function is better than that under precautionary loss function for all sample sizes and different values of θ and b .

3. From Table (3)…the performance of mini max estimators corresponding to Jeffreys' prior under quadratic loss function is better than that under precautionary loss function for all sample sizes and different values of θ.

4. From Table (4)…the performance of mini max estimators corresponding to extension of Jeffreys' prior under precautionary loss function is better than that under quadratic loss function for all sample sizes and different values of θ. The MSE and MAPE values are increases as extension constant (k) increases.

5. Informative Gumbel type II prior doesn't record any appearance as the best prior with θ=1,1.5. While with θ=3, record appearance when b=5 for one time under quadratic loss function.

6. Non-informative prior distributions didn't record any appearance as the best prior.

7. The MSE and MAPE values associated with each estimator under each prior and every loss function, reduce with the increase in the sample size and this conforms to the statistical theory. For large sample size (n=100), all the estimators have approximately the same MSE values and the same MAPE values.

9. Bayes estimators under quadratic and precautionary loss functions have been introduced semi-mini max estimators corresponding to informative priors and introducedminimax estimators corresponding to non-informative priors.

10. The simulation experiment results show a convergence between most of the estimators to true values of the parameter (θ) with increasing the sample size.

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Table (1): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Inverted Gamma Prior when θ =1

n	α	β	.Est		MSE		MAPE	
			Quadratic	Prec.	Quadratic	Prec.	Quadratic	Prec.
10	4	4	0.9311527	1.0353245	0.0479455	0.0546613	0.1778571	0.1806594
	3	2	0.8548065	0.9581501	0.0706794	0.0640672	0.2227816	0.2021860
	6	10	1.1745465	1.2888831	0.0641041	0.1239587	0.1983000	0.2947493
15	4	4	0.9468996	1.0240494	0.0381407	0.0418895	0.1596565	0.1620730
	3	2	0.8914733	0.9682806	0.0509149	0.0471774	0.1867137	0.1758494
	6	10	1.1335451	1.2168494	0.0470252	0.0806627	0.1698251	0.2309833
25	4	4	0.9650478	1.0159962	0.0287544	0.0307725	0.1375326	0.1394416
	3	2	0.9293598	0.9802142	0.0344544	0.0331686	0.1519210	0.1468509
	6	10	1.0922323	1.1461027	0.0327055	0.0479906	0.1407612	0.1734556
30	4	4	0.9685032	1.0119817	0.0253298	0.0267155	0.1279867	0.1295192
	3	2	0.9381650	0.9815811	0.0296140	0.0285720	0.1399297	0.1352774
	6	10	1.0783138	1.1239879	0.0279108	0.0390347	0.1303549	0.1559226
50	4	4	0.9830651	1.0106718	0.0169812	0.0177591	0.1043422	0.1054629
	3	2	0.9642330	0.9918274	0.0185977	0.0183906	0.1099594	0.1081444
	6	10	1.0538347	1.0823607	0.0184416	0.0231796	0.1060375	0.1193418

