Weibull Lindley Burr XII Distribution

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Abstract: In this paper, we derive the new special distribution is Weibull Lindley Burr XII distribution (WLBD) . Several properties are included, PDF, CDF, moment generating function, hazard rate, reversed hazard function, odd function, quantile function, order statistics and moments are proved of the WLBD . An estimation procedure by the method of maximum likelihood, fisher matrix and a simulation are studied of WLBD. Two real data applications are introduced to prove that the our model fits better than of Burr and Lindley distributions. Finally, we give graph of PDF, CDF, mean squared error and bias at different values of parameters..

1 INTRODACTION

The fundamental reason for parametric statistical modeling is to identify the most appropriate model that adequately describes a data set obtained from experiment, observational studies, surveys, and so on. Most of these modeling techniques are based on finding the most suitable probability distribution that explains the underlying structure of the given data set. However, there is no single probability distribution that is suitable for different data sets. Thus, this has triggered the need to extend the existing classical distributions or develop new ones. Barrage of methods for defining new families of distributions have been proposed in literature for extending or generalizing the existing classical distributions in recent time. The Weibull distribution is a very popular model and it has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is generally adequate for modeling monotone hazard rates. The new compound class of log-logistic Weibull Poisson distribution is introduced by [8]. A new family of distributions is presented by [13], which is called the generalized Burr-G (GBG) family of distributions. In this case, some mathematical properties of this family are proved. Odd Burr-G Poisson family is considered in [14]. Odd generalized exponential family is introduced by [15]. Odd Frechet general family is discussed in [16]. Some of these methods include Weibull general familiy is studied in [17]. Odd Lindley general family is obtained by [10]. Topp Leone odd log logistic general family [9]. Odd Frechet general family [16], odd gamma general family [7], alpha power transformed family [1], based on W-G family, some particular distributions have been studied by [3]. A new family from Burr XII distribution is called T-Burr family is presented in [12]. In this family is considered the quantile functions of three well-known distributions, namely, Lomax, logistic and Weibull. Weibull inverse Lomax (WIL) is presented and studied by [4]. In this side, some structural properties are derived and estimation of the model parameters is performed based on Type II censored sample. Transmuted geometric general family is discussed in [5]. A new class of continuous distributions is called the Kumaraswamy transmuted-G family is introduced by [4]. In this family, some mathematical properties are proved. The new distributions is proposed capable of modeling medical data with unimodal failure rate function [11]. The alpha power exponential (APE) and alpha power transformed Weibull (APTW) distributions are introduced in [19]. A new family of univariate distributions generated from the Weibull random variable is proposed in [20], which is called a new Weibull-X family of distributions. In this approach, general expressions for some statistical properties are discussed. These methods are developed with the motivation of defining new models with different kinds of failure rates (monotonic and non monotonic), constructing heavy tailed distributions for modeling different kinds of data sets, developing distributions with symmetric, right skewed, left skewed, reversed J shape, and consistently providing a reasonable parametric fit to given data sets. The Weibull Lindley distribution is presented by Asgharzadeh [6], he study the compounding approach between Weibull and Lindley distributions to give a new distribution is called Weibull Lindley (WL) distribution. [18] Nabeel J. Hassan , Hazim Ghdhaib Kalt, Habeeb A. Aal-Rkhais , Ayed E. Hashoosh , are presented new family from probability distributions by using arbitrary distribution fun is called general weibull lindley distributions. In this paper, we drive a new distribution is called Weibull Lindley Burr distribution (WLBD) by using special cumulative distribution function (CDF) of any random variable. In this approach, we derive density probability function (PDF) and cumulative distribution function (CDF) of the WLBD. We prove Several properties PDF, CDF, moment generating function, hazard rate, reversed hazard function, odd function, quantile function, order statistics, Lorenz curve, k-moments and variances. An estimation procedure by the method of maximum likelihood, fisher matrix and a simulation study to assess its performance are given. In this case, we solve the maximum likelihood equations by using numerical method, also, the mean squared error (MSE) and biases are calculated numerically and graph it. We show that WLBD is better than of Burr and Lindley distributions based on data set 1 and data set 2. This paper is unfolded as follows. In Section 2, we provides special distribution is WLBD obtained by the Weibull Lindley general family (WLG), also some general mathematical properties of the WLBD are discussed in this section. In Section 3, the estimation by the method of maximum likelihood and simulation procedures for WLBD are derived. In section 4, we present application on real data sets to show that WLBD is good fit. The plot of CDF, PDF, bias and mean square error of both distributions is put in appendix. Finally, we give the conclusion.

The pdf of Weibull Lindley distribution WLD [6] is given as

$$f(x,\beta,\lambda)=\frac{[\lambda\beta x+\beta(\lambda+1)+\lambda^{2}(1-x)]\exp(-[\lambda+\beta]x)}{\lambda+1} \qquad (1)$$

And the cdf of WLD is

$$F(x,\beta,\lambda)=1-\frac{[1+\lambda+\lambda x]\exp(-[\lambda+\beta]x)}{\lambda+1} \qquad (2)$$

The cdf and pdf of general WLD [18] can be calculated as

$$\begin{aligned}H_{GWL}(x)&=\int_{0}^{G(x)}\frac{[\lambda\beta t+\beta(\lambda+1)+\lambda^{2}(1+t)]e^{-[\lambda+\beta]t}}{\lambda+1}dt\\&=\frac{[\beta(\lambda+1)+\lambda^{2}]}{\lambda+1}\int_{0}^{G(x)}e^{-[\lambda+\beta]t}dt+\frac{\lambda(\beta+\lambda)}{\lambda+1}\int_{0}^{G(x)}te^{-[\lambda+\beta]t}dt\end{aligned} \qquad (3)$$

We find first integral as

$$\int_{0}^{G(x)}\exp(-(\lambda+\beta)t)\,dt=\frac{-1}{\lambda+\beta}[\exp(-(\lambda+\beta)G(x))-1] \qquad (4)$$

