Co Compressible Acts

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الخلاصة

الهدف الرئيسي من البحث، وسعنا مفهوم الموديولات رديف المضغوطة، الموديولات رديف المضغوطة حرجا، الى المؤثرات رديف المضغوطة، المؤثرات رديف المضغوطة حرجا، بالتتابع، كذلك اعطينا بعض الخواص الاساسية والبراهين والامثلة. في الختام، درسنا مفهوم المؤثرات رديف المسحوبة كأعمام لمفهوم الموديولات رديف المسحوب، بالإضافة، قدمنا علاقته بالمؤثرات رديف المضغوطة.

الكلمات المفتاحية

الموديلات رديف المضغوطة، الموديلات رديف المضغوطة حرجاً، المؤثرات رديف المسحوبة، المؤثرات رديف المضغوطة.

Abstract

The goal of this paper, we expand the notion of co compressible modules, critically co compressible modules, to co compressible acts, critically co compressible acts, respectively, likewise we giveseveral of their basic properties, theorems and examples.Lastly, we studied the coretractable acts as a generalization of the concept of coretractable modules. Overtime, we introduce relation between it and co compressible acts.

Keywords

co compressible acts, critically co compressible acts, corational acts, fully corational acts, coretractable acts.

1. Introduction and Priliminaries:

Throughout this paper Swill denote a semi-group with multiplication. An element x∈S is called a left identity (a right identity) of S, if ∀ y∈S: x·y=y

(y·x=y). If x is both a left and a right identity of S, then x is called an identity of S [1]. A semigroup is a monoid; if it has an identity, for a semigroup Swe shall use the notion S^1 defined byS^1=S if it has an identity and S^1=S $\dot{\cup}$ {1} otherwise [2]. An element x∈S is called a left zero (a right zero)of S, if ∀ y∈S:x·y=x(y·x=x)[1]. A zero element is both a left and a right zero of S, for a semigroup S we shall define S^0 byS^0=S if it has a zero element,S^0=S $\dot{\cup}$ 0 otherwise[2].

A monoidSis said to be zero divisor free (ZDF) if for each a,b∈S,

$a \neq 0 \neq b$ implies $a \cdot b \neq 0$ [3].

Let S be a monoid and A≠∅ be a set. If we have a mapping μ:A×S→Asuch that (a,s)↦as:=μ(a,s) and the following properties hold:

(a) a·1=a.(b) a(st)=(as)t for all a∈A and s,t∈S, we call A a right

S-act,or a right act over S and write A_S. A subact B of anS-actA_S is a non-empty subset of A such that bs ∈B for all b∈B,s∈Swhich is designatebyB$_S \leq A_S$.

A subact B_S of A_S is called a proper subact (designatebyB$_S < A_S$), if B≠A[4].

We call A_S a simple act if it has no proper subacts. We call A_S a θ-simple act

if A_S has no subacts other than A_S and one element subact. Let S be a monoid and A_S be an S-act, then A_S is simple if and only if A_S= aS for each0 ≠ a∈A_S[5].

A proper subact B_S of an S-actAis called maximal if for each subact C_S of A_S with $B_S \subseteq C_S \subseteq A_S$ implies either B_S=C_S or A_S=C_S [5].

An element θ∈A_S is called a zero of A_S (a fixed element, a sink) if θs=θ for all s∈S.

If A_S and B_S are right S-acts, then a mapping f:$A_S \to B_S$ is called a homomorphism of right S-acts, if f(as)= f(a)s for all a∈A and for all s∈S, theS-homomorphism f:$A_S \to A_S$is called an endomorphism of A_S and the set of allendomorphism of A_S is designateby End(A_S), it is a monoid with the composition operation.

An equivalence relation ρ on A_S is called a congruence onA_S, if aρa´implies(as) ρ (a´s) for a,a´ ∈A_S, s∈S.

Let S ρbe a congruenceonA_S, define a right multiplication on the factor set A_S /ρ={[a]_ρ,a∈A_S }by elements of S by: $[a]_\rho s=[as]_\rho$for every s∈S, which is called a factor act of A_S by ρ[4]. Any homomorphism f:$A_S \to B_S$, the kernel congruence on A_S is defined by aρa´, if and only if f(a)= f (a´).

