piecewise 3-monotone approximation for 3-monotone functions in L_P-spaces for P<1

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الخلاصة

في هذا البحث قمنا بتخمين درجة التقريب 3 - رتيب على الفضاءات L_p عندما P<1.

الكلمات المفتاحية

درجة التقريب، فضاءات Lp.

Abstract

In this paper we estimate the degree of 3-monotone shape preserving approximationon L_P-spaces for <1.

Keywords

The degree of monotone, L_P-spaces.

1. Introduction

In [1] K.A.Kopotun, studied approximation of k-monotone functions, in [2] E. S. Bhaya,andR. R. Mohsin, studied approximation of 3-monotone functions by 3-monotone functions in L_p-spaces, and in [3] E.S. Bhaya, and M. S. Al-Muhjastudiedk-monotone approximation in L_p-spaces, in [4] G.A.Dzyubenko,K. A. Kopotun, and A.V.Prymark, studied Three-monotone spline approximation, in [5] A.Bondarenko , D.Leviatan, and A.Prymark, studiedpointwise estimates for 3-monotone approximation .

In this paper we introduce shape preserving approximation theorems for 3-monotone functions in L_p-spaces for <1 .

2. The auxiliary results :

Let us introduce some auxiliary lemmas that we need in our work.

2.1. Lemma

suppose $f,s \in \Delta^2_{[a,b]} \cap L_p[a,b]$ for <1 . Then either $s'(b-) \le f[a,b]$ or $s'(a+) \ge f[a,b]$, satisfying

$\|f-l\|_{L_p[a,b]} \le c(p)\|f-s\|_{L_p[a,b]}$,

wheref define on[a,b] and l is linear Lagrange interpolation f at a and b

Proof:

Suppose that $s'(b-) \le f[a,b]$, and $s'(a+) \ge f[a,b]$ is symmetrical. If $x_1 = \sup\{f'(x) \le f[a,b], \forall x \in (a,b)\}$, such that $s'(x) \le s'(b-) \le f[a,b]$, and $x_1 \le x \le b$.

$\|f-l\|_{L_p[a,b]}) = l(x_1) - f(x_1)$

$= \int^b_{x_1} (f'(x) - l'(x))\,dx$

$\le \int_{x_1}^{b} (f'(x) - s'(x))dx$

$\le f(b) - s(b) - (f(x_1) - s(x_1))$

$\le c(p)\|f-s\|_{L_p[a,b]}$ ■

2.2. Lemma [6]:

Suppose that f is defined on $[a_1,b_1]$, and that s is a piecewise polynomial of degree $\le k-1$, which knot a and b, $a_1 \le a \le b \le b_1$, such that $s'(a+) \le f[a,b] \le s'(b-)$. If, $s \in \Delta_{[a,b]}^2$, then there exist a piecewise polynomial $s_1 \in \Delta[a_1,b_1]2$ of degree $\le k-1$, with knot a and b, satisfying (1)$s'(a+) \le s_1'(a+)$, $s'(b-) \le s_1'(b-)$.(2)$s_1(a) = f(a)$, $s_1(b) = f(b)$

2.3. Lemma:

Let f define on $[a_0,b_0]$ and let $s \in \Delta_{[a,b]}^2 \cap L_{p[a,b]}$ for P<1, of degree $\le k-1$ at knots a and b, $a_0 \le a \le b \le b_0$, $s'(a+) \le f[a,b] \le s'(b-)$ .If f, and s are convex polynomials on $[a_0, b_0]$,then there is s' be convex a piecewise polynomial on $[a_0,b_0]$ of degree $\le k-1$, at same knots, satisfying

(1) $s'(a+) \le s_1'(a+)$, $s_1'(b-) \le s'(b-)$

(2)$s_1(a) = f(a)$, $s_1(b) = f(b)$

(3)$\|f-s_1\|_{L_p[a,b]} \le c(p)\|f-s\|_{L_p[a,b]}$

(4)$\|f-s_1\|_{LP[a_0,b_0]} \le c(p)\|f-s\|_{L_p[a_0,b_0]}$

Note if $[a,b]=[a_0,b_0]$, such that s, s_1 are convex a piecewise polynomials on $[a_0,b_0]$ of degree $\le k-1$.