Now we find another integral as

$$\begin{aligned}\int_{0}^{G(x)}t\exp(-(\lambda+\beta)t)\,dt&=\frac{-G(x)}{\lambda+\beta}\exp(-(\lambda+\beta)G(x))\\&\quad-\frac{1}{(\lambda+\beta)^{2}}[\exp(-(\lambda+\beta)G(x))-1]\end{aligned} \qquad (5)$$

Butting equation (4) and equation (5) in equation (3), we obtain as

$$H_{GWL}(x)=1-\left(\frac{1+\lambda+\lambda G(x)}{\lambda+1}\right)\exp(-(\lambda+\beta)G(x)) \qquad (6)$$

And the pdf of general WLD is

$$h_{GWL}(x)=\frac{\lambda\beta G(x)+\beta(\lambda+1)+\lambda^{2}(1+G(x))}{\lambda+1}g(x)\exp(-(\lambda+\beta)G(x)) \qquad (7)$$

2 THE WEIBULL LINDLEY BURR XII DISTRIBUTION (WLBD)

In this section, we present new special distribution is called Weibull Lindley Burr distribution WLBD (λ, β, c, k). We prove mathematical properties of WLBD, which is including: PDF, CDF, hazard function, reserved hazard function, odd function, quantile function, moment generating function (MGF), k-moments, Lorenz curve, mean and variance. The Burr XII distribution is defined as

$$f(x;c,k) = ck\frac{x^{c-١}}{(١ + x^c)^{k+١}} c, k, x > ٠, \qquad (٨)$$

and corresponding cumulative distribution function

$$F(x;c,k) = ١ - (١ + x^c)^{-k} \qquad (٩)$$

By putting equation (9) in equation (6), we get

$$H_{WLBD}(x) = ١ - \left(\frac{(\lambda + ١) + \lambda(١ - (١ + x^c)^{-k})}{\lambda + ١}\right) \exp\left(-(\lambda + \beta)(١ - (١ + x^c)^{-k})\right)$$

Then

$$H_{WLBD}(x) = ١ - \left(\frac{(٢\square + ١) - \lambda(١ + x^c)^{-k}}{(\lambda + ١)}\right) \exp(-(\lambda + \beta)) \exp\left((\lambda + \beta)(١ + x^c)^{-k}\right) \qquad (١٠)$$

Equation (10) represent CDF of Weibull Lindley Burr distribution.

The plot of cdf of WLBD is

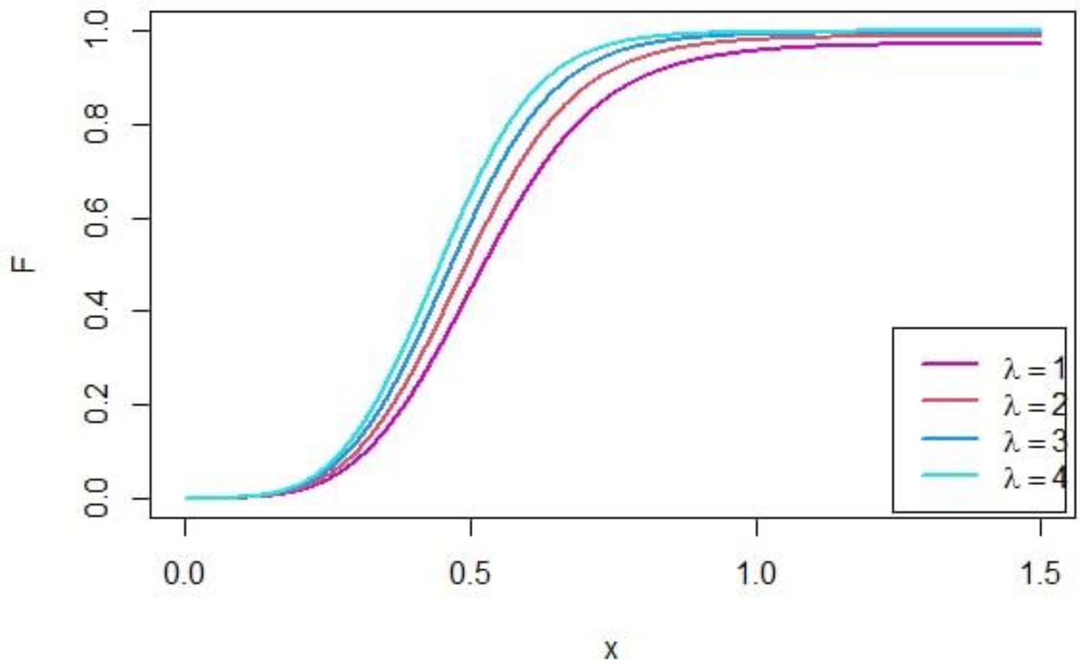


Figure 1: Graph of cdf of WLBD , when $\beta = k = 3 \; and \; c = 4 \; with \; different \; values$

The PDF of WLBD can be found as

$$h_{WLBD} = \left[\frac{\lambda\beta[١ - (١ + x^{c})^{-k}] + \beta(\lambda + ١) + \lambda^{٢}(٢ - (١ + x^{c})^{-k})}{\lambda + ١}\right] \frac{ckx^{c-١}}{(١ + x^{c})^{k+١}} \exp\left(-(\lambda + \beta)\right) \exp\left((\lambda + \beta)(١ + x^{c})^{-k}\right) \qquad (١١)$$