If Bis a subact of A, the Rees congruence ρ_B is defined by:aρ_B (a)´if eithera,a´∈Bor a=a ´.We denote the resulting factor act byA_S/B_S.

We say A_S is faithful, if s,t∈S, as=atfor all a∈A, it follows that s=t.

A right S-actA is called quasi-projective if for any right S-actB and any S-homomorphismsf and g from A_S to B_S with f being surjective, there exists an S-homomorphism h:$A_S \to A_S$ such that f= gh [6].

If$A_S$satisfies the ascending chain condition for subacts, it is called, Noetherian [7]. In [8], Alex and James defined thatA_S is Noetherian if every congruence on A_S is finitely generated, and we say that a monoid S is Noetherian if it is Noetherian as anS-actover itself.

A subact B_S is large (or essential) in A_S if for any P_S and any S-homomorphism $f:A_S\to P_S$ whose restriction to B_S is one-to-one, then f is itself one-to-one [9].

A nonzero subact B of $A_S$is intersection large if for all nonzero subacts Xof A, $X\cap B\neq\theta$. This will be designateby $B_S \subseteq' A_S$ [9]. Feller and Gantos in [10] proved that every large subact of A_S is intersection large.

The singular congruence $\psi_A$on S-actA_S is a right congruence on A_S defined by $\psi=\psi(A_S)=\{(a,b)\in A_S\times A_S \mid aD\doteq bD$ for some intersection large ideal D of S$\}$. When $\psi=i_{A_S}$, we say that A_S is non-singular [11].

A right S-act A is called retractable if for any subact B_S ofA_S, $Hom(A_S, B_S)\neq\emptyset$. It is clear that for a right S-act A to be retractable it is enough that $Hom(A_S, aS) \neq \emptyset$ for each $a\in A_S$[12].

A subact B_S of A_S is called small (designate$B_S\ll A_S$) by if it satisfies one of the conditions: (1) for each S- homomorphism $f:C_S\to A_S$, if $\pi_B\circ f$ is epimorphism then f is epimorphism where $\pi_B:A_S\to A_S/B_S$ is the natural epimorphism. Or (2) $A_S=B_S\cup C_S$ implies $C_S=A_S$ for each subact C_S of A_S.

An S-act A is called hollow if every subact of A_S is small.

The purpose of this work is to introduce and investigate the concept of co compressible acts as a converted concept of co compressible modules [13]. Related properties and results are investigated. An affirmative answer to analogues problem, named Zelmanowitz problem, in the case of modules is given.

2. CO-Compressible Acts

2.1. Definition.

LetA_S be an S-act over a monoidS. A_S is co compressible if for each nonuniversal congruence ρ on A, $Hom_S(A/\rho,A)$ has at least one surjective element.That is,A_S is co compressible if and only if for all congruence $\rho\neq A\times A$ on A_S, there existsan epimorphism $\alpha:A_S/\rho\to A_S$.

2.2. Remark.

If A_S is a co compressible S-act which is not simple, then A_S must have a zero element.

Proof:

Let B_S be a proper subact of A_S, then A_S/B_S has a zero element. If $\alpha:A_S/B_S\to A_S$ is an epimorphism, then $\alpha(0')$ is a zero elementofA_S.⊛

2.3. Remark.

If A_S is not simple and has no zero we say A_S co compressible ifA_S^0 is co compressible.

2.4. Definition.

A_S is critically co compressible if it is co compressible and for proper subact B_S, $Hom_S(B,A)$ has no surjective element. That is A_S critically co compressible if and only if A_S is co compressible and

$\nexists\ \alpha:B_S\to A_S, \forall\ B_S\leq A_S$, where α is epimorphism.

2.5. Examples.

1. If A_S is a simple act, then A_S is co compressible (trivially).

2. If S=(N,·) is a monoid where N is the set of natural numbers and (·)is the usual multiplication, then $S_S$is not co compressible.

Proof:

Let S=(N,·), and the subact of $S_S$are B=2N, according to (Remark 2.3), let f:S⁄B→S^0 and a∈Im(f), then a=f(x ̅) for somex ̅∈S⁄B, this implies

2a=f(x ̅2)=f(0). If b∈Im(f)and b≠a, then b=f(y ̅)→2b=f(y ̅2)=f(0). HenceIm(f) has at most one element. Hence Hom(S⁄B,S^0)={0} and ∄ epimorphism from S⁄B onto S^0.⊛

2.6. Remarks.

Analogous to the casein modules, it easy to prove the following statements for acts. (See [14])

1. $B_S$is a maximal subact of $A_S$if and only if A_S/B_S is θ-simple.

