δ-Divisor Graphs

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الخلاصة

في هذا البحث ، قدمنا نتائج لنوع من بطاقات الرسومات البيانية ، والتي أطلقنا عليها رسم بياني δ-divisor وهو عبارة عن رسم بياني مقسّم معدّل.

الكلمات الفتاحية

بطاقات الرسومات البيانية، رسم بياني δ-divisor ، رسم بياني مقسّم معدّل.

Abstract

In this paper, we introduce results for a kind of labelings of graphs, which we named it the δ-divisor graph which is a modified divisor graph.

Keywords

labelings of graphs, δ-divisor graph, modified divisor graph

1. Introduction

G. Santhosh and G. Singh [1,4] called a graph G(V,E) with vertex set Vand edge set E a divisor graph if Vis labeled by a set of integers and for each edge uv∈E either the label assigned to u divides the label assigned to v or vice versa.Here, we studythe notion "divisor graph" in the sense thatits vertices can be labeled with distinct integers 1,2,…,|V| such that for each edge uv∈E either the label assigned to u divides the label assigned to v or vice versa and we named it δ-divisor graph. A graph which is not aδ-divisor is called a non-δ-divisor graph.

We introduce a method to calculate the number of vertices of degree 2 in the maximal δ-divisor graph of n vertices. We prove the following graphs are non-δ-divisor graphs: the $\langle S_{n_1}, S_{n_1}, S_{n_1}\rangle$is a non-δ -graph if and only if $d_l > \left\lfloor \frac{n}{3} \right\rfloor - 1$ or $n_l - \left| \frac{\left\lfloor \frac{n}{3} \right\rfloor - 1}{2} \right| + n_k + 3 > \left\lfloor \frac{n}{2} \right\rfloor$, where $n = n_1 + n_2 +$, where $n=n_1+n_2+n_3+5$, n_j, n_k, n_l are the number of the pendant vertices of the star S_{n_i} ,i=1,2,3 where the degrees of their central vertices ared_j, d_k, d_l respectively, $d_j \geq d_k \geq d_l$.(G=$\langle S_{n_1}, S_{n_2}, \ldots, S_{n_t}\rangle$ is the graph obtained by joining the central vertices of each star$S_{n_{m-1}}$ and the starS_m to a new vertex x_{m-1},where $2 \leq m \leq t$);P_n except P_1 ,P_2 ,P_3 ,P_4 and P_6; G=w S_m ,m >1,w ≥ 4 (the union of w stars each of m vertices);and hence every graph can be embedded as an induced subgraph of a δ-divisor graph.

Any notion or definition which is not found here could be found in [1], [2].

1.1. Definition [2]

Let x be a non-negative real number .The Gauss function π(x) is defined to be the number of primes not exceeding x . i.e.,π(x)=|{p:pisprime ,p ≤ x}|.

1.2. Lemma [5]

The number of vertices of degree 1 in the maximal divisor graph is

$\pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$, where π is the Gauss's function.

2 δ-divisor graphs

2.1. Definition

A graph G(V,E)with vertex set V is said to be δ-divisor if its vertices can be labeled with distinct integers 1,2,…,|V| such that for each edge uv∈E either the label assigned to u divides the label assigned to v or vice versa. A graph which is not δ-divisor is called a non-δ-divisor graph.

2.2. Definition

A maximal δ-divisor graph of n vertices is a δ-divisor graph such that adding any new edge yields a non -δ-divisor graph.

2.3. Method

A method to calculate the number of vertices of degree 2 in the maximal δ-divisor graph of n vertices:

Explanation of method: Let the number of vertices of degree 2 in the maximal δ-divisor graph of n vertices be M(n). There are two kinds of vertices of degree 2:

Kind1. Let p_i be the prime less than or equal to $\left\lfloor \frac{n}{2} \right\rfloor$, i = 1,2,…,k, where

$$k = \pi\left(\left|\frac{n}{2}\right|\right), p_j < p_{j+1}, j = 1,2,\ldots,k-1. \text{ If } 3p_i > n \ldots (1)$$

, then the vertex which is labeled by p_i has degree 2, because p_i is joined only with 1 and $2p_i$. Let p_{k-u_1}, $0 \le u_1 \le k$, be the smallest prime number satisfying (1), then the number of vertices of degree 2 in this case isu_1+1.