The plot of pdf of WLBD is

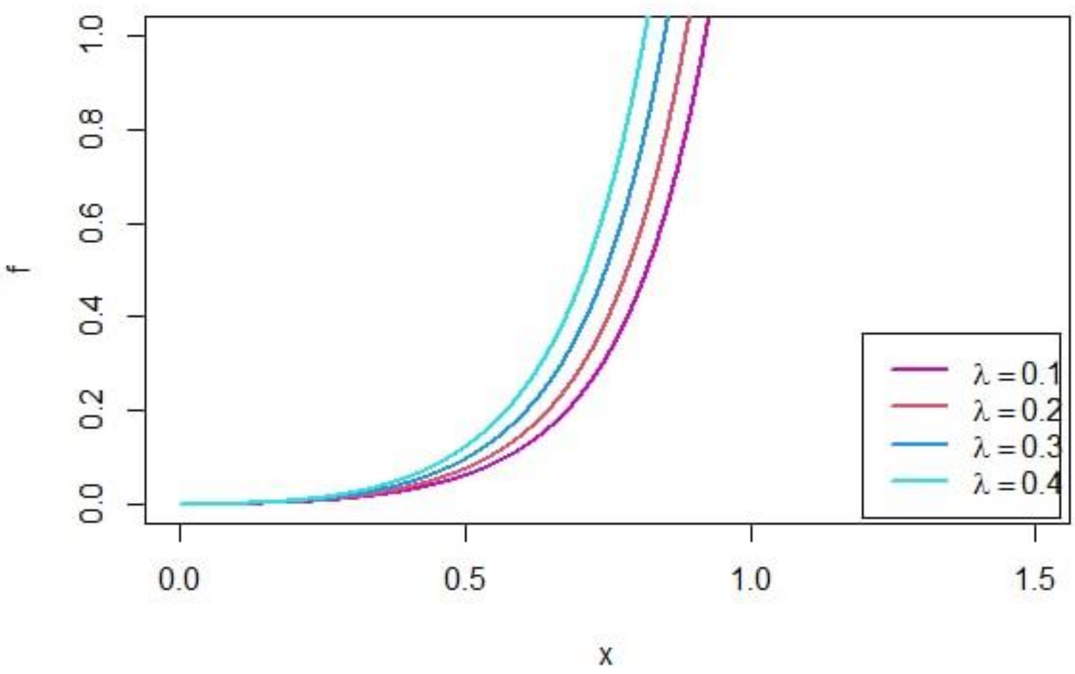


Figure 2: Graph pdf of WLBD when $\beta = 0.1, k = 1$ *and* $c = 4$ *at different values of* λ.

2.1 The Mathematical Properties of WLBD

2.1.1 Survival function

$$S(x) = \left(\frac{(2\square + 1) - \lambda(1 + x^c)^{-k}}{(\lambda + 1)}\right) exp(-(\lambda + \beta)) exp\left((\lambda + \beta)(1 + x^c)^{-k}\right) \qquad (12)$$

2.1.2 hazard rate

The hazard function of WLBD can be calculated as

$$Z(x) = \frac{[\lambda\beta[1 - (1 + x^c)^{-k}] + \beta(\lambda + 1) + \lambda^2(2 - (1 + x^c)^{-k})]ckx^{c-1}}{\left((2\square + 1) - \lambda(1 + x^c)^{-k}\right)(1 + x^c)^{k+1}} \qquad (13)$$

2.1.3 Odd Function

The odd function of WLBD is considered

$$O(x) = \frac{1 - \left(\frac{(2\square + 1) - \lambda(1 + x^c)^{-k}}{(\lambda + 1)}\right) exp(-(\lambda + \beta)) exp\left((\lambda + \beta)(1 + x^c)^{-k}\right)}{\left(\frac{(2\square + 1) - \lambda(1 + x^c)^{-k}}{(\lambda + 1)}\right) exp(-(\lambda + \beta)) exp\left((\lambda + \beta)(1 + x^c)^{-k}\right)}$$

After calculations, the odd function is

$$O(x) = \frac{(\lambda + 1)exp\left((\lambda + \beta)\right)exp\left(-(\lambda + \beta)(1 + x^{c})^{-k}\right)}{\left((2\square + 1) - \lambda(1 + x^{c})^{-k}\right)} - 1 \qquad (14)$$

2.1.4 Moment Generating Function

The moment generating function of WLBD can be derived as

$$\begin{aligned} M(x) &= E(\exp(xt)) \\ &= \int_{0}^{\infty} \exp(xt) h_{WLBD}\, dx \\ &= \int_{0}^{\infty} \exp(xt)\frac{ckx^{c-1}\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)(1+x^{c})^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1-x^{c})^{-k}]\exp\left((\lambda+\beta)(1-x^{c})^{-k}\right)dx \\ &= \frac{ck\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_{0}^{\infty} \exp(xt)\frac{x^{c-1}}{(1+x^{c})^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1-x^{c})^{-k}]\exp\left((\lambda+\beta)(1-x^{c})^{-k}\right)dx \end{aligned}$$

by transformation $p = x^{c}$

By transformations $p = x^{c}$ we get

$$M(x) = \frac{k\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_{0}^{\infty} \exp\left(tp^{\frac{1}{c}}\right)\frac{1}{(1+p)^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1+p)^{-k}]\exp\left((\lambda+\beta)(1+p)^{-k}\right)dp$$

By transformations $y = (1+p)^{-k}$ we obtain

$$M(x) = \frac{\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_{0}^{1} \exp\left(t\left(y^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}}\right)[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)y]\exp\left((\lambda+\beta)y\right)dy$$

By transformations $z = y^{-\frac{1}{k}}$ we obtain

$$M(x) = \frac{k\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_{0}^{1}\sum_{j=0}^{\infty}\frac{t^{j}}{j!}(1-z)^{\frac{j}{c}}z^{-k-1}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)z^{-k}]\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^{n}}{n!}z^{-nk}dz$$

then

$$\begin{aligned} M(x) = \frac{k\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\sum_{j=0}^{\infty}\frac{t^{j}}{j!}\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^{n}}{n!}\Big[&(2\lambda(\beta+\lambda)+\beta)B\left(-k(1+n),\frac{j}{c}+1\right) \\ &- \lambda(\beta+\lambda)B\left(-k(2+n),\frac{j}{c}+1\right)\Big] \qquad (15) \end{aligned}$$

2.1.5 The r−moment

The r−moment can be found as

$$m^r = E(x^r)$$
$$= \int_0^{\infty} x^r h_{WLBD}\,dx$$
$$= \int_0^{\infty} x^r \frac{ckx^{c-1}\exp(-\eta_\tau)}{(\lambda+1)(1+x^c)^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1-x^c)^{-k}]\exp\left((\lambda+\beta)(1-x^c)^{-k}\right)dx$$
$$= \frac{ck\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_0^{\infty}\frac{x^{r+c-1}}{(1+x^c)^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1-x^c)^{-k}]\exp\left((\lambda+\beta)(1-x^c)^{-k}\right)dx$$

