Bivariate Exponentiated Exponential Pareto distribution

Kareema Abed Al-Kadim and Ashraf Alawi Mahdi
Mathematics Department, College of Education of Pure Sciences, University of Babylon, Iraq

Received Date: 29 / 3 / 2016
Accepted Date: 21 / 6 / 2017

الخلاصة

في هذه البحث، نقدم التوزيع المعمم للأسي باريتو بمتغيرين. اشتققنا بعض خصائص التوزيع، كدالة الكثافة الاحتمالية والهامشية، الدالة المولدة للعزوم الهامشية، دالة الموثوقية ومعكوس داله الخطر. أخيراً، قدمنا تقدير الامكان الاعظم.

الكلمات المفتاحية

التوزيع المعمم للأسي باريتو للمتغيرين، الدالة الموثوقية، معكوس دالة الخطر.

Abstract

In this search, a bivariate exponentiated exponential Pareto distribution is presented. We derived some properties of the distribution, as probability density function and its marginal, marginal moment generating function, reliability function and reversed hazard function. Finally, we presented the maximum likelihood estimation.

Keywords

Exponentiated exponential Pareto distribution, reliability function, reversed hazard function.

1. Introduction

Al-Kadim and Boshi [1] introduced the exponential Pareto (EP) distribution that is the Weibull distribution is special case of the EP distribution.

The statistical multivariate analysis is important in many fields such as data reduction and hypotheses testing (see Johnson, R. A., Wichern, D. W.,2007) [2]. The object of this search is to construct a 2- dimensional exponentiated exponential Pareto (BEEP) distribution by using the similar method to those used by Sherpieny et al [3] in finding a bivariate distribution with generalized Gompertz, bivariate generalized exponential marginal distributions of Kundu and Gupta [4]. A new family of 2- dimensional distributions was introduced by Sarhan and Balakrishnan [5]. Block, Langberg and Stoffer [6] presented a 2- dimensional exponential and geometric autoregressive and autoregressive moving average models, another class of bivariate Gompertz distributions was studied by Al-Khedhairi and El-Gohary [7], Kundu [8] presented a bivariate geometric (maximum) generalized exponential distribution.

In section (2), we describe the models and discuss some properties. In section (3) we present the marginal of moment and moment generating functions of proposed 2- dimensional distribution. In section (4) we introduce some reliability functions. In section (5) we obtain the parameter estimation using MLE. Finally, some conclusions for the results are given in Section (6).

2. Bivariate Exponentiated Exponential Pareto distribution

In this section we introduce the BEEP distribution. We discuss its distribution function (CDF), probability density function (PDF) and some properties of this distribution.

The random variable Y is distributed Exponentiated Exponential Pareto (EEP) distribution with parameters λ and p, that are scale parameters and θ and α are shape parameters, if its cdf is defined as follows:

$$F_{EEP}(y;\alpha,\lambda,p,\theta)=(1-\exp[-\lambda(y/p)^{\theta}])^{\alpha},\quad y,\alpha,\lambda,p,\theta>0 \qquad (1)$$

The pdf of EEP distribution is

$$f_{EEP}(y;\alpha,\lambda,p,\theta)=\alpha\lambda\ \theta\ p^{-\theta}\ y^{\theta-1}\ \exp[-\lambda(y/p)^{\theta}]\ (1-\exp[-\lambda(y/p)^{\theta}\ )^{\alpha-1} \qquad (2)$$

2.1. The Cumulative Distribution Function

In the following theorem introduce the cumulative distribution function of the 2-dimentional vector (Y_1,Y_2).

2.1.1. Theorem

The cdf of the 2-dimentional vector (Y_1,Y_2) that has BEEP $(\alpha_1,\alpha_2,\alpha_3,p,\lambda,\theta)$, where $Y_1=\max(D_1,D_3)$ and $Y_2=\max(D_2,D_3)$, and the independent random variables distributed EEP with the shape parameters $\alpha_1,\alpha_2,\alpha_3,\theta$ and the scale parameters p,λ, $j=1,2,3$, $D_j\sim(\alpha_j,p,\lambda,\theta)$, is as follows

$$F_{Y_1,Y_2}(y_1,y_2)=(1-\exp[-\lambda(y_1/p)^{\theta}])^{\alpha_1}(1-\exp[-\lambda(y_2/p)]^{\theta})^{\alpha_2}\ (1-\exp[-\lambda(u/p)]^{\theta})^{\alpha_3} \qquad (3)$$

where $u=\min(y_1,y_2)$

Proof: We know that $F_{Y_1,Y_2}(y_1,y_2)=P(Y_1\le y_1,Y_2\le y_2)$, and since $Y_1=\max(D_1,D_3)$ and $Y_2=\max(D_2,D_3)$