100	4	4	0.9910747	1.0054499	0.0092333	0.0094508	0.0763532	0.0768837
	3	2	0.9813735	0.9957469	0.0096774	0.0096239	0.0784208	0.0778333
	6	10	1.0286247	1.0432615	0.0096340	0.0109388	0.0773143	0.0824785
when θ =1.5								
10	4	4	1.2613952	1.4025125	0.1554513	0.1312993	0.2215559	0.1970315
	3	2	1.2086377	1.3547585	0.1979878	0.1631898	0.2517320	0.2214860
	6	10	1.4659369	1.6086390	0.0778619	0.1041640	0.1498185	0.1656851
15	4	4	1.3272076	1.4353433	0.1193708	0.1088747	0.1914698	0.1770340
	3	2	1.2917975	1.4030958	0.1425323	0.1264017	0.2106424	0.1922306
	6	10	1.4792796	1.5879920	0.0744075	0.0929936	0.1446640	0.1568105
25	4	4	1.3898604	1.4632362	0.0748710	0.0708913	0.1485077	0.1435135
	3	2	1.3688211	1.4437228	0.0843497	0.0778580	0.1581488	0.1509428
	6	10	1.4904941	1.5640073	0.0552332	0.0648134	0.1252406	0.1335964
30	4	4	1.4004515	1.4633213	0.0666228	0.0632645	0.1419200	0.1358748
	3	2	1.3828177	1.4468113	0.0738297	0.0686182	0.1498729	0.1420644
	6	10	1.4869136	1.5498947	0.0509188	0.0576270	0.1210643	0.1260191
50	4	4	1.4380260	1.4784090	0.0410075	0.0397496	0.1098316	0.1070809
	3	2	1.4276190	1.4684747	0.0437950	0.0417882	0.1136652	0.1100731
	6	10	1.4928321	1.5332411	0.0346556	0.0376080	0.0996362	0.1028308
100	4	4	1.4696062	1.4909223	0.0207271	0.0204643	0.0771494	0.0762171
	3	2	1.4645063	1.4859558	0.0214457	0.0209788	0.0785981	0.0772507
	6	10	1.4982117	1.5195304	0.0190731	0.0199979	0.0734731	0.0749515
when θ =3								
10	4	4	2.2718121	2.5259688	0.9333990	0.7230947	0.2783443	0.2361123
	3	2	2.2912273	2.5682299	0.9651487	0.7678794	0.2807246	0.2412329
	6	10	2.3574813	2.5869710	0.7266947	0.5485381	0.2455979	0.2056542
15	4	4	2.4694196	2.6706183	0.6099523	0.4926288	0.2193933	0.1909807
	3	2	2.4941259	2.7090140	0.6198274	0.5140019	0.2196254	0.1937068
	6	10	2.5176542	2.7026768	0.5040928	0.4011980	0.1994484	0.1723515
25	4	4	2.6301009	2.7689534	0.4008607	0.3460326	0.1757846	0.1607414
	3	2	2.6518285	2.7969361	0.4037821	0.3555628	0.1758139	0.1623901
	6	10	2.6532196	2.7840800	0.3523190	0.3021396	0.1647980	0.1502015
30	4	4	2.6890147	2.8097313	0.3081752	0.2670780	0.1522526	0.1401843
	3	2	2.7092798	2.8346590	0.3086035	0.2726432	0.1519009	0.1413889
	6	10	2.7058247	2.8204351	0.2757594	0.2378329	0.1440227	0.1322869
50	4	4	2.7962339	2.8747585	0.1896393	0.1722400	0.1180807	0.1119844
	3	2	2.8109790	2.8914238	0.1893843	0.1743647	0.1178365	0.1125170
	6	10	2.8033836	2.8792676	0.1765648	0.1600501	0.1139375	0.1079490

100	4	4	2.8927754	2.9347340	0.0953449	0.0905575	0.0831168	0.0804804
	3	2	2.9013598	2.9438538	0.0951979	0.0911423	0.0829207	0.0806259
	6	10	2.8947796	2.9359706	0.0918139	0.0871566	0.0815632	0.0789547

Table (2): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Gumbel Type II Prior