By transformations $p = x^c$ we get

$$m^r = \frac{k\exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_0^{\infty}\frac{p^{\frac{r}{c}}}{(1+p)^{k+1}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)(1+p)^{-k}]\exp\left((\lambda+\beta)(1+p)^{-k}\right)dp$$

By transformations $y = (1+p)^{-k}$ we obtain

$$m^r = \frac{exp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_0^{1}\left(y^{-\frac{1}{k}}-1\right)^{\frac{r}{c}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)y]exp\left((\lambda+\beta)y\right)dy$$

By transformations $z = y^{-\frac{1}{k}}$ we obtain

$$m^r = \frac{kexp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\int_0^{1} z^{-k-1}(1-z)^{\frac{r}{c}}[(2\lambda(\beta+\lambda)+\beta) - \lambda(\beta+\lambda)z^{-k}]\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^n}{n!}z^{-nk}dz$$

then

$$m^r = \frac{kexp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^n}{n!}\Big[(2\lambda(\beta+\lambda)+\beta)B\left(-k(1+n),\frac{r}{c}+1\right)$$
$$-\lambda(\beta+\lambda)B\left(-k(2+n),\frac{r}{c}+1\right)\Big] \qquad (16)$$

Putting r=1, then the mean of WLBD

$$E_{WLBD}(x) = \frac{\mathrm{kexp}\left(-(\lambda+\beta)\right)}{(\lambda+1)}\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^n}{n!}\Big[(2\lambda(\beta+\lambda)+\beta)B\left((-k(1+n))k+1,\frac{1}{c}+1\right)$$
$$-\lambda(\beta+\lambda)B\left(-k(2+n)2,\frac{1}{c}+1\right)\Big] \qquad (17)$$

Putting r=2, then we the second moment is

$$E_{WLBD}(x^2) = \frac{kexp\left(-(\lambda+\beta)\right)}{(\lambda+1)}\sum_{n=0}^{\infty}\frac{(\lambda+\beta)^n}{n!}\Big[(2\lambda(\beta+\lambda)+\beta)B\left(-k(1+n),\frac{2}{c}+1\right)$$
$$-\lambda(\beta+\lambda)B\left(-k(2+n),\frac{2}{c}+1\right)\Big]$$

Then the variance is

$$V_{WLBD}(x) = E_{WLBD}(x^2) - (E_{WLBD}(x))^2$$

$$V_{WLBD}(x)=\frac{k\exp(-(\lambda+\beta))}{(\lambda+١)}\sum_{n=٠}^{\infty}\frac{(\lambda+\beta)^n}{n!}\left[(٢\lambda(\beta+\lambda)+\beta)B(-k(١+n))k+١,\frac{٢}{c}+١\right)-\lambda(\beta+\lambda)B\left(-k(٢+n),\frac{٢}{c}+١\right)\Big]$$
$$-\left[\frac{k\exp(-(\lambda+\beta))}{(\lambda+١)}\sum_{n=٠}^{\infty}\frac{(\lambda+\beta)^n}{n!}\left[(٢\lambda(\beta+\lambda)+\beta)B\left(-k(١+n),\frac{١}{c}+١\right)-\lambda(\beta+\lambda)B\left(-k(٢+n),\frac{١}{c}+١\right)\right]\right]^٢ \quad (١٨)$$

2.1.6 Lorenz curve

Lorenz curve WLBD for a positive random variable X is defined as

$$L(H_{WLBD}(x))=\frac{\int_٠^a xh_{WLBD}(x)d(x)}{\int_٠^\infty xh_{WLBD}(x)d(x)}$$

Now we find the numerator is

$$\int_٠^a xh_{WLBD}\,dx=\int_٠^\infty x\frac{ckx^{c-١}\exp(-(\lambda+\beta))}{(\lambda+١)(١+x^c)^{k+١}}[(٢\lambda(\beta+\lambda)+\beta)-\lambda(\beta+\lambda)(١-x^c)^{-k}]\exp\left((\lambda+\beta)(١-x^c)^{-k}\right)dx$$
$$=\frac{ck\exp(-(\lambda+\beta))}{(\lambda+١)}\int_٠^a\frac{x^{١+c-١}}{(١+x^c)^{k+١}}[(٢\eta_٢+\beta)-\lambda(\beta+\lambda)(١-x^c)^{-k}]\exp\left((\lambda+\beta)(١-x^c)^{-k}\right)dx$$

For the denominator can be found as

Let $p=x^c$ then we get

$$\int_٠^\infty xh_{WLBD}(x)d(x)=\frac{ck\exp(-(\lambda+\beta))}{(\lambda+١)}\int_٠^{a^c}\frac{p^{\frac{١}{c}(c)}}{(١+p)^{k+١}}[(٢\lambda(\beta+\lambda)+\beta)-\lambda(\beta+\lambda)(١+p)^{-k}]\exp\left((\lambda+\beta)(١+p)^{-k}\right)$$
$$\times\frac{١}{c}p^{\frac{١}{c}-١}dp$$

$$=\frac{k\exp(-(\lambda+\beta))}{(\lambda+١)}\int_٠^{a^c}\frac{p^{\frac{١}{c}}}{(١+p)^{k+١}}[(٢\lambda(\beta+\lambda)+\beta)-\lambda(\beta+\lambda)(١+p)^{-k}]\exp\left((\lambda+\beta)(١+p)^{-k}\right)dp$$

By transformations $y=(١+p)^{-k}$ one can obtain

$$\int_٠^\infty xh_{WLBD}(x)d(x)=\frac{\exp(-(\lambda+\beta))}{(\lambda+١)}\int_{(١+a^c)^{-k}}^{١}\left(y^{-\frac{١}{k}}-١\right)^{\frac{١}{c}}[(٢\lambda(\beta+\lambda)+\beta)-\lambda(\beta+\lambda)y]\exp((\lambda+\beta)y)dy$$