That is $F_{Y_1,Y_2}(y_1,y_2)=P(\max(D_1,D_3)\le y_1,\max(D_2,D_3)\le y_2)$

$$=P(D_1\le y_1,D_2\le y_2,D_3\le \min(y_1,y_2))$$

As $D_j\sim(\alpha_j,p,\lambda,\theta)$ are independent random variables, we get

$$F_{Y_1,Y_2}(y_1,y_2)=P(D_1\le y_1)P(D_2\le y_2)P(D_3\le \min(y_1,y_2))$$

$$=F_{D_1}(y_1;\alpha_1,p,\lambda,\theta)\ F_{D_2}(y_2;\alpha_2,p,\lambda,\theta)\ F_D(u;\alpha_3,p,\lambda,\theta)$$

$$=(1-\exp[-\lambda(y_1/p)^\theta])_{\alpha_1}\ (1-\exp[-\lambda y_2/p)]^\theta)^{\alpha_2}\ (1-\exp[-\lambda(u/p)]^\theta)^{\alpha_3}\ \blacksquare$$

2.2. The Probability Density Function

We can derive the pdf of the (Y_1,Y_2) in the following theorem.

2.2.1. Theorem If the cdf of (Y1,Y2) is as in (3), the pdf is

$f_{Y_1,Y_2}(y_1,y_2)=$

1) $$f_1(y_1,y_2)=(\alpha_1+\alpha_3)\,\lambda\,\theta\,p^{-\theta}{y_1}^{\theta-1}e^{-\lambda\left(\frac{y_1}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1+\alpha_3-1}$$
$$\times\,\alpha_2\,\lambda\,\theta\,p^{-\theta}{y_2}^{\theta-1}e^{-\lambda\left(\frac{y_2}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta]\right)^{\alpha_2-1} \quad (4)$$
for $y_1<y_2$

2) $$f_2(y_1,y_2)=\alpha_1\,\lambda\,\theta\,p^{-\theta}{y_1}^{\theta-1}e^{-\lambda\left(\frac{y_1}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1-1}$$
$$\times\,(\alpha_2+\alpha_3)\,\lambda\,\theta\,p^{-\theta}{y_2}^{\theta-1}e^{-\lambda\left(\frac{y_2}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta]\right)^{\alpha_2+\alpha_3-1} \quad (5)$$
for $y_2<y_1$

$$f_3(y,y)=\alpha_3\,\lambda\,\theta\,p^{-\theta}y^{\theta-1}e^{-\lambda\left(\frac{y}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1} \quad (6)$$
for $y_1=y_2=y$

Proof. For $y_1<y_2$

$$F_{Y_1,Y_2}(y_1,y_2)=F_1(y_1,y_2)=\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1+\alpha_3}\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta]\right)^{\alpha_2} \quad (7)$$

Then, $\frac{\partial^2 F_{Y_1,Y_2}(y_1,y_2)}{\partial Y_1\partial Y_2}=\frac{\partial^2 F_1(y_1,y_2)}{\partial Y_1\partial Y_2}=f_1(y_1,y_2)$

Also, $y_2<y_1$.

$$F_2(y_1,y_2)=\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1}\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta]\right)^{\alpha_2+\alpha_3} \quad (8)$$

And then $\frac{\partial^2 F_2(y_1,y_2)}{\partial Y_1\partial Y_2}=f_2(y_1,y_2).$

But when to find $f_3(y,y)$, we use the following formula to derive $f_3(y,y)$

$$\int_0^\infty\int_0^{y_2} f_1(y_1,y_2)\,dy_1\,dy_2+\int_0^\infty\int_0^{y_1} f_2(y_1,y_2)\,dy_2\,dy_1+\int_0^\infty f_3(y,y)dy=1$$

let $J_1=\int_0^\infty\int_0^{y_2} f_1(y_1,y_2)\,dy_1\,dy_2$ and $J_2=\int_0^\infty\int_0^{y_1} f_2(y_1,y_2)\,dy_2\,dy_1$

Then

$$J_1=\int_0^\infty\int_0^{y_2}(\alpha_1+\alpha_3)\,\lambda\,\theta\,p^{-\theta}{y_1}^{\theta-1}exp[-\lambda\left(\frac{y_1}{p}\right)^\theta\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1+\alpha_3-1}$$
$$\times\,\lambda\alpha_2\,\theta\,p^{-\theta}{y_2}^{\theta-1}\,exp\left[-\lambda\left(\frac{y_1}{p}\right)^\theta\right]\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta\right)^{\alpha_2-1}dy_1dy_2$$
$$=\int_0^\infty\ \lambda\alpha_2\,\theta\,p^{-\theta}{y_2}^{\theta-1}\,exp\left[-\lambda\left(\frac{y_1}{p}\right)^\theta\right]\left(1-exp[-\lambda\left(\frac{y_2}{p}\right)^\theta\right)^{\alpha_1+\alpha_2+\alpha_3-1}dy_2 \quad (9)$$