2. If α:A_S→$B_S$is an epimorphism and C_S≤$B_S$thenα($α^{-1}$(C_S))=C_S.

3. If B_S is a proper subact of a finitely generated S-act A_S then B_S is contained in a maximal subact ofA_S.

4. Let S be a monoid and $A_S$be a Noetherian S-act then A_S is finitely generated.

5. Iff:A_S→B_S is a right S-act homomorphism thenA_S⁄(Ker(f))≅Im(f).

6. Let B_S, C_S≤A_S, then ((B_S∪C_S))⁄C_S ≅BS ⁄(B_S∩C_S).

2.7. Proposition.

If A_S is co compressible S-act which is not simple, then A_S has no maximal nonzero subact.

Proof:

Assume that A_S is a co compressibleS-act, and assume that 0≠$B_S$is a maximal subact of $A_S$then A_S/B_S is a θ-simple act (by Remarks 2.6 (1)). If α: A_S/B_S →A_S is an S-epimorphism, then $α^{-1}$ (B_S)is a subact of A_S/B_S, and hence, either $α^{-1}$ (B_S)=A_S/B_S, which implies B_S=α(A_S⁄B_S)=A_S (since α is onto) (by Remarks 2.6 (2)), not possible or $α^{-1}$ (B_S)=0 ̅, that is B_S= α(0̅)=(0) ̅contradicts the assumption, therefore A_S has no maximal subact.⊛

2.8. Corollary.

Any finitely generated co compressible act is simple.

Proof:

If $A_S$is not simple, then it has a proper subact say B_S and (by Remarks2.6 (3)), B_S is contained in a maximal subact which contradicts Proposition 2.7. Therefore $A_S$is simple.⊛

2.9. Corollary.

Any Noetherian co compressible act is simple.

Proof:

Since A_S is Noetherian then A_S is finitely generated (by Remarks2.6 (4)), henceA_S is simple act (by Corollary 2.8).⊛

2.10. Proposition.

A homomorphic image of a co compressible S-act is co compressible.

Proof:

Assume that f:A_S→A'_S is an epimorphism and A_S is a co compressibleS-act, let δ be a congruence on A'_S define a relation ρ on A_S by

$(a,b)\in\rho\leftrightarrow(f(a),f(b))\in\delta$, then ρ is a congruence on A_S, hence there exists $\alpha:{}^{A_S}/\rho\to A_S$ epimorphism (since A_S is co compressible) which induces a $\beta:{}^{A'_S}/\delta\to A_S/\rho$ in this way, by $\delta f(a)\mapsto\rho a$. β is well defined since if

$\delta f(a)=\delta f(a')$ then $(a,a')\in\rho$, that is $\rho a=\rho a'$. That is β is an epimorphism. Now the composition $f\circ\alpha\circ\beta$ is an epimorphism of ${}^{A'_S}/\delta$ onto A_S'. Therefore A'_S is co compressible.⊛

At the following we give more conditions, with which a co compressible act become simple.

2.11. Proposition.

Let A_S be a co compressible actsatisfying one of the following statements:

1. A_S is finite.
2. There are only finite subacts in A_S.
3. A_S has a finitenontrivial factor,i.e. ∃ a congruence ρ on A_S such that A_S/ρ is finite.
4. There is a simple factor of A_S.

Then, A_S is simple act.

Proof:

1. and 2. imply that A_S has a maximal subact, consequently A_S is simple (by Proposition 2.7).

3. If ρ is a congruence on A_S such that A_S/ρ is finite factor of A_S since A_S is co compressible thenthere exists $f:{}^{A_S}/_\rho\to A_S$. Then A_S must be finite too, so A_S is simple by means of (1.)