By transformations $z=y^{-\frac{١}{k}}$ we obtain

$$\int_0^\infty x h_{WLBD}(x)d(x) = \frac{k\exp(-(\lambda+\beta))}{(\lambda+1)} \int_{(1+a^c)}^1 z^{-k-1}(1-z)^{\frac{1}{c}}\left[(2\eta_2+\beta) - \lambda(\beta+\lambda)z^{-k}\right] \sum_{n=0}^\infty \frac{(\lambda+\beta)^n}{n!} z^{-nk}dz$$

then

$$\int_0^\infty x h_{WLBD}(x)d(x) = \frac{k\exp(-(\lambda+\beta))}{(\lambda+1)} \sum_{n=0}^\infty \frac{(\lambda+\beta)^n}{n!}\Bigg[(2\lambda(\beta+\lambda)+\beta)\int_{(1+a^c)}^1 z^{-k(1+n)-1}(1-z)^{\frac{1}{c}}dz$$

$$-\eta_2 \int_{(1+a^c)}^1 z^{-k(2+n)-1}(1-z)^{\frac{1}{c}}dz\Bigg]$$

After some simplifications, then the denominator is

$$\int_0^\infty x h_{WLBD}(x)d(x) = \frac{k\exp(-(\lambda+\beta))}{(\lambda+1)} \sum_{n=0}^\infty \frac{(\lambda+\beta)^n}{n!}\Bigg[(2\lambda(\beta+\lambda)+\beta)B_{(1+a^c)}\left(-k(1+n),\frac{1}{c}+1\right)$$

$$-\lambda(\beta+\lambda)B_{(1+a^c)}\left(-k(2+n),\frac{1}{c}+1\right)\Bigg]$$

where $B_\tau(\alpha,\beta)$ is incomplete beta function.

$$B_\tau(\alpha,\beta) = \int_\tau^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

Then the Lorenz is

$$L(H_{WLR}(x)) = \frac{\sum_{n=0}^\infty \frac{(\lambda+\beta)^n}{n!}\gamma\left(\frac{3}{2},x\right)\left[\frac{\lambda\beta+\beta(\lambda+1)+2\lambda^2}{[\theta^2(n+1)]^{\frac{3}{2}}} - \frac{(\lambda\beta+\lambda^2)}{[\theta^2(n+2)]^{\frac{3}{2}}}\right]}{\sum_{n=0}^\infty \frac{(\lambda+\beta)^n}{n!}\Gamma\left(\frac{3}{2}\right)\left[\frac{\lambda\beta+\beta(\lambda+1)+2\lambda^2}{[\theta^2(1+n)]^{\frac{3}{2}}} - \frac{\lambda\beta+\lambda^2}{\theta^2(n+2)^{\frac{3}{2}}}\right]}$$

$$L = \frac{\sum_{n=0}^\infty \frac{(\lambda+\beta)}{n!}\left[(2\lambda(\beta+\lambda)+\beta)B_{(1+a^c)}\left(-k(1+n),\frac{1}{c}+1\right) - \lambda(\beta+\lambda)B_{(1+a^c)}\left(-k(2+n),\frac{1}{c}+1\right)\right]}{\sum_{n=0}^\infty \frac{(\lambda+\beta)}{n!}\left[(2\lambda(\beta+\lambda)+\beta)B\left(-k(1+n),\frac{1}{c}+1\right) - \lambda(\beta+\lambda)B\left(-k(2+n),\frac{1}{c}+1\right)\right]} \qquad (19)$$

2.1.7 Quantile function

$$H = 1 - \frac{\left(1+2\lambda-\lambda\left(1+x_p^c\right)^{-k}\right)e^{-(\lambda+\beta)}}{\lambda+1} * e^{(\lambda+\beta)\left(1+x_p^c\right)^{-k}}$$

The p–th quantile say x_p of a WLBD random variable defined by $F(x_p) = p$ where $0 < p < 1$

is the root of.

$$\frac{\left(١ + ٢\lambda - \lambda\left(١ + x_p^c\right)^{-k}\right) e^{-(\lambda+\beta)}}{\lambda + ١} * e^{(\lambda+\beta)(١+x_p^c)^{-k}} = ١ - P$$

$$e^{(\lambda+\beta)(١+x_p^c)^{-k}} = \frac{(\lambda + ١)(١ - P)e^{(\lambda+\beta)}}{١ + ٢\lambda - \lambda\left(١ + x_p^c\right)^{-k}}$$

$$\left(١ + x_p^c\right)^{-k} = \frac{١}{\lambda+\beta} \ln\left[\frac{(\lambda + ١)(١ - P)e^{(\lambda+\beta)}}{١ + ٢\lambda - \lambda\left(١ + x_p^c\right)^{-k}}\right]$$

$$١ + x_p^c = \frac{١}{-k}\ln\left[\frac{\ln\left[\frac{(\lambda + ١)(١ - P)e^{(\lambda+\beta)}}{١ + ٢\lambda - \lambda\left(١ + x_p^c\right)^{-k}}\right]}{\lambda + \beta}\right]$$

$$x_p = \left[\ln\left[\frac{\ln\left[\frac{(\lambda + ١)(١ - P)e^{(\lambda+\beta)}}{١ + ٢\lambda - \lambda\left(١ + x_p^c\right)^{-k}}\right]}{\lambda + \beta}\right]^{\frac{١}{-k}} - ١\right]^{\frac{١}{c}} \qquad (٢٠)$$

We find the value of the quantile function numerically by using Newton Raphson.

3 THE ESTIMATION AND SIMULATION OF WLBD

Let $x_١, x_٢, \dots, x_n$ be observed values from the Weibull Lindley Burr Distribution (WLBD) with parameters λ, β, c and k. The likelihood function for $\theta = (\lambda, \beta, c, k)$ is given by

$$\prod_{i=١}^{n} h(x_i;\lambda,\beta,c,k) = \frac{١}{(\lambda+١)^n}\prod_{i=١}^{n}\left[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})\right]$$
$$\times c^n k^n \prod_{i=١}^{n} \frac{x_i^{c-١}}{(١+x_i^c)^{k+١}} \exp\left(-(\lambda+\beta)\sum_{i=١}^{n}(١-(١+x_i^c)^{-k})\right)$$