Similarly

$$J_2=\int_0^\infty\ \alpha_1\,\lambda\,\theta\,p^{-\theta}{y_1}^{\theta-1}e^{-\lambda\left(\frac{y_1}{p}\right)^\theta}\left(1-exp[-\lambda\left(\frac{y_1}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1}dy_1 \quad (10)$$

From (9) and (10), we get

$$\int_0^\infty f_3(y,y)dy=(\alpha_1+\alpha_2+\alpha_3)\,\lambda\,\theta p^{-\theta}$$
$$\times\int_0^\infty y^{\theta-1}\,exp[-\lambda\left(\frac{y}{p}\right)^\theta\left(1-exp[-\lambda\left(\frac{y}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1}dy$$
$$-\int_0^\infty\ \alpha_2\,\lambda\,\theta\,p^{-\theta}y^{\theta-1}exp[-\lambda\left(\frac{y}{p}\right)^\theta]\left(1-exp[-\lambda\left(\frac{y}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1}dy$$
$$-\int_0^\infty\ \alpha_1\,\lambda\,\theta\,p^{-\theta}y^{\theta-1}exp[-\lambda\left(\frac{y}{p}\right)^\theta]\left(1-exp[-\lambda\left(\frac{y}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1}dy$$

hence

$$f_3(y,y)=\alpha_3\,\lambda\,\theta\,p^{-\theta}y^{\theta-1}exp[-\lambda\left(\frac{y}{p}\right)^\theta e^{-\lambda\left(\frac{y}{p}\right)^\theta}]\left(1-exp[-\lambda\left(\frac{y}{p}\right)^\theta]\right)^{\alpha_1+\alpha_2+\alpha_3-1}$$

2.3. Marginal Probability Density Functions

We derive the marginal probability density function of (Y_1,Y_2) in this subsection.

2.3.1. Theorem The marginal probability density functions of Y_j (j=1,2) is given by

$$f_{Y_j}(y_j)=(\alpha_j+\alpha_3)\,\lambda\,\theta\,p^{-\theta}y_j^{\theta-1}exp[-\lambda\left(\frac{y_j}{p}\right)^\theta]\left(1-exp[-\lambda\left(\frac{y_j}{p}\right)^\theta]\right)^{\alpha_i+\alpha_3-1}$$
$$=f_{Y_i}(y_j;\alpha_j+\alpha_3,p,\lambda,\theta) \quad (11)$$

proof:

The marginal cumulative distribution function of Y_j, say $F(Y_j)(y_j)$, as follows:

$$F_{Y_j}(y_j)=F_j(y_j)=P(Y_j\le y_j)=P(\max(D_j,D_3)\le y_j)=P(D_j\le y_j,D_3\le y_j) \quad (12)$$

and since D_j is independent of D_3, then

$$F_j(y_j) = \left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j}\left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_3} = \left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j+\alpha_3}$$
$$= F_{Y_j}(y_j, \alpha_j + \alpha_3, p, \lambda, \theta) \qquad (13)$$

By differentiation w.r.t. Y_j, then we get $fY_j(y_j)$ as in (11)

2.4. Conditional Probability Density Functions

We present the conditional probability density functions of $(Y_1,Y_2)$by using the marginal probability density functions in the following theorem.

2.4.1. Theorem The conditional probability density functions of Y_j, given $Y_k=y_k$, $f_{Y_j/Y_k}(y_j y_k)$,j,k=1,2,j≠k, is given by

for $y_j<y_k$

$$f^{(1)}_{Y_j/Y_k}\left(y_j/y_k\right) = \frac{(\alpha_j+\alpha_3)\,\alpha_k\lambda\,\theta\,p^{-\theta}{y_j}^{\theta-1}e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\left(1-exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j+\alpha_3-1}}{(\alpha_k+\alpha_3)\left(1-exp[-\lambda\left(\frac{y_k}{p}\right)^{\theta}]\right)^{\alpha_3}} \qquad (14)$$

for $y_k < y_j$

$$f^{(2)}_{Y_j/Y_k}\left(y_j/y_k\right) = \alpha_j\,\lambda\,\theta\,p^{-\theta}{y_j}^{\theta-1}e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j-1} \qquad (15)$$

and for $y_j = y_k = y$

$$f^{(3)}_{Y_j/Y_k}\left(y_j/y_k\right) = \frac{\alpha_3\left(1-exp[-\lambda\left(\frac{y}{p}\right)^{\theta}]\right)^{\alpha_1-1}}{(\alpha_2+\alpha_3)} \qquad (16)$$

Proof.