Proof:

If (b)-f(a)=s(b)-s(a), let $s_1(x) = s(x) + f(a) - s(a)$, for each $x \in [a_0,b_0]$. Then by Lemma 2.2 we get (1) and (2),then

$\|f-s_1\|_{L_p[a,b]} = \|f-(s(x)+f(a)-s(a))\|_{L_p[a,b]}$

$\le c(p)\|f-s\|_{L_p[a,b]}$

And

$\|f - s_1\| L_p[a_0,b_0] = \| f - (s(x) + f(a) - s(a))\|L_p[a_0,b_0]$

$\le c(p)\|f\text{-}s\|L_p[a_0,b_0]$

Suppose f(b)-f(a)<s(b)-s(a) and this case f(b)-f(a)>s(b)-s(a) is symmetrical.

First defines_1 on the interval [a,b] and then extended to interval $[a_0,b_0]$if $[a,b]=[a_0,b_0]$.

Assume $\ddot{s}(x)= s(x) - s'(a+)(x-a)$, for each $x \in [a,b]$, and $\ddot{f}(x)=f(x)-s'(a+)(x-a)$, for each $x \in [a,b]$, such that $\|\ddot{f}-\ddot{s}\|L_p[a,b]=\|f-s\|L_p[a,b]$, then by our hypothesisf(b)-f(a)<$\ddot{s}(b)-\ddot{s}(a)$, and $\ddot{s}(b)-\ddot{s}(a)>0$. Thus set $\ddot{s}_1(x)=\ddot{f}(a)+\lambda(\ddot{s}(x)-\ddot{s}(a))$, for each x∈[a,b] and $\lambda=(\ddot{f}(b)-\ddot{f}(a))/(\ddot{s}(b)-(s(\ddot{a}))$ where $0\le\lambda\le1$, note that $\ddot{S}$ is non decreasing and $\|\ddot{s}(.)-\ddot{s}(a)\|(_{LP[a,b]})=\ddot{s}(b)-\ddot{s}(a)$, $\ddot{s}_1(a)=\ddot{f}(a)$, $\ddot{s}_1(b)=\ddot{f}(b)$, so that $\ddot{s}_1$ is convex in [a,b] . Hence

$\|f-s_1\|L_p[a,b] = \|\ddot{f}-\ddot{s}_1\|L_p[a,b]$

$=\|\ddot{f}(.)-\ddot{s}(.)+\ddot{s}(a)-\ddot{f}(a)+\ddot{s}(.)-\ddot{s}(a)+\ddot{f}(a)-\ddot{s}_1(.)\|L_{P[a,b]}$

$\le c(p)\ \|\ddot{f}-\ddot{s}\|_{LP[a,b]} + \|\ddot{s}(.)-\ddot{s}(a)+\ddot{f}(a)-\ddot{f}(a)-\lambda(\ddot{s}(.)-\ddot{s}(a))\|_{LP[a,b]}$

$\le c(p)(\|\ddot{f}-\ddot{s}\|_{LP[a,b]} + \|\ddot{s}(.)-\ddot{s}(a)-\lambda(\ddot{s}(b)-\ddot{s}(a))_{\|LP[a,b]})$

$\le c(p)(\|\ddot{f}-\ddot{s}\|_{LP[a,b]} +(1-\lambda)\|\ddot{s}(b)-\ddot{s}(a)\|_{LP[a,b]}$

$\le c(p)(\|\ddot{f}-\ddot{s}\|_{LP[a,b]}+\|\ddot{s}(b)-\ddot{s}(a)-((\ddot{f}(b)-\ddot{f}(a))/(\ddot{s}(b)-(s(\ddot{a}))\ (\ddot{s}(b)-s(a))\ddot{}\|(_{LP[a,b]})$

$\le c(p)\ \|\ddot{f}-\ddot{s}\|_{LP[a,b]}+c(p)\|\ddot{f}-\ddot{s}\|_{LP[a,b]}$

$\le c(p)\ \|\ddot{f}-\ddot{s}\|_{LP[a,b]})=c(p)\|f-s\|_{LP[a,b]}$

Further, if $[a,b]\neq[a_0,b_0]$, then extendS_1 either to right or to left or both.