$$\ln h(x_i;\lambda,\beta,c,k) = -n\ln(\lambda+١) + \sum_{i=١}^{n} \ln\left[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})\right]$$
$$+n\ln c + n\ln k + (c-١)\sum_{i=١}^{n} \ln x_i$$
$$-(k+١)\sum_{i=١}^{n} \ln(١+x_i^c) - (\lambda+\beta)\sum_{i=١}^{n} (١-(١+x_i^c)^{-k})$$

$$U_\lambda = \frac{-n}{\lambda+١} + \sum_{i=١}^{n} \frac{\beta(١-(١+x_i^c)^{-k}) + \beta + ٢\square(٢-(١+x_i^c)^{-k}))}{[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$-\sum_{i=١}^{n} (١-(١+x_i^c)^{-k}) \qquad (٢١)$$

$$U_\beta = \sum_{i=١}^{n} \frac{\lambda(١-(١+x_i^c)^{-k}) + \lambda + ١}{[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})]} - \sum_{i=١}^{n} (١-(١+x_i^c)^{-k}) \qquad (٢٢)$$

$$U_c = \sum_{i=١}^{n} \frac{\lambda k(\beta+\lambda)(١+x_i^c)^{-k-١} x_i^c \log(x_i)}{[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$+\frac{n}{c} + \sum_{i=١}^{n} \log(x_i) - (k+١)\sum_{i=١}^{n} \frac{x_i^c \log(x_i)}{١+x_i^c} - (\lambda+\beta)k\sum_{i=١}^{n} (١+x_i^c)^{-k-١} x_i^c \log(x_i) \qquad (٢٣)$$

$$U_k = \sum_{i=١}^{n} \frac{-(\lambda\beta+\lambda^٢)(١+x_i^c)^{-k}\log(١+x_i^c)}{[\lambda\beta(١-(١+x_i^c)^{-k}) + \beta(\lambda+١) + \lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$+\frac{n}{k} - \log(١+x_i^c) + (\lambda+\beta)\sum_{i=١}^{n} (١+x_i^c)^{-k}\log(١+x_i^c) \qquad (٢٤)$$

The Fisher information matrix I of WLBD is given by

$$i(\lambda,\beta,c,k) = -E\big(I(\lambda,\beta,c,k)\big) = -\begin{pmatrix} E(U_{\lambda\lambda}) & E(U_{\lambda\beta}) & E(U_{\lambda c}) & E(U_{\lambda k}) \\ E(U_{\beta\lambda}) & E(U_{\beta\beta}) & E(U_{\beta c}) & E(U_{\beta k}) \\ E(U_{c\lambda}) & E(U_{c\beta}) & E(U_{cc}) & E(U_{ck}) \\ E(U_{k\lambda}) & E(U_{k\beta}) & E(U_{kc}) & E(U_{kk}) \end{pmatrix} \qquad (٢٥)$$

$$U_{\lambda\lambda} = \frac{n}{(\lambda+1)^2} + \sum_{i=1}^{n} \frac{2(2-(1+x_i^c)^{-k})}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{[\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square(2-(1+x_i^c)^{-k})]^2}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \quad (٢٦)$$

$$U_{\lambda\beta} = \sum_{i=1}^{n} \frac{[(1-(1+x_i^c)^{-k}+1)]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{[\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square(1+(1-(1+x_i^c)^{-k}))][\lambda(1-(1+x_i^c)^{-k}) + \lambda + 1]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \quad (٢٧)$$

$$U_{\lambda c} = \sum_{i=1}^{n} \frac{\left(\beta k(1+x_i^c)^{-k-1} x_i^c \log(x_i) + 2\square\square(1+x_i^c)^{-k} x_i^c \log(x_i)\right)}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square(1+(1-(1+x_i^c)^{-k}))}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \times [\lambda\beta k(1+x_i^c)^{-k-1} x_i^c \log(x_i) + \lambda^2 k(1+x_i^c)^{-k-1} x_i^c \log(x_i)] - k\sum_{i=1}^{n} (1+x_i^c)^{-k-1} \log(x_i)(x_i^c) \quad (٢٨)$$

$$U_{\lambda k} = \sum_{i=1}^{n} \frac{-\beta(1+x_i^c)^{-k}\log(1+x_i^c) - 2\square(1+x_i^c)^{-k}\log(1+x_i^c)}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square(1+(1-(1+x_i^c)^{-k}))}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \times [\lambda\beta(1+x_i^c)^{-k}\log(1+x_i^c) - \lambda^2(1+x_i^c)^{-k}\log(1+x_i^c)] + \sum_{i=1}^{n} (1+x_i^c)^{-k}\log(1+x_i^c) \quad (٢٩)$$

$$U_{\beta\lambda} = \sum_{i=1}^{n} \frac{[(1-(1+x_i^c)^{-k}+1)]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{[\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square(1+(1-(1+x_i^c)^{-k}))][\lambda(1-(1+x_i^c)^{-k}) + \lambda + 1]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \quad (٣٠)$$

$$U_{\beta\beta} = \sum_{i=1}^{n} \frac{-[\lambda(1-(1+x_i^c)^{-k}) + (\lambda+1)]^2}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \quad (٣١)$$

$$U_{\beta c} = \sum_{i=1}^{n} \frac{\lambda k(1+x_i^c)^{-k-1} x_i^c (\log x_i)}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} - \sum_{i=1}^{n} \frac{[\lambda(1-(1+x_i^c)^{-k}) + \lambda + 1][-k\lambda\beta(1+x_i^c)^{-k-1} x_i^c \log(x_i)]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} - \sum_{i=1}^{n} k(1+x_i^c)^{-k-1} x_i^c \log(x_i) \quad (٣٢)$$

$$U_{\beta k}=\sum_{i=١}^{n}\frac{-\lambda(١+x_i^c)^{-k}\log(١+x_i^c)}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$+\sum_{i=١}^{n}\frac{[\lambda(١-(١+x_i^c)^{-k})+\lambda+١][\lambda\beta(١+x_i^c)^{-k}\log(١+x_i^c)+\lambda^٢(١+x_i^c)^{-k}\log(١+x_i^c)]}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]^٢}$$
$$+\sum_{i=١}^{n}(١+x_i^c)^{-k}\log(١+x_i^c)\qquad(٣٣)$$