n	*b*	Est.		MSE		MAPE	
		Quadratic	Prec.	Quadratic	Prec.	Quadratic	Prec.
when θ =1							
10	3	1.0806075	1.2363826	0.0740063	0.1442518	0.2080071	0.2932024
	5	1.2472742	1.4270752	0.1286533	0.2707683	0.2809250	0.4356989
15	3	1.0551760	1.1578924	0.0519316	0.0837984	0.1791007	0.2253779
	5	1.1728231	1.2869919	0.0787550	0.1412327	0.2197736	0.3056608
25	3	1.0352383	1.0963455	0.0352328	0.0474047	0.1480871	0.1692578
	5	1.1093124	1.1747919	0.0459403	0.0686745	0.1668281	0.2074985
30	3	1.0280504	1.0787552	0.0299018	0.0382602	0.1364269	0.1528273
	5	1.0905504	1.1443378	0.0373143	0.0528912	0.1507229	0.1815009
50	3	1.0205496	1.0509145	0.0190985	0.0223964	0.1089521	0.1169694
	5	1.0590111	1.0905204	0.0221586	0.0279981	0.1162334	0.1311610
100	3	1.0104201	1.0255137	0.0098086	0.0106429	0.0782232	0.0812884
	5	1.0300279	1.0454144	0.0106017	0.0120544	0.0811042	0.0865823
when θ =1.5							
10	3	1.4934106	1.7086934	0.1539794	0.2450691	0.2085206	0.2527303
	5	1.6600773	1.8993859	0.1795607	0.3610253	0.2168429	0.3116558
15	3	1.5025972	1.6488679	0.1239010	0.1713510	0.1853802	0.2119789
	5	1.6202442	1.7779674	0.1383529	0.2264552	0.1908256	0.2445856
25	3	1.5072523	1.5962211	0.0775098	0.0961297	0.1479660	0.1620731
	5	1.5813264	1.6746676	0.0840711	0.1173799	0.1518473	0.1787416
30	3	1.5004939	1.5745002	0.0678453	0.0802528	0.1390423	0.1483617
	5	1.5629939	1.6400828	0.0718133	0.0943257	0.1403658	0.1590391
50	3	1.5017582	1.5464407	0.0415819	0.0462466	0.1089070	0.1138163
	5	1.5402198	1.5860466	0.0431965	0.0514939	0.1101012	0.1193977
100	٣	1.5030260	1.5254781	0.0209945	0.0222661	0.0770138	0.0790427
	٥	1.5226339	1.5453789	0.0214976	0.0236762	0.0776885	0.0812787
when θ =3							
10	3	2.7564318	3.1537855	0.6892339	0.8482574	0.2238300	0.2367366
	5	2.9230985	3.3444780	0.6358223	0.9432726	0.2107902	0.2457874
15	3	2.8463760	3.1234572	0.4781840	0.5626362	0.1839184	0.1942883
	5	2.9640230	3.2525567	0.4558780	0.6111794	0.1773281	0.2010251

25	3	2.8852973	3.0556081	0.3391264	0.3686798	0.1568511	0.1596662
	5	2.9593714	3.1340546	0.3276203	0.3835581	0.1525753	0.1611544
30	3	2.9098598	3.0533780	0.2610966	0.2813897	0.1377458	0.1412217
	5	2.9723598	3.1189606	0.2537353	0.2926921	0.1350829	0.1430739
50	3	2.9383243	3.0257496	0.1695062	0.1763725	0.1105019	0.1114259
	5	2.9767859	3.0653555	0.1662412	0.1799808	0.1089866	0.1119340
100	3	2.9680531	3.0123897	0.0898732	0.0916805	0.0797486	0.0799560
	5	2.9876610	3.0322905	0.0890048	0.0925696	0.0791090	0.0801167

Table (3): Estimated, MSE and MAPE Values for Bayes Estimator of θ with JeffreysPrior

n	Est.		MSE		MAPE	
	Quadratic	Prec.	Quadratic	Prec.	Quadratic	Prec.
when θ =1						
10	0.9061173	1.0506447	0.0891549	0.1105789	0.2425324	0.2569280
15	0.9336245	1.0308194	0.0595948	0.0682280	0.1995706	0.2068347
25	0.9596706	1.0186381	0.0382825	0.0416465	0.1586915	0.1622167
30	0.9644391	1.0136240	0.0322882	0.0344543	0.1445012	0.1470861
50	0.9817369	1.0115381	0.0197494	0.0207456	0.1125259	0.1139861
100	0.9907213	1.0056695	0.0099791	0.0102259	0.0793771	0.0799746
when θ =1.5						
10	1.3564480	1.5728039	0.2038037	0.2515979	0.2465808	0.2623490
15	1.4090095	1.5556943	0.1481443	0.1736039	0.2073554	0.2170554
25	1.4498390	1.5389251	0.0860461	0.0956256	0.1577799	0.1635494
30	1.4521227	1.5261787	0.0745850	0.0805397	0.1478349	0.1504071
50	1.4723809	1.5170759	0.0439882	0.0461811	0.1127834	0.1144385
100	1.4882045	1.5106588	0.0215420	0.0221673	0.0782363	0.0790295
when θ =3						
10	2.7342892	3.1704133	0.8202453	1.0368942	0.2441782	0.2617293
15	2.8367745	3.1320966	0.5398249	0.6430432	0.1954133	0.2077062
25	2.8808857	3.0579033	0.3657147	0.3994059	0.1628838	0.1661859
30	2.9069520	3.0552020	0.2782132	0.3007974	0.1421892	0.1450104
50	2.9371150	3.0262729	0.1762187	0.1835715	0.1126686	0.1136772
100	2.9677368	3.0125146	0.0916617	0.0935326	0.0805381	0.0807596