Let

$$s^1(x)= \begin{cases} s(x)+f(x)-s(a) & ,\text{for each } x\in[a_0,a) \\ s(x)+f(x)-s(b) & ,\text{for each } x\in(b,b_0] \end{cases}$$

Then see a piecewise polynomial $s_1\in\Delta[a_0,b_0]^2$ of degree ≤ k-1 at knots aand b.

Now

$\|f-s_1\|_{LP[b,b0]} = \|f-s(x) - (f(b) - s(b))\|_{LP[b,b0]})$

$\le c(p)\|f-s\|_{LP[b,b0]}$

and similarly

$\|f-s_1\|_{LP[a0,a]} = \| f-s(x)-(f(a)-s(a))\|_{LP[a0,a]}$

$\le c(p)\ \|\ f-s\ \|_{LP[a0,a]}$

Combined these with (3), we get (4) ■

2.4. Lemma:

Let$f ,s\in\Delta^2_{[a,b_0]}\cap L_P [a,b]$, for P<1 , $a<b<b_0$, and $s'(b-)-f[b,b_0]>0$. Then

$(s'(b-)-f[b,b_0])\ (b_0-b) \le c(p)\ \|f-s\|_{LP[b,b_0]}$

Symmetric, if $f, s\in\Delta^2_{[a_0,a]}\cap L_P [a,b]$, for P<1 , $a_0<a<b$, and $f[a_0,a]-s'(a+)\le c(p)\|f-s\|_{LP[a_0,a]}$

Proof:

The proofs of the first statement ,and the second are similar, let $x_0=\sup\{x\in(b,b_0):f'(x) \le s'(b-)\}$, then

$$(s'(b-)-f[b,b_0])(b_0-b)=\int_b^{b_0}(s'(b-)-f[b,b_0])dx$$

$$=\int_b^{b_0}(s'(b-)-f'(x))dx$$

$$\le\int_b^{x_0}(s'(b-)-f'(x))dx$$

$$\le\int_b^{x_0}(s'(x)-f'(x))dx$$

$=s(x_0)-f(x_0)-(s(b)-f(b))$

$\le c(p)\|f-s\|_{Lp[b,b_0]}$

wheres' be no decreasing such that $f'(x) \le s'(b-) \le s'(x)$, for each $x\in(b,x_0)$. ■

2.5. Lemma [6]:

Let$a_1 < a < b < b_1$, and $f\in\Delta^2_{[a1,b1]}$, and suppose that $s\in\Delta^2_{[a1,b1]}$is a piecewise polynomial of degree ≤ k-1 with knots a and b , satisfying f(a)=s(a),f(b)=s(b) . Then ,there is a polynomial $s_1\in\Delta^2_{[a,b]}$of degree ≤ k-1 , such that

(1) $s'(a+) \leq s_1'(a+), s_1'(b-) \leq s'(b-)$.

(2) $f[a,a_1] = k_a \leq s_1'(a+)$, $s_1'(b-) \leq k_b = f[b,b_1]$.

(3) $s_1(a) = f(a)$, $s_1(b) = f(b)$.

2.6. Lemma:

Suppose $a_0 < a < b < b_0$, and let m=max $\{\frac{b-a}{b_0-b}, \frac{b-a}{a-a_0}\}$, and f is convex polynomial on $[a_0,b_0]$, and lets convex piecewise polynomial on $[a_0,b_0]$ of degree $\leq$ k-1, at knots a and b, we take f(a)= s(a), and f(b) =s(b). Then there exist s_1 is convex polynomial on [a,b] of degree $\leq$ k-1, satisfying

(1) $s'(a+) \leq s_1'(a+), s_1'(b-) \leq s'(b-)$.