Kind 2. Let $p_i \le \left\lfloor \frac{n}{2} \right\rfloor$, such that $\left\lfloor \frac{n}{2} \right\rfloor < p_i^2 \le n$, i =1,2,…,k.… (2).

It is clear that the degree of the vertices labeled by p_i^2 is 2, since p_i^2 is joined with 1 and p_i ($2p_i^2 > n$). Let u_2 be the number of the prime numbers which are satisfying (2),$0 \le u_2 \le k$, therefore

(n)=u_1+u_2+1.

□

2.4. Example

G(V,E),|V|=n=10

Prime numbers are 2,3,5,7

Kind 1: $\pi\left(\left\lfloor \frac{10}{2} \right\rfloor\right)$=3,$p_i \le \left\lfloor \frac{10}{2} \right\rfloor$, $i.e.\ p_i \le 5$, thenthe prime satisfying condition (1) is $p_3=5$, then $p_3 = p_{3-0}$, therefore $u_1=0$.

Then the number of vertices of degree 2 in this case is $u_1+1=1$

Kind 2: If $5 < p_i^2 \le 10$, the only prime satisfying condition(2) is 3 , so $u_2=1$. Therefore ,M(n)= u_1+u_2+1= 0+1+1=2.

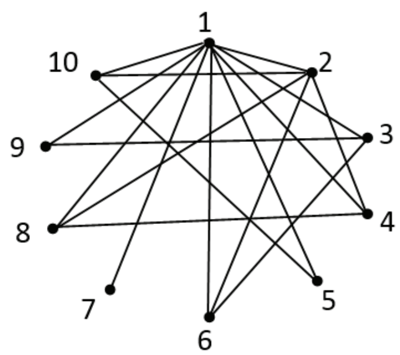

Fig. (1): Maximal δ-divisor graph of order 10

2.5.Remark

If G(V,E) is a connected graph of n vertices and degree $(v) < \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$,

for every v∈V, then G is a non-δ-divisor graph.

Proof.By hypothesis, there is no vertex v∈G such that degree (v) $\ge \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$, so there is at least one isolated vertex whose label is a prime number, since all $\pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$, vertices of prime labels can be joined with only the vertex of label one. Thus, we get the result.

2.6. Theorem

The pathP_n with n vertices is a non-δ-divisor graph exceptP_1 ,P_2 ,P_3 ,P_4 andP_6.

proof.

(i) It is clear thatP_1 ,P_2 ,P_3 ,P_4 andP_6 are δ-divisor graphs.

(ii) For all P_n ;n=5,or 10 or $n \ge 7$,it is clear that P_5 and $P_1$0are a non- δ-divisor graphs. For allP_n ;$n \ge 7$ except n=10, $\pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \ge 2$, there are at least two vertices prime numbered labels such that their labels greater than $\left\lfloor \frac{n}{2} \right\rfloor$and less than or equal to n. So we must put these vertices as pendantvertices and join them with a vertex of label one and this is impossible. Thus, we get the result.

2.7.Theorem G=w S_m is a non-δ-divisor graph , w≥4,m>1.

Proof. Let v_i ,i=1,…,w be the central vertices of the stars.The labeling of the centers of the stars must be labeled from 1 to w, since the vertex labeled1 can be joined with any other vertex, and the vertex labeled2 can be joined with$\left\lfloor \frac{n}{2} \right\rfloor$-1 vertices, where n=w(m+1), the remainingvertices labeled 3,…,n are joined with less than $\left\lfloor\frac{n}{2}\right\rfloor$-1vertices.Now suppose that 1,2,…,r-1 be the labels of the centers ofthe stars S_m^i,i=1,2,…,r-1, and let s be the label of the center of the starS_m^r,where n ≥ s > r and r ≤ w ,s > w. The number of vertices that can be joined with the vertex labeledr is greater than or equal to the number of vertices that can be joined with the vertex labeled s since:

1) $|M_1| \geq |M_2|$, where M_1 is a set of the multiples of r other than r from r to sand M_2 is a set of the δ-divisor s of s other than s from r to s, i.e.