4. Let ρ be a congruence on A_S such that A_S/ρ is simple since A_S is co compressible act then, ∃ $f:{}^{A_S}/_\rho\to A_S$ epimorphism. Assume that B_S is a subact of A_S, then f^{-1} B_S is a subact of A_S/ρ, either equal to ${}^{A_S}/\rho$, then

$B_S=f\ {}^{A_S}/\rho=A_S$, not possible, or f^{-1} $B_S=$, then $B_S=f(\bar{0})=0$. Therefore A_S is simple.⊛

2.12. Definition.

A_S is epiform if for each $\rho\neq A\times A$ congruence on A_S then each homomorphism $\alpha:A_S\to A_S/\rho$ is an epimorphism.

2.13. Proposition.

Let A_S be a co compressible act, and B_S be a subact of A_S, then the following statements are equivalent:

1. A_S is critically co compressible
2. A_S is epiform.

Proof:

1.⇛ 2.

Let $\alpha:A_S\to{}^{A_S}/\rho$ be a nonzero homomorphism, let $\ker(\alpha)=\delta$, if $Im(\alpha)=B_S/\rho'$, $\rho'=\rho\cap(B_S\times B_S)$, which is a congruence on B_S then ${}^{A_S}/\delta\cong{}^{B_S}/\rho'$ (by Remarks 2.6 (5)).Since A_S is co compressible then there exists $\beta:{}^{A_S}/_\delta\xrightarrow{epi.}A_S$,

Consider $B_S\xrightarrow{\pi}{}^{B_S}/_{\acute{\rho}}\cong{}^{A_S}/_\delta\xrightarrow{\beta}A_S$, here π is the natural mapping. Hence $B_S\to A_S$ is an epimorphism a contradiction to critically co compressible so $Im(\alpha)=A_S/\rho$ and α is an epimorphism.

2. ⇛ 1.

Assume that, A_S is co compressible, if $\alpha:B_S\to A_S$ is an epimorphism then,

$A_S\cong{}^{B_S}/\ker(\alpha)$. Let $\varphi:A_S\to B_S/\ker(\alpha)$ be an isomorphism.Let

$\varphi':A_S\to A/\delta$ where $\delta=\ker(\alpha)\cup\{(a,a):a\in A_S\}$, such that $\varphi'(a)=\varphi(a)$, for each $a\in A_S$ then φ

´is not epimorphism a contradiction with epiform.⊛

2.14. Definition.

A subact B_S is called corational in A_S, if $Hom(A_S, {}^{B_S}/\rho)=0$ for each ρ congruence on B_S.

2.15. Definition.

Let A_S be an S-act, we say that A_S is fully corational if every proper subact of A_S is corational.

The following proposition gives the relevance between epiform act and corational act,

2.16.Proposition.

Let A_S be an S-act. Then A_S is fully corational act if and only, if A_S is epiform.

Proof:

$\Rightarrow$

Assume that $0 \neq \alpha: A_S \to {}^{A_S}/(\rho)$ is not an epimorphism

Let $\alpha': A_S \to Im(\alpha) = {}^{B_S}/\rho, B_S < A_S, \alpha' \neq 0$ where $\alpha'(a)=\alpha(a)$ for each $a \in A_S$.

Then B_S is not corational hence A_S is not fully corational.

$\Leftarrow$

Let $\alpha: A_S \to B_S/\rho, \alpha \neq 0$ then $\alpha': A_S \to A_S/\rho$ define by

$\alpha(a)=\alpha'(a)$, it is not epimorphism.⊛

2.17. Definition.

Let A_S be an act, we call A_S a copolyform act if for any small subact B_S of A_S, $Hom(A_S, {}^{B_S}/\rho)=0$ for all congruence ρ on B_S.

It is clear that if A_S is copolyform and hollow, then it is fully corational, and thus (by Proposition 2.16) epiform.

In the following lemma, the converse will be proved,

2.18. Lemma.

If the S-act A is epiform, then it is hollow.

Proof:

Assume that A_S is epiform, and let B_S, C_S be two proper subacts of A_S such that $B_S \cup C_S = A_S$ then ${}^{A_S}/B_S \cong {}^{C_S}/(B_S \cap C_S)$ (by Remarks2.6 (6)).

$A_S \xrightarrow{\rho_B} {}^{A_S}/_{B_S} \xrightarrow{\theta} {}^{C_S}/_{(B_S \cap C_S)} \xrightarrow{i} {}^{A_S}/_{(B_S \cap C_S)}$

since $i \circ \theta \circ \rho_B \neq 0$, the composition $i \circ \theta \circ \rho_B$ is not epimorphism; this contradicts the assumption that A_S is epiform, therefore A_S is hollow.