(2) $f[a,a_1] = k_a \leq s_1'(a+), s_1'(b-) \leq k_b = f[b,b_1]$.

(3) $s_1(a) = f(a)$, $s_1(b) = f(b)$.

(4) $\|f-s_1\|_{LP[a,b]} \leq c(p,m)\|f-s\|_{LP[a0,b0]}$

Proof:

Suppose that f(a)= f(b). If S be constant on interval [a,b], put $s_1(x)= s(x)$, for each x∈[a,b]. Otherwise let s(b)=s(a) and sisconvex, such that s^' (b-)>0>s' (a+). let

$$\lambda = \min\{\frac{k_b}{s'(b-)}, \frac{k_a}{s'(a+)}\} \geq 0.$$

If λ≥0, then put $s_1(x)=s(x)$, for each x∈[a,b]. Otherwise <0, suppose that $\lambda = \frac{k_b}{s'(b-)} < 0$. Then assume $s_1(x)=s(a)+\lambda(s(x)-s(a))$, for each x∈[a,b], since s_1 is convex polynomial on [a,b] of degree $\leq$ k-1.We get (1), (2) and (3) from Lemma 2.5.

Let $x_1=\sup\{s'(x) \leq 0: x\in(a,b)\}$, and 0= s(b)-s(a)= $\int_a^b S'(x)dx$, so that

$$\|s - s_1(a)\|_{Lp[a,b]} = \int_{x_1}^{a} s'(x)dx = \int_{x_1}^{b} s'(x)dx \leq (b-a)s'(b-)$$

Then by Lemma 2.4.,

$\|s-s_1\|_{LP[a,b]} = \|s(x)-s(a)-\lambda(s(x)-s(a))\|_{LP[a,b]}$
$=(1-\lambda)\|s-s(a)\|_{LP[a,b]} \leq (1-\lambda)s'(b-)(b-a)$
$=(1-\lambda)s'(b-)(b_0-b)$
$\leq c(p,m)\|f-s\|_{LP[b,b0]}$

Hence

$\|f-s_1\|_{LP[a,b]} = \|f-s+s-s_1\|_{LP[a,b]} \leq C(p)(\|f-s\|_{LP[a,b]} + \|f-s_1\|_{LP[a,b]})$

$\leq c(p)\|f-S\|_{LP[a0,b0]}$

The proof of Lemma2.6 is complete■

2.7. Lemma [6]:

Let $a<b<b_1$, $\tilde{m} = \frac{b-a}{b_1-b}$, and $f\in\Delta^2_{[a,b_1]}$, and suppose that $s\in\Delta^2_{[a,b_1]}$ is a piecewise polynomial of degree $\leq$ k-1 with knot b, satisfying f(a)=s(a) and f(b)=s(b). Then, there is a polynomial $s_1\in\Delta^2_{[a,b]}$ of degree $\leq$ k-1, such that

(1) $s_1'(b-) \leq s'(b-)$,

(2) $s_1'(b-) \leq k_b = f[b,b_1]$,

(3) $s_1(a)= f(a), s_1(b)= f(b)$,

Symmetrically, let $a_1 < a < b$, and $f\in\Delta^2_{[a1,b]}$, and suppose that $s\in\Delta^2_{[a1,b]}$ is a piecewise polynomial of degree ≤k-1 with knot a, satisfying f(a)=s(a) and f(b)=s(b). Then, there is a polynomial $s_1\in\Delta^2_{[a,b]}$ of degree $\leq$ k-1,such that

(1) $s_1'(a+) \leq s'(a+)$,

(2) $f[a,a_1]=k_a \leq s_1'(a+)$,

(3) $s_1(a)= f(a), s_1(b)= f(b)$,

2.8. Lemma:

Suppose $a < b < b_0$, and f be convex polynomial on $[a,b_0]$ of degree $\leq$ k-1, and assume S be convex a piecewise polynomial on $[a,b_0]$ of degree $\leq$ k-1 at the knot b, put f(a)=s(a), f(b)=s(b).Then, there exist s_1 convex polynomial on [a,b], of degree $\leq$ k-1, satisfying