$M_1 = \left\{jr: 2 \leq j \leq \left\lfloor\frac{s}{r}\right\rfloor\right\}$ and

$M_2 = \left\{\frac{s}{k} : \frac{s}{k} \textit{ is anintegerand } 2 \leq k \leq \left\lfloor\frac{s}{r}\right\rfloor\right\}$

2) From s+1 to n, the number of the multiples of r is greater than or equal tothe number of the multiples of s, since the nearest multiple of s is 2s and in this range there is at least one multiple of r. Therefore, we must label the center of the star S_m^r by label r. We continue with the same manner to other labels. So that let $f(v_i)=i$,i=1,…,w.

Case 1. If w is even, then w/2 of the central vertices are labeled by even numbers, so all vertices of these stars must have even labels, and the number of these vertices is, where n is the number of vertices of G, n = $\left\lfloor\frac{n}{2}\right\rfloor$ wm + w. The other adjacent vertices with v_i would be labeled by odd numbers, but this means that one vertex of these vertices would be labeled by (2m+1)(w-1) > n, this is impossible.

2.8.Definition

Consider t of stars namely S_{n_1},S_{n_2} ,…,S_{n_t} then

$G=\langle S_{n_1}, S_{n_2}, \ldots, S_{n_t}\rangle$ is the graph obtained by joining the central vertices of each S_{m-1} and S_m to a new vertex $x_{m-1}$where 2 ≤ m ≤ t.

2.9.Lemma

The graph $\langle S_{n_1}, S_{n_2}, S_{n_3}\rangle$ is a δ-divisor graph if $n_l \leq \left\lfloor\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rfloor$, where $n=n_1+n_2+n_3+5$, n_j, n_k, n_l are the number of the pendant vertices of the star S_{n_i},i=1,2,3 where the degrees of their central vertices are d_j ,d_k,d_l respectively, $d_j \geq d_k \geq d_l$.

Proof. Let c_i be the central vertex of S_{n_i}fori = 1,2,3. Now c_1 and c_2 are adjacent tox_1, c_2 and c_3 are adjacent to x_2. Let d_i= deg c_i, i=1,2,3, where deg $c_i=n_i+1$,i=1,3 and deg $c_2=n_2+2$. Let d_j, d_l be the maximum and the minimum numbers of the set $\{d_i$,i=1,2,3$\}$ respectively, and the third bed_k. Let n_j, n_k, n_l be the number of pendant vertices of the stars where the degrees of their central vertices are d_j,d_k, d_l respectively.

We will label the central vertices of degrees d_j, d_k,d_l by the labels 1,2,3 respectively, (since any label which is greater than 3 can be joined with a number of vertices less than or equal to the number of vertices which can be joined with the vertex labeled 3).

If n_l is less than or equal to the number of the odd multiples of 3,other than 3 which is equal to $\left\lfloor\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rfloor$, then we assign the odd multiples of 3,other than 3 to the pendant vertices of S_{n_l} , and the even labels to the pendant vertices of S_{n_k} and the vertices x_1 and x_2 ,the remaining labels are assigned to the vertices of S_{n_j}.Hence, the graph is a δ-divisor graph.

2.10. Example

In Fig.(2)we give labeling for:

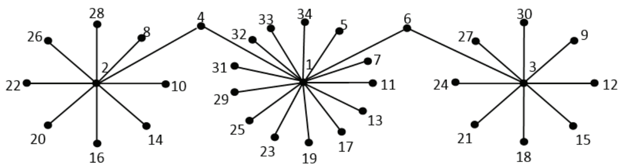

Fig. (2): (S_8,S_{13}, S_8)

2.11. Theorem

The graphs

(i) G=$\langle S_{n_1}, S_{n_2}\rangle$is a δ-divisor graph if and only if $n_1 \neq n_2$.