We can prove this theorem by using the relation

$$f_{Y_j,Y_k}\left(y_j/y_k\right) = \frac{f_{Y_j,Y_k}(y_j,y_k)}{f_{Y_k}(y_k)} \quad ,j \neq k = 1,2 \qquad (17)$$

Using (4) where j,k are replaced instead of 1,2 respectively and using (11) to prove (14), Therefore we use (5),(6), (11) to prove (15), and(16). ■

3. The Marginal of Moment and Moment Generating Functions

We present the marginal of moment and moment generating functions of Y_j.

3.1. The Marginal Moment

We present the marginal moment of Y_j,j=1,2.

3.2. Theorem

The r^th moments of Y_j is given by

$$E\left(Y_j^r\right) = (\alpha_j + \alpha_3)\,\lambda^{\frac{-r}{\theta}}p^r\sum_{i=0}^{\infty}(-1)^i\binom{\alpha_j+\alpha_3-1}{i}\frac{\Gamma\left(\frac{r}{\theta}+1\right)}{(i+1)^{\left(\frac{r}{\theta}+1\right)}}, \quad r=1,2,.... \qquad (18)$$

Proof

$$E\left(Y_j^r\right) = \int_0^{\infty} y_j^r f_{Y_j}(y_j)\,dy_j = (\alpha_j + \alpha_3)\,\lambda\,\theta\,p^{-\theta}I$$

where

$$I = \int_0^{\infty} y_j^{r+\theta-1}\exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j+\alpha_3-1} dy_j$$

Since $0 < (1-\exp[-\lambda (y_j/p)^{\theta}])<1$ for $y_j > 0$, then by using the binomial series expansion

$$\left(1 - exp[-\lambda\left(\frac{y_j}{p}\right)^{\theta}]\right)^{\alpha_j+\alpha_3-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha_j+\alpha_3-1}{i}\exp[-i\lambda\left(\frac{y_j}{p}\right)^{\theta}] \qquad (19)$$

That is

$$E\left(Y_j^r\right) = (\alpha_j + \alpha_3)\lambda\,\theta\,p^{-\theta}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha_j+\alpha_3-1}{i}\int_0^{\infty} y_j^{r+\theta-1}\exp[-i\lambda\left(\frac{y_j}{p}\right)^{\theta}]\,dy_j$$

Then

■

3.2.1. The Marginal Moment Generating Function

We find the marginal moment generating function of Y_j, (j=1,2) in the following lemma

.

3.2.2. Proposition

If Y_j~BEEPD, then the marginal moment generating function of

Y_i (i=1.2) is:

$$M_{Y_j}(t_j) = (\alpha_j + \alpha_3)\,\lambda^{\frac{-r}{\theta}} p^r \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} (-1)^i \frac{t_j^r}{r!} \binom{\alpha_j+\alpha_3-1}{i} \frac{\Gamma\left(\frac{r}{\theta}+1\right)}{(i+1)^{\left(\frac{r}{\theta}+1\right)}} \quad (20)$$

Proof

$$M_{Y_j}(t_j) = E(e^{t_j y_j}) = \int_0^{\infty} e^{t_j y_j} f_j(y_j)\, dy_j$$

$$= \sum_{r=0}^{\infty} \frac{t_j^r}{r!} \int_0^{\infty} y_j^r f_j(y_j) dy_j = \sum_{r=0}^{\infty} \frac{t_j^r}{r!} E(Y_j^r)$$

3.2.3. Using Theorem

$$M_{Y_j}(t_j) = (\alpha_j + \alpha_3)\,\lambda^{\frac{-r}{\theta}} p^r \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} (-1)^i \frac{t_j^r}{r!} \binom{\alpha_j+\alpha_3-1}{i} \frac{\Gamma\left(\frac{r}{\theta}+1\right)}{(i+1)^{\left(\frac{r}{\theta}+1\right)}}$$

4. Reliability Functions

We discuss some reliability functions, the reliability function, hazard function and the reversed hazard functions in this section.

4.1. Reliability Function

We introduce the reliability function of (Y_1,Y_2) in following theorem .

4.1.1. Theorem The reliability function of the random vector (Y_1,Y_2) is:

$R_{Y_1,Y_2}(y_1,y_2)$

1) $$R_1(y_1,y_2) = 1 - \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_3} - \left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^{\theta}}\right)^{\alpha_2+\alpha_3} + \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_3}\left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^{\theta}}\right)^{\alpha_2} \quad (21)$$

for $y_1 < y_2$

2) $$R_2(y_1,y_2) = 1 - \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_3} - \left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^{\theta}}\right)^{\alpha_2+\alpha_3} + \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^{\theta}}\right)^{\alpha_1}\left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^{\theta}}\right)^{\alpha_2+\alpha_3} \quad (22)$$

for $y_2 < y_1$

and

$$R_3(y,y) = 1 - \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_3} - \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}\