2.19. Proposition.

Any critically co compressible act A_S is hollow, and hence indecomposable.

Proof:

It is clear that A_S is epiform through (Proposition 2.13) and through (Lemma 2.18) it follows that A_S is hollow, and hence indecomposable⊛

In the rest of this section, two results about co compressible and critically co compressible acts will be added. For the first result the following two statements are needed,

2.20. Lemma.

A faithful simple finite S-act is nonsingular.

Proof:

Assume that A_S is faithful simple (and finite)S-act over a monoid S. Let

$\psi_{A_S}=\{(a,b)\in A\times A \mid as=bs\ \forall s$ in some intersection large ideal in S$\}$.Let$(a,b)\in\psi_{A_S}$, then there exists an intersection large ideal of S such that $aI\doteq bI$ (where $I\neq 0$ since it is intersection large ideal), since A_Sis faithful then there exists$x\in A_S, x\neq 0$ and $xI\neq 0$. But xI is a subact of the simple act A_S,so $xI=A_S=xS$. But A_S is finite, implies$I=S$.Now $a=a\cdot 1=b\cdot 1=b$, that is $\psi_{A_S}=i_{A_S}$, then A_S is nonsingular. ⊛

2.21.Lemma.

If A_S if an act and B_S is finite subact of A_S, then $A_S/(A_S\backslash B_S)S$ is finite. Since $(A_S\backslash B_S)S\supseteq A_S\backslash B_S$ it follows that$|(A_S\backslash B_S)S|\geq|A_S\backslash B_S| \rightarrow |A_S/(A_S\backslash B_S)S|=|A_S\backslash(A_S\backslash B_S)S|$. Hence $|A_S\backslash(A_S\backslash B_S)S|\leq|A_S\backslash(A_S\backslash B_S)=|B_S|$.

2.22.Theorem.

A faithful co compressible act A_S over a monoid S is nonsingular in each of the following cases:

1. $A_S$is finite;
2. Sis finite;
3. There are only finite numbers of right ideals in S,
4. Thereare only finite numbers of right essential ideals in S,
5. S is a right Artinian;
6. The intersection of all essential right ideals is nontrivial.

Proof:

1. Since A_S is finite and co compressible implies A_S is simple (by Proposition 2.11), hence (by lemma 2.20), $A_S$is nonsingular.

2. Let $0\neq a\in A_S$ such that $aS\neq 0$ (A_S faithful).

Let$M=A_S\backslash aS$, and$N=MS$, then A/N is a finite factor of A_S(by Lemma 2.21). Since $A_S$is co compressible, it is simple. Now,A_S is simple and S is finite imply that $A_S$is finite, then by (1.) it is nonsingular.

3. if S has a finite number of ideal, then it has a minimal ideal, say I, let

$0\neq a\in A_S$ such that $aI\neq 0$ ($A_S$is faithful), let $M=A_S\backslash aI$, and $N=MS$, then N is a maximal subact of A_S, for if L is a subact of $A_S$containing N, then it must contain an element if aI (and hence all of aI, since it is minimal), that is $L=A$.Then A/N is a simple factor of A_S, hence $A_S$itself is simple. Since $A_S$is simple implies $A_S=aS$ for each $0\neq a$ inA_S hence S is simple (since if S has a proper ideal I, then aI is a proper subact ofA_S).

Now, if$(a,b)\in\psi_{A_S}$, then $aS=bS$ (S is the only intersection large ideal of S), then $a=b$, that is $\psi_{A_S}=i_{A_S}$ and A_S is nonsingular.

4. If S has a finite number of intersection large ideals, then it possess a minimal one, then we complete the proof as in (3.).

5. If S is Artinian, then it has a minimal ideal, then we proceed as in (3.).

6. In this case also we have a minimal intersection large ideal then as (4.)the proof is completed.⊛

For the second result, we recall the definition of quasi-projective act from section 1.

2.22.Proposition.

LetA_S is a critically co compressible act.A_S

is simple if and only if A_S is quasi-projective.