(ii) G= $\langle S_{n_1}, S_{n_2}$),$S_{n_3}\rangle$ is a non-δ-divisor graph if and only if $d_l > \left\lfloor\frac{n}{3}\right\rfloor$- 1 or $n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil + n_k + 3 > \left\lfloor\frac{n}{2}\right\rfloor$,where n= $n_1+n_2+n_3+5$, n_j, n_k, n_l are the number of the pendant vertices of the star S_{n_i} ,i=1,2,3 where the degrees of their central vertices are d_j ,d_k,d_l respectively, $d_j \geq d_k \geq d_l$.

Proof.(i) If $n_1 > n_2$, let $v_1^{(i)}$,$v_2^{(i)}$,…,$v_n^{(i)}$ be the pendant vertices of the starS_{n_i} and let c_i be the central vertex of S_{n_i} fori = 1,2. Now $c_1$and c_2 are adjacent tox, where x is the label of a vertex join the centers vertices of $S_{n_1}^{(1)}$ and $S_{n_2}^{(2)}$ .We define the labeling functionf:V$\langle S_{n_1}$ $S_{n_2}\rangle \rightarrow$ {1,2,…,n_1+n_2+3}as followsf(c_1)=1,f(C_2)=2 ,f(x)=4 , the vertices $v_1^{(2)}$,$v_2^{(2)}$,…,$v_{n_2}^{(2)}$ will be labeled by even numbers, and the remaining labels are assigned to the vertices $v_1^{(1)}$,$v_2^{(1)}$,…,$v_{n_1}^{(1)}$.

Conversely, If $n_1 = n_2$, we have two vertices of degree n_1+1, but we have only one label "1" divides n_1+1 numbers, since the number of vertices of this graph is

$2n_1 + 3$ and $|A_i| - 2 \leq |A_2| - 2 = (\left\lceil\frac{2n_1+3}{2}\right\rceil$-1) < n_1+1, where A_i={k:k|i or i|k :k ≤ $2n_1+3$} ,i ≥ 2. So the graph is a non-δ-divisor graph.

(ii) Let c_i be the central vertex of S_{n_i} fori = 1,2,3. Now c_1 and c_2 are adjacent to x_1, c_2 and c_3 are adjacent to x_2. Let d_i=deg c_i, i=1,2,3, where deg c_i= n_i +1, i=1,3 and deg c_2=n_2 +2. Let d_j , d_l be the maximum and the minimum numbers of the set {d_i,i=1,2,3} respectively, and the third bed_k. Let n_j, n_k, n_l be the number of pendant vertices of the stars where the degrees of their central vertices are d_j ,d_k, d_l respectively.

We will label the central vertices of degrees d_j ,d_k,d_l by the labels 1,2,3 respectively, (since any label which is greater than 3 can be joined with a number of vertices less than or equal to the number of vertices which can be joined with the vertex labeled 3).

Now if$d_l > \left\lfloor\frac{n}{3}\right\rfloor - 1$ or$n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil + n_k + 3 > \left\lfloor\frac{n}{2}\right\rfloor$, then there are two conditions:

Condition1. If $d_l > \left\lfloor\frac{n}{3}\right\rfloor - 1$, $\left\lfloor\frac{n}{3}\right\rfloor - 1$is the maximum number of labels which can be joined with the central vertex of S_{n_l}.since label 1 is used to label the central vertex of S_{n_j}. Thus G is a non- usual δ-divisor graph.

Condition 2.If $n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil + n_k + 3 > \left\lfloor\frac{n}{2}\right\rfloor$, $\left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil$is the number of odd labels which can be joined with the central vertex of S_{n_l}. If$n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil \leq 0$, then by Lemma 2.9 the graph is δ-divisor , which is a contradiction, so $n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil > 0$. Let all odd multiples of 3, other than 3 be assigned to the pendant vertices of S_{n_l}, then $n_l - \left\lceil\frac{\left\lfloor\frac{n}{3}\right\rfloor-1}{2}\right\rceil$is the minimum number of even labels which are assigned to the remaining pendant vertices of

S_{n_l}. Therefore, we need $n_l - \left[\frac{\lfloor \frac{n}{3} \rfloor - 1}{2}\right] + n_k + 3$ even labels to label the vertices of the graph,since the vertices of $S_{n_k}$and the vertices x_1 and x_2 must be even labels, hence the result.