(1)$s_1'(b-) \le s'(b-)$,

$s_1'(b-) \le k_b = f[b,b_0]$,

$s_1(a)=f(a), s_1(b)=f(b)$,

(2) $\|f-s_1\|_{LP[a,b]} \le c(p,\hat{m})\|f-s\|_{LP[a,b_0]}$

, suppose $a_0 < a < b$, $\hat{m} = \frac{b-a}{a-a_0}$, and f convex polynomial on $[a_0,b]$ of degree $\le$ k-1 at knot a , we put f(a)= s(a) , f(b)=s(b) . Then , there exist s_1 convex polynomial on [a,b] of degree $\le$ k-1, satisfying

(1) $s_1'(a+) \le s'(a+)$,

$f[a,a_1] = k_a \le s_1'(a+)$,

$s_1(a)= f(a), s_1(b)= f(b)$,

(2)$\|f-s_1\|_{LP[a,b]} \le c(p,\hat{m})\|f-s\|_{LP[a0,b]}$

Proof:

We prove the first case and the second case is symmetric.

By Lemma 2.7. , we get (1).

And prove (2) by Lemma 2.6 ,

$\|f-s_1\|_{LP[a,b]} = \|f-s(a)-\lambda(s(x)-s(a))\|_{LP[a,b]}$

$\le c(p)\|f-s\|_{LP[a,b]} + c(p)\|s-f(a)\|_{LP[a,b]}$

$\le c(p,\hat{m})\|f-s\|_{LP[a,b0]}$ ■

2.9. Proposition [7]:

Letk ≥ 1 and $r \ge 1$, be integers such that either $r \ge 2$ or $2 \le k+r \le 3$. Then for each f $f \in L^{(r)}_{P[-1,1]} \cap \Delta^2_{[-1,1]}$ there exist piecewise polynomials s_1 ,$s_2 \in \Delta^2_{[-1,1]} \cap_{LP[-1,1]}$ of degree $\le$ k + r-1 such that s_1 has n equidistant knots , and satisfies

$$\|f-s_1\|_{L_{P[-1,1]}} \le \frac{c(p,k,r)}{n^r}\omega_k(f^{(r)},\tfrac{1}{n};[-1,1])(1)$$

ands$_2$ has knots on the Chebyshev partition, and satisfies

$$\|f-s_2\|_{L_{P[-1,1]}} \le \frac{c(p,k,r)}{n^r}\omega_k^{\varphi}(f^{(r)},\tfrac{1}{n};[-1,1])(2)$$

Moreover, s_1 and s_2 interpolate f at the respective knots.

As a direct consequent of Lemma 11 in [2] ,p.167 , we get the following :

2.10. Lemma:

AssumeB>1and= $\max_{0<i<j\le n}\frac{(j-i)(x_{i+1}-x_i)}{x_j-x_i}$. .Then for each step-function $g(x) = \sum_{j=1}^{n-1}\alpha_j(x-x_j)_+^0$, for each x∈[a,b], where $\alpha_i \ge 0$, there is a polygonal-line $p(x) = \sum_{j=1}^{n-1}\frac{\beta_j}{(x_{i+1}-x_i)}(x-x_j)_+$. Satisfying

$|\beta_j| < \frac{\alpha_j}{B}, j = 1,2,3,\dots,n-1,$ (3)

And

$\|g(x)-p(x)\|_{LP[a,b]} \le 8\,\mu\,A\,B$ (4)

where

A= max┬(j=1,2,…,n-1)α_j .

The following auxiliary Lemma is an improvement of Lemma introduced by D.Leviatan , and A.V.Prymark [Lemma 12 , p.167] , and it can be proved in the same way and get.