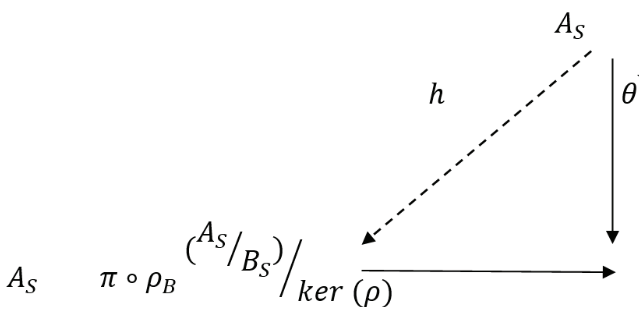
Proof:

$\Rightarrow$

It is clear that every simple act A_S is quasi-projective.

$\Leftarrow$

Let B_S be a proper subact of A_S since A_S is co compressible there exists $\rho: {}^{A_S}/B_S \to A_S$, epimorphism hence $A_S \cong (({}^{A_S}/B_S)/\ker(\rho))$. Consider the composition $A_S \overset{\rho_B}{\to} {}^{A_S}/_{B_S} \overset{\pi}{\to} \left({}^{A_S}/_{B_S}/_{\ker(\rho)}\right)$, since A_S is quasi-projective $\exists$ $h: A_S \to A_S$ such that $(\pi \circ \rho_B \circ h) = \theta$ which is isomorphism



that is h is injective, hence $A_S \cong Im(h)$, then if Im(h)=As, then h itself is an isomorphism and hence $\pi \circ \rho_B$ is an isomorphism (not possible), or Im(h) is a proper subact of A_S which contradicts the assumption that A_S is critically co compressible therefore A_S has no proper subact, that is, A_S is simple.⊛

3. Coretractable Acts

3.1. Definition.

An S-act A is coretractable if there exists a nonzero homomorphism of every nonzero factor of A_S into A_S, i.e.

$\exists f: A_S/\rho \to A_S$, $f \neq 0$ $\forall$ $\rho \neq A \times A$ congruence on A_S .

3.2. Examples.

(1) Let S= (N,·)be a semigroup with identity, then S_S is not coretractable. (see Examples2.5 (2)).

(2)If A_S is simple, then A_S is coretractable (trivially).

3.3.Remark.

Every co compressible act is coretractable act.

In the next lemma, a condition, makes the converse of Remark 3.3 is correct, is given,

3.4.Lemma.

Let an act A_S be a coretractable act, if each $0 \neq f \in$ End A_S is an epimorphism, then each nonzero element of Hom ${}^{A_S}/\rho, A_S$ is an epimorphism, for every congruence ρ of A_S. In particular A_S is co compressible.

Proof:

Let ρ be a congruence on A_S and $0 \neq g: {}^{A_S}/\rho \to A_S$, which exist since A_S is coretractable. A nonzero homomorphism considering $\upsilon: A_S \to A_S/\rho, g \circ \upsilon$

is epimorphism and obviouslyg is an epimorphism. In particular A_S is co comperessible.⊛

The next theorem is generalization to proposition 2.13.

3.5. Theorem.

Let A_S be a coretractable act, then A_S is critically co compressible if and only if A_S is epiform.

Proof:

⇛ (by Theorem 2.13).

⇚

SinceA_S is epiform implies each f∈ End_S (A) is an epimorphism since

$f:A_S \to A_S$, and $\varphi:A_S \to {}^{A_S}/i_{A_S})$ is an isomorphism $\varphi \circ f:A_S \to {}^{A_S}/ i_{A_S})$ is an epimorphism, then f is epimorphism) (by Lemma 3.4),A_S is co compressible, (by Theorem 2.12), A_S is critically co compressible.⊛

3.6. Lemma.

Let A_S be an act with zero, then the next cases are equivalent:

1. Every $0 \neq f \in End_S$ (A) is an epimorphism;

2. For each B_S proper subact A_S, for each h∈ Hom $({}^{A_S}/B_S, A_S)$is an epimorphism;

3.For each $0 \neq B_S$ proper subactA_S, $Hom(A_S,B_S)=0$.

Proof:

1. ⇛2.The same proof of (Lemma 3.4).

2. ⇛1. Let $0 \neq f \in End(A_S)$ takeB_S= 0 and define,

$h:{}^{A_S}/ B_S \to A_S$, by $h([a]_B)=f(a)$ thenh is well defined and h≠0 , through (2) h is an epimorphism, it follows that f is an epimorphism also.