$$U_{c\lambda}=\sum_{i=١}^{n}\frac{\left(\beta k(١+x_i^c)^{-k-١}x_i^c\log(x_i)+٢\square\square(١+x_i^c)^{-k}x_i^c\log(x_i)\right)}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$-\sum_{i=١}^{n}\frac{\beta(١-(١+x_i^c)^{-k})+\beta+٢\square\left(١+(١-(١+x_i^c)^{-k})\right)}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]^٢}$$
$$\times[\lambda\beta k(١+x_i^c)^{-k-١}x_i^c\log(x_i)+\lambda^٢k(١+x_i^c)^{-k-١}x_i^c\log(x_i)]-k\sum_{i=١}^{n}(١+x_i^c)^{-k-١}\log(x_i)(x_i^c)\qquad(٣٤)$$

$$U_{c\beta}=\sum_{i=١}^{n}\frac{\lambda k(١+x_i^c)^{-k-١}x_i^c(\log x_i)}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$-\sum_{i=١}^{n}\frac{[\lambda(١-(١+x_i^c)^{-k})+\lambda+١][-k\lambda\beta(١+x_i^c)^{-k-١}x_i^c\log(x_i)]}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)}$$
$$+\lambda^٢(٢-(١+x_i^c)^{-k})]^٢-\sum_{i=١}^{n}k(١+x_i^c)^{-k-١}x_i^c\log(x_i)x_i\qquad(٣٥)$$

$$U_{cc}=\sum_{i=١}^{n}\frac{\lambda k(\beta+\lambda)\log x_i^c[(-k-١)(١+x_i^c)^{-k-٢}x_i^{٢\square}\log x_i^c+(١+x_i^c)^{-k-١}x_i^c\log x_i^c]}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]}$$
$$-\sum_{i=١}^{n}\frac{[\lambda\beta k(١+x_i^c)^{-k-١}x_i^c\log(x_i)+\lambda^٢k(١+x_i^c)^{-k-١}x_i^c\log x_i]^٢}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]^٢}$$
$$-\frac{n}{c^٢}-(k+١)\sum_{i=١}^{n}\frac{x_i^c(\log(x_i))^٢)(١+x_i^c)-x_i^{٢\square}(\log(x_i))^٢}{(١+x_i^c)^٢}$$
$$-(\lambda+\beta)k\sum_{i=١}^{n}x_i^c(\log(x_i))^٢[(-k-١)(١+x_i^c)^{-k-٢}x_i^c+(١+x_i^c)^{-k-١}]\qquad(٣٦)$$

$$U_{ck}=\sum_{i=١}^{n}\frac{\lambda(\beta+\lambda)x_i^c\log(x_i)[(١+x_i^c)^{-k-١}+k(١+x_i^c)^{-k-١}\log(١+x_i^c)]}{[\lambda\beta(١-(١+x_i^c)^{-k})+\beta(\lambda+١)+\lambda^٢(٢-(١+x_i^c)^{-k})]^٢}$$
$$-\sum_{i=١}^{n}\frac{x_i^c\log(x_i)}{١+x_i^c}-(\lambda+\beta)\sum_{i=١}^{n}x_i^c\log(x_i)[(١+x_i^c)^{-k-١}+k(١+x_i^c)^{-k-١}log(١+x_i^c)]\qquad(٣٧)$$

$$
\begin{aligned}
U_{k\lambda} = & \sum_{i=1}^{n} \frac{-\beta(1+x_i^c)^{-k}\log(1+x_i^c) - 2\square(1+x_i^c)^{-k}\log(1+x_i^c)}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} \\
& - \sum_{i=1}^{n} \frac{\beta(1-(1+x_i^c)^{-k}) + \beta + 2\square\left(1+(1-(1+x_i^c)^{-k})\right)}{[\lambda - \beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \\
& \times [\lambda\beta(1+x_i^c)^{-k}\log(1+x_i^c) - \lambda^2(1+x_i^c)^{-k}\log(1+x_i^c)] \\
& + \sum_{i=1}^{n} (1+x_i^c)^{-k}\log(1+x_i^c) \qquad (38)
\end{aligned}
$$

$$
\begin{aligned}
U_{k\beta} = & \sum_{i=1}^{n} \frac{-\lambda(1+x_i^c)^{-k}\log(1+x_i^c)}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]} \\
& + \sum_{i=1}^{n} \frac{[\lambda(1-(1+x_i^c)^{-k}) + \lambda + 1][\lambda\beta(1+x_i^c)^{-k}\log(1+x_i^c) + \lambda^2(1+x_i^c)^{-k}\log(1+x_i^c)]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \\
& + \sum_{i=1}^{n} (1+x_i^c)^{-k}\log(1+x_i^c) \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
U_{kc} = & \sum_{i=1}^{n} \frac{\lambda(\beta+\lambda)x_i^c\log(x_i)[(1+x_i^c)^{-k-1} + k(1+x_i^c)^{-k-1}\log(1+x_i^c)]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \\
& - \sum_{i=1}^{n} \frac{x_i^c\log(x_i)}{1+x_i^c} - (\lambda+\beta)\sum_{i=1}^{n} x_i^c\log(x_i)[(1+x_i^c)^{-k-1} + k(1+x_i^c)^{-k-1}\,log(1+x_i^c)] \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
U_{kk} = & \sum_{i=1}^{n} \frac{-(\lambda\beta+\lambda^2)(1+x_i^c)^{-k}(\log(1+x_i^c))^2[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \\
& - \sum_{i=1}^{n} \frac{[(\lambda\beta+\lambda^2)(1+x_i^c)^{-k}\log(1+x_i^c)]^2}{[\lambda\beta(1-(1+x_i^c)^{-k}) + \beta(\lambda+1) + \lambda^2(2-(1+x_i^c)^{-k})]^2} \\
& - \frac{n}{k^2} + (\lambda+\beta)\sum_{i=1}^{n} (1+x_i^c)^{-k}(\log(1+x_i^c))^2 \qquad (41)
\end{aligned}
$$