2.11. Lemma:

Given the partition $x_0<x_1<x_2<\dots<x_n$, and the sequence $\delta_1,\delta_2,\dots,\delta_{n-1}$ are nonnegative-numbers , such that $\delta_j \le \frac{1}{(x_{j+1}-x_{j-1})^2}\Omega$, $1 \le j \le n-1$,withΩ is a positive-constant . Then there is a piece-wise polynomial q of degree ≤ 3 ,at the knots $x_1,x_2,\dots,x_{n-1}$, such that $q \in L^{(1)}_{P[a,b]}$,

$q''(x_j+)-q''(x_j-) = -\delta_j$, j=1,2,3,…,n-1(5)

$q \in \Delta x^3_{(j-1),x_j)} \cap L_P(x_{j-1},x_j)$, j=1,2,3,…,n(6)

$\|q\|_{LP[a,b]} \le c(p,m,\mu)\Omega$(7)

wherec(p,m,μ) is constant depending on m and μ , μ= $\max_{0<i<j\le n}\frac{(j-i)(x_{i+1}-x_i)}{x_j-x_i}$

and = $\max_{1\le j\le n-1}\left\{\frac{x_{j+1}-x_j}{x_j-x_{j-1}},\frac{x_j-x_{j-1}}{x_{j+1}-x_j}\right\}$

2.12. Lemma [8]:

Let $f \in L_p$ [a,b], $0 < P < \infty$. Then there exist q_{k-1} a polynomial of degree $\leq$ k-1 , such that

$\|f-q_{k-1}\|_{Lp}$ [a,b] $\leq c\, \omega_k$ (f,b-a,[a,b]$)_p$.

2.13. Lemma [8]:

Let Pbe a piecewise polynomial of degree $\leq$ k ,such that

$\|P_k^{(s)}\|_{LP\,[a,b]} \leq c\, k^s \|P_k\|_{LP\,[a,b]}$

where c is a constant and s is the order of derivative .

3. The main result

In this section we introduce our main theorems.

3.1.Theorem:

Let $\in\Delta^2_{[a,b]} \cap L_p[a,b]$, for P < 1, with the partition $x_{-1}=a=x_0 < x_1 < x_2 <\cdots< x_n = b = x_{n+1}$, and $k \geq 2$. Then for all convex piecewise polynomial S on interval [a,b] of degree $\leq$ k-1, at the knots x_j and j= 1,2,3,…,n-1, there exist s_1 is a convex a piecewise polynomial on interval [a,b] of degree $\leq$ k-1 at the knots x_j andj=1,2,3,…,n-1, satisfying

(1) f $(x_j)=s_1(x_j)$, and j= 0,1,…,n [1]

(2) $\|f-s_1\|_{LP\,[xj-1,xj]} \leq c(p,m) \|f-s\|_{LP[x_{j-2},x_{j}+1]}$

wherec(p,m) be constant depended on p and m , and m = $\max_{1\leq j\leq n-1}\left\{\frac{x_{j+1}-x_j}{x_j-x_{j-1}}, \frac{x_j-x_{j-1}}{x_{j+1}-x_j}\right\}$

Proof :

Let

l_u (.)=L(.;x_{u-1}, x_u), u = 0,1,…,n+1. Assume $A\subset\{1,2,3,\ldots,n\}$ is the set of each, such that s' $(x_{i-1}+) \leq l_i' \leq$ s' (x_i-). For each $\notin$A , satisfying $s_1(x)=l_i(x)$, $x\in[x_{i-1},x_i]$. By using Lemma 2.8.

$\|f-s_1\|_{LP[x_{i}-1,x_{i}]} \leq c\,(p)\, \|f-s\|_{Lp\,[x_{i}-2,x_{i}+1]}$ (8)

So as to define s_1 on interval $[x_{i-1},x_i]$, i∈A, we first suppose 1< i < n, and use the interval $[x_{i-2},x_{i+1}]$. first Lemma 2.3 and use Lemma 2.6 ,at a= x_{i-1} and b= x_i. We achieve the existence a convex polynomial $s_1$on $[x_{i-1},x_i]$, then