1. ⇛3.Ifh $\in Hom(A_S,B_S)$and h≠0 then define $f:A_S \to A_S$ by f(a)=h(a) for each $a \in A_S$.In this case $Im(f) = Im(h) = B_S \neq A_S$ it follows that f is not epimorphism, a contradiction with(1) hence h=0.

3.⇛1. Assume $0 \neq f \in End_S$ (A) and f is not epimorphism $0 \neq B_S = Im(f) \neq A_S$ define $h:A_S \to B_S$ by h(a)=f (a) then $h \in Hom(A_S,B_S)$ and h≠ 0 a contraction with (3) hence f is epimorphism.⊛

3.7. Note.

If A_S is an epiform, then the casesof (Lemma3.6)are satisfied, since each $f:A_S \to A_S$, can be writtenaccordinglyf:$A_S \to A_S / i_{A_S})$ and hence it is an epimorphism if A_S is epiform.

3.8. Theorem.

Let A_S be an act, the nextcases are equivalent:

1. A_S is co compressible and every $0 \neq f \in End_S$ (A) is an epimorphism;

2. A_S is co compressible and End_S (A) is ZDF;

3. A_S is coretractable and every 0≠ End_S (A)is an epimorphism;

4. A_S is coretractable and End_S (A) is ZDF.

Proof:

1. ⇛2. Since f≠0 and h≠0. f,h are epimorphisms

,then h∘fis epimorphism, henceh∘f ≠ 0.

2. ⇛ 4.It is clear that, since every co compressible act is coretractable act.

4. ⇛3. Let $f:A_S \to A_S$ be not epimorphism, then Im(f) ≠ 0 since A_S is coretractable there exists $\varphi:{}^{A_S}/Im(f) \to A_S$, if $\pi:A_S \to A_S/Im(f)$ is the natural epimorphism, then $\varphi \circ \pi = h \in End_S$ (A) it is clear that h≠0 (since π is onto and φ≠0), (h∘f) (a) = h (f(a)) = φ∘π(f(a)) = φ(0) = 0΄. Now h∘f = 0, a contradiction with End_s (A)is ZDF.

3. ⇛1.(by Lemma 3.4).⊛

References:

[1] T. Harju, Lecture Notes on Semigroups, Department of Mathematics, University of Turku, Finland, (1996).

[2] M. Kilp, U. Knauer, A. V. Mikhalev, Monoids, Act and Categories, Walter de Gruyter, Berlin. New York, (2000).

[3] U. Hebissch, Semirings Without Zero Divisors, Mathmatica Pannonica 1/1 73-94, (1990).

[4] H. Oltmanns, Homological classification of monoids by projectivities of right acts, Ph.D. Thesis, University of Oldenburg (Germany), (2000).

[5] M.S. Abbas and A. Shaymaa, Principally quasi injective system over monoid, Mustansiriyah University journal of advances in mathematics, Vo.10. 3152-3162, (2015).

[6] J. Ahsan and L. Zhongkui, On relative quasi-projective acts over monoids, The Arabian journal for science and engineering, Vo.35, 225-233, (2010).

[7] K. Ahmadi and A. Madanshekaf, Nakayama's Lemma for acts over monoids, Semigroup Forum, Vo.91 (2), 321-337, (2015).

[8] A. Bailey and J. Renshaw, Covers of acts over monoids II, Semigroup Forum, Vo.87(1),1-18, (2012).

[9] A. M .Lopez , Jr. and J. K. Luedeman , Quasi-injective S-systems and their S-endomorphism Semigroup, Czechoslovak Math .J., 29(104) ,97-104, (1979).

[10] E. H. FELLER and R. L. Gantos of Milwaukee, Indecomposable and Injective S-systems with Zero, Math. Nachr, 38-48, (1967).

[11] C. V. HINKLE,The Extended Centralizer Of An S-set, Pacific Journal of Mathematics,Vo.53, No.1.163-170, (1974).

[12] [12]R. KHOSRAVI, On Retractable S-Acts, Journal of Mathematical Research with Applications, Vo.32. 392-398, (2012).

[13] A.M.AL-Hosainy and H.A. kadhim, Co compressible modules, International Research Journal of Scientific Findings .Vo1, (2014).

[14] F. Kasch, Modules and Rings, Academic Press Inc. London, (1982).