Table (4): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Extension of JeffreysPrior

n	k	.Est		MSE		MAPE	
		Quadratic	Prec.	Quadratic	Prec.	Quadratic	Prec.
when θ =1							

10	1	0.8306075	0.9503439	0.0962025	0.0908408	0.2599119	0.2407388
	3	0.6229557	0.6878084	0.1801361	0.1437553	0.3864170	0.3352891
15	١	0.8787054	0.9642433	0.0635996	0.0601469	0.2086800	0.1985509
	٣	0.7113330	0.7663029	0.1153658	0.0917944	0.3008867	0.2603757
25	١	0.9241272	0.9786758	0.0397478	0.0385770	0.1631744	0.1583709
	٣	0.8048850	0.8459329	0.0638550	0.0522189	0.2170437	0.1923466
30	١	0.9343004	0.9803813	0.0334314	0.0324427	0.1486753	0.1441493
	٣	0.8304892	0.8666885	0.0517383	0.0428255	0.1937189	0.1728843
50	١	0.9628573	0.9915056	0.0200558	0.0198763	0.1141886	0.1124276
	٣	0.8940818	0.9187282	0.0273222	0.0236087	0.1376884	0.1261581
100	١	0.9810083	0.9956625	0.0100607	0.0100108	0.0799584	0.0793823
	٣	0.9439891	0.9575502	0.0121189	0.0110436	0.0895179	0.0847681
when θ =1.5							
10	1	1.2434106	1.4226546	0.2197740	0.2074984	0.2617065	0.2455746
	3	0.9325580	1.0296418	0.4085794	0.3267929	0.3869717	0.3360311
15	1	1.3261266	1.4552187	0.1541262	0.1511947	0.2160998	0.2069680
	3	1.0735310	1.1564906	0.2630672	0.2122235	0.3025139	0.2652806
25	1	1.3961412	1.4785514	0.0882438	0.0873312	0.1607375	0.1576935
	3	1.2159940	1.2780077	0.1394173	0.1141843	0.2124475	0.1880813
30	1	1.4067439	1.4761263	0.0765418	0.0752725	0.1515724	0.1475531
	3	1.2504390	1.3049429	0.1158867	0.0964282	0.1937967	0.1741877
50	1	1.4440659	1.4870319	0.0447075	0.0442581	0.1144959	0.1127298
	3	1.3409184	1.3778823	0.0611581	0.0527677	0.1372123	0.1261092
100	1	1.4736143	1.4956270	0.0216815	0.0216361	0.0787911	0.0782868
	3	1.4180062	1.4383768	0.0261544	0.0237911	0.0878961	0.0833443
when θ =3							
10	1	2.5064318	2.8677468	0.8735180	0.8420984	0.2584778	0.2447265
	3	1.8798239	2.0755227	1.6091181	1.2865955	0.3851769	0.3333847
15	1	2.6699054	2.9298081	0.5635461	0.5523214	0.2035726	0.1957630
	3	2.1613520	2.3283753	1.0012323	0.7968025	0.2957070	0.2554565
25	1	2.7741862	2.9379385	0.3769615	0.3694391	0.1674893	0.1629612
	3	2.4162267	2.5394505	0.5880669	0.4852460	0.2193661	0.1951234
30	1	2.8161098	2.9550041	0.2867869	0.2805651	0.1451324	0.1422221
	3	2.5032087	2.6123182	0.4466802	0.3679800	0.1889287	0.1679385
50	1	2.8806320	2.9663407	0.1799510	0.1768424	0.1143918	0.1125327
	3	2.6748726	2.7486086	0.2485838	0.2140593	0.1376879	0.1260811
100	1	2.9386414	2.9825386	0.0926175	0.0918319	0.0813357	0.0804162
	3	2.8277493	2.8683718	0.1119436	0.1019800	0.0912363	0.0863490