$\|f-s_1\|_{LP[xi-1,xi]} \leq c\,(p)\, \|f-s\|_{LP\,[xi-2,xi+1]}$ (9)

and $f(x_{i-1})= s_1(x_{i-1})$,$f(x_i)= s_1(x_i)$

Finally, we dealwith the possible that either i=1 or i= n , and 1,n∈A . To this suppose 1∈A, and the second case n∈A is symmetrically, since s' $(a+) \leq f[a,x_1] \leq$ s' (x_1-) . Then by using Lemma 2.3 we have a piecewise polynomial

$\ddot{s}_1\in\Delta^2_{[a,x_2]}$, which f interpolate with a and x_1. Since $\ddot{s}_1'(x_1-) \leq s'(x_1-)$, then

$\|f-\ddot{s}_1\|_{Lp[a,x_2]} \leq c\,(p)\, \|f-s\|_{Lp\,[a,x_2]}$ (10)

We now using Lemma 2.8 and get $s_1\in\Delta^2_{[a,x1]}$of degree $\leq$ k-1, which f interpolate with aand x_1, since $\ddot{s}_1'(x_1-) \leq s'(x_1-)$, then

$\|f-s_1\|_{Lp[a,x_1]} \leq c\,(p)\|f-s\|_{LP\,[a,x_2]}$ (11)

so ,s_1is a convex and piecewise polynomial –function of degree ≤k-1 , and $s_1(x_i)$= $f(x_i)$, i= 0,1,2,…,n , and from (1)-(3) include

$\|f-s_1\|_{Lp[xj-1,xj]} \leq c(p,m) \|f-s\|_{LP[xj-2,xj+1]}$ ■

3.2. Theorem:

Let F be a 3-monotone function in L_P [a,b], satisfying f (x)=F' (x), for each x∈(a,b), and take k >2 , x(-1)= a= $x_0 < x_1 <\cdots< x_n = b = x_{n+1}$ be partition for the interval [a,b]. Also let s be convex a piecewise polynomial on[a,b] of degree $\leq$ k-1 ,at knots x_j , j=1,2,…,n-1 ,then , there is a piecewise polynomial 3-monotone P in L_P [a,b]

of degree ≤ k at the same knots, satisfying

$\|F-P\|_{LP[a,b]} \le c\,(p,m)\|f-s\|_{Lp\,[x(i-2),xi+1]}$

wherec(p,m)is constant depended on p and m ,and m= $\max_{1\le j\le n-1}\left\{\frac{x_{j+1}-x_j}{x_j-x_{j-1}},\frac{x_j-x_{j-1}}{x_{j+1}-x_j}\right\}$.

Proof :

By using Theorem 3.1 ,and using Theorem 2.3 ,we have

$\|F-P\|_{Lp[a,b]} \le \|F-P\|_{LP\,[a,b]}+\|f-s_1\|_{LP[xj-1,xj]} \le c\,(p)\,\|f-s\|_{LP[xi-1,xi]}+c(p)\|f-s\|_{Lp[xj-2,xj+1]}$

$\le c(p,m)\|f-s\|_{Lp[xi-2,xi+1]}$■

3.3.Theorem:

Given the integers k > 1 and > 0, such that either r >3 , or 3<k+r ≤ 4 , and (k,r)≠(4,0) . Then for all $\in L_P^{(r)}{}_{[-1,1]}\cap\Delta^3_{[-1,1]}$, there exist 3-monotone a piecewise polynomials S_1 and S_1 on [-1,1] of degree ≤ k + r-1 , such that

$$\|F-S_1\|_{L_{P[-1,1]}} \le \frac{c(p,k,r)}{n^r}\omega_k(F^{(r)},\frac{1}{n};[-1,1])(12)$$

Where S_1 has n equal –distant knots.

And such that

$$\|F-S_2\|_{L_{P[-1,1]}} \le \frac{c(p,k,r)}{n^r}\omega_k^{\varphi}(F^{(r)},\frac{1}{n};[-1,1])(13)$$

Where S_2 has n knot on chebyshev partition.

Proof :

We prove the first case and the second case is symmetrically.