Conversely, let G be a non- δ-divisor graph, the vertices which are joined with the central vertex of S_{n_j} can be labeled by any labels. The central vertex of $S_{n_k}$and the vertices which are joined with it need at most n_k+3 even labels and $n_k + 3 \leq \lfloor \frac{n}{2} \rfloor$, so there is no problem to label all the vertices which are joined with the central vertex of S_{n_k}. Thus, we discuss the problem that could occur when we label the adjacent vertices of the central vertex of S_{n_l} , which is labeled 3. Again if $n_l - \left[\frac{\lfloor \frac{n}{3} \rfloor - 1}{2}\right] \leq 0$,then by Lemma 2.9 the graph is δ-divisor which is a contradiction. Let all odd multiples of 3, other than 3 be assigned to the pendant vertices of S_{n_l}, then we need $n_l - \left[\frac{\lfloor \frac{n}{3} \rfloor - 1}{2}\right]$ even labels to label the remaining pendant vertices of S_{n_l}, so there are two cases that depend ond_l:

Case 1. If$d_l > \lfloor \frac{n}{3} \rfloor - 1$, hence the result.

Case 2. If$d_l \leq \lfloor \frac{n}{3} \rfloor - 1$ so we have n_l even labels, if $n_l - \left[\frac{\lfloor \frac{n}{3} \rfloor - 1}{2}\right] + n_k + 3$

$n_l - \left[\frac{\lfloor \frac{n}{3} \rfloor - 1}{2}\right] + n_k + 3 \leq \lfloor \frac{n}{2} \rfloor$, then the graph is a δ-divisor graph, which is a contradiction, hence the result.□

2.12. Corollary

$\langle S_{n_1}, S_{n_2}, S_3 \rangle$is a non-δ-divisor graph if

(i)$n_1 = n_2 = n_3$

(ii)$d_j = d_k = d_l$, where d_j, d_k, d_l are the degree of their central vertices respectively, $d_j \geq d_k \geq d_l$.

2.13. Theorem

Every graph G(n,q) can be embedded as an induced subgraph of a δ-divisor graph.

Proof .Let G(n,q) be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$. We shall establish an embedding of G in H, where $V(H)=\{v_1,v_2,\ldots,v_n,v_{n+1},\ldots,v_{2^{n-1}}\}$. Let $f(v_{i+1})=2^i, i=0,1,\ldots,n-1$, other vertices are labeled from the set $\{\{1,2,\ldots,2^{n-1}\}-2^i\}, i=0,1,\ldots,n-1$ and join all vertices of V(H)-V(G) with a vertex of label one. It is clear that H is a δ-divisor graph and$E(H)=q+2^{n-1}-n$.

2.14.Corollary

Every bipartite graph can be embedded into a bipartite δ-divisor graph.

as an example, see Fig (3)

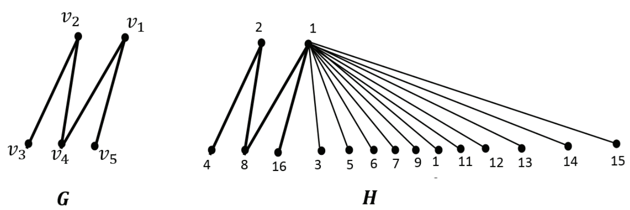

Fig. (3)

Reference

[1] J.A.Gallian, A Dynamic Survey of Graph Labeling,The Electronic Journal of Combinatorics, 18,#DS7, (2015)

[2] F. Harary, Graph Theory, Addison-Wesely, Reading, Massachusetts, (1969).

[3] G.H.Hardy, and E.M.Wright,An Introduction to the Theory of Numbers, 5 th ed., Clarendon Press.,Oxford, (2002).

[4] G. Santhosh and G. Singh, On divisor graphs, preprint.