The simulation of WLBD is assessing the performance of the maximum likelihood estimators given by

equations (29), equation (30) and equation (31) with respect to sample size n. The assessment was based

on The first, generate ten thousand samples of size n from equation (13). The second, we compute the

maximum likelihood estimates for the ten thousand samples, say

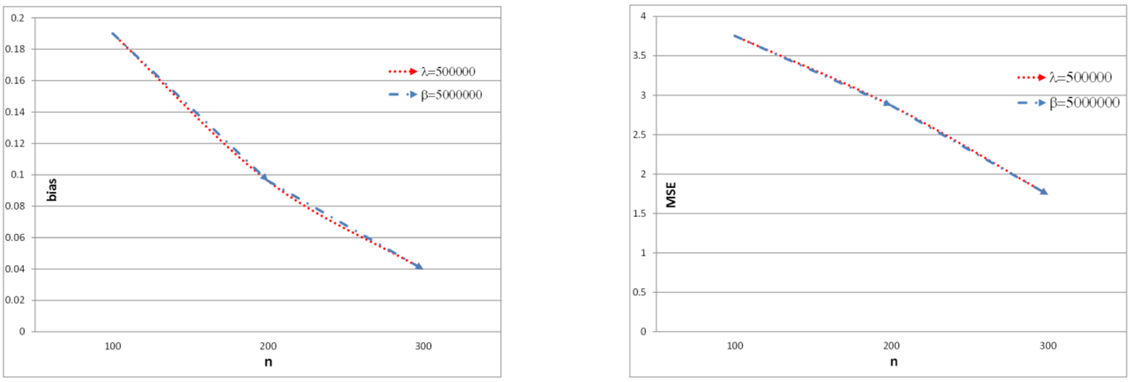
$\hat{\lambda}_i, \hat{\beta}_i, \hat{c}_i, \hat{k}_i$ for $i = 1, 2, \ldots, 10000$. Finally. we calculate the biases and mean squared errors given by

$$\text{bias}_h(n) = \frac{١}{١٠٠٠٠}\sum_{i}^{١٠٠٠٠}\left(\hat{h}_i - h_i\right) \qquad (٤٢)$$

and

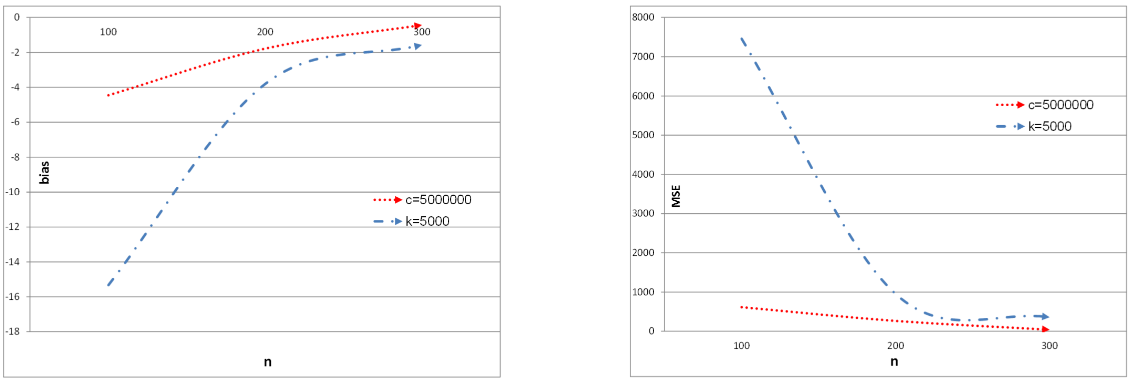
$$MSE_h(n) = \frac{١}{١٠٠٠٠}\sum_{i}^{١٠٠٠٠}\left(\hat{h}_i - h_i\right)^{٢} \qquad (٤٣)$$

where $h = \lambda, \beta, c, k$ under assumption n = $100,150,200$. The plot of bias and MSE of λ, β, c and k with respect to WLBD (λ, β, c, k) respectively in Figure 1 and Figure 2.



(a) (b)

FIGURE 1. (a) The graph of bias of WLBD for (β,λ) (b) control the layout mean square error of WLBD for (β,λ)



(a) (b)

FIGURE 2. (a) The graph of bias of WLBD for (c,k) (b) control the layout mean square error of WLBD for (c,k)

5 Application

In this section, we calculate the AIC and BIC of the WLRD, Rayleigh distribution, Lindley distribution and new general Lindley distribution (NGLD) to show that which these models is good fit with respect to real data, in this side the AIC and BIC are defined as

$$AIC = 2k - 2\log L \tag{52}$$

$$BIC = k\log n - 2\log L \tag{53}$$

Where k is parameters number , n is sample size and L is likelihood function.

4.1 Data set 1

The data set represent remission times (months) of a random sample of 128 bladder cancer patients reported in [21].

Table 1: The AIC, BIC and −2logL of the models based on data set 1

Models	AIC	BIC	−2logL
WLBD	−7920.75	7909.342	−7928.75
Burr XII	867.4807	873.1847	863.4807
Lindley	841.06	843.892	839.04

In above table 1, we find that AIC of our model (WLBD) is negative value , therefore it is smallest value comparison to it value of Burr distribution and Lindley distribution. So, the WLBD is good fit better than of Burr and Lindley distributions.

4.2 Data set 2

The data set represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by [22].

Table 2: The AIC, BIC and −2logL of the models based on data set 2

Models	AIC	BIC	−2logL
WLBD	−5243.004	− 5233.898	−5251.004

Burr	200.8386	205.392	196.8386
Lindley	215.857	218.133	213.857

In above table 2, we find that that AIC and BIC of our model (WLBD) are negative, therefore both are smallest comparison to their values in Burr and Lindley distributions. So, the WLBD is good fit better than of Burr and Lindley.

5 CONCLUSION

In this paper, we derive new special distribution is called Weibull Lindley Burr XII distribution WLBD(λ, β, c, k). We have studied of probability density and hazard rate functions, moments, moment generating function, Lorenz curve, numerical maximum likelihood estimators and make simulation. we draw the bias and mean square error for WLBD for every parameter and estimators. Our model WLBD based on data set 1 is good fit better than of Burr and Lindley distributions, while based on data set 2, the WLBD is also good fit better than of Burr and Lindley distributions.

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