Byusing Theorem 3.2

$\|F-P\|_{Lp}[a,b]\le c(p,m)\|\,f-s\|_{Lp}\,[x_{i-2},x_{i+1}]$

andalso we have ,

$$\|f-s_1\|_{L_{P[-1,1]}} \le \frac{c(p,k,r)}{n^r}\omega_k(f^{(r)},\frac{1}{n};[-1,1])$$

Hence ,

$$\|F-S_1\|_{L_{P[-1,1]}} \le c(p,m)\|f-s\|_{L_P[x_{i-2},x_{i+1}]}+\frac{c(p,k,r)}{n^r}\omega_k(f^{(r)},\frac{1}{n};[-1,1]) \le \frac{c(p,k,r)}{n^r}\omega_k(F^{(r)},\frac{1}{n};[-1,1])$$■|

3.4. Theorem:

Let S be a 3-monotone a piecewise polynomial in L_P [a,b], P<1, of degree ≤ k and k >3, at knots $x_{-1}=a=x_0<x_1<x_2<\cdots<x_n=b=x_{n+1}$. Then there exist a piecewise polynomial S_1 of degree ≤ k at the knots $x_{-1}=a=x_0<x_1<x_2<\cdots<x_n=b=x_{n+1}$, such that $S_1\in\Delta^3_{[a,b]}\cap{}^{(2)}{}_{Lp[a,b]}$, for P < 1.And

$\|S-S_1\|_{Lp[a,b]}\le c(p,k,m,\mu)\omega_{k+1}\,(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}]\,)$ (14)

Where c(p,k,m,μ) is a constant depending on p,k,mand , and m=

$\max_{1\le j\le n-1}\left\{\frac{x_{j+1}-x_j}{x_j-x_{j-1}},\frac{x_j-x_{j-1}}{x_{j+1}-x_j}\right\}$, and

$\mu=\max_{0<i<j\le n}\frac{(j-i)(x_{i+1}-x_i)}{x_j-x_i}$(15)

Proof :

Let δ_j=S'' (x_j+)-S'' (x_j-), j=1,2,…,n-1. Since S be a 3-monotone function in L_P [a,b]for <1 , on interval$[x_0,x_n]$, $\delta_j\ge 0$, 1 ≤ i ≤ n-1 . By Lemma 2.12 there exist polynomial U_k of degree ≤ k , such that

$\|S-U_k\|LP[xj-1,xj+1]\le c(p,k)\omega_{k+1}(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}])_{LP}$

This transition by Lemma2.13,

$$\|U_k''-S''\|_{L_P[x_{j-1},x_{j+1}]} \le \frac{c(p,k)}{(x_{j+1}-x_j)^2}\omega_{k+1}(S,(x_{i+1}-x_{i-1});[x_{i-1},x_{i+1}])_{L_P}$$

Thus,

$$\delta_j \le \frac{c(p,m,k)}{(x_{j+1}-x_j)^2}\omega_{k+1}(S,(x_{i+1}-x_{i-1});[x_{i-1},x_{i+1}])$$

Let

$\Omega=c(p,m,k)\omega_{k+1}\,(S,(x_{i+1}-x_{i-1});[x_{i-1},x_{i+1}])$

And by Lemma 2.11 to get the apiecewise-polynomial . The set S_1 (x)=S(x)+g(x) , for each $x\in[x_0,x_n]$. Clearly ,apiecewise-polynomial S_1 of degree ≤ k at knots $x_0,x_1,\ldots,x_n$,such that

$S_1''(x_j-)=S_1''(x_j+)$, j=1,2,…,n-1,(16)

so that $S_1\in{}^{(2)}{}_{LP[a,b]}$.Also, since $S\in\Delta^3_{[x0,xn]}\cap L_P$

[a,b]for P<1, by (6) that S_1'' be nodecreasing in(x_{j-1},x_j) , $1 \leq j \leq n$. Combine with (16) ,and that S_1'' be non decreasing on(x_{j-1},x_j), so as to S_1 is 3-monotone function on [a,b]. At last, (14)from (7). This completes the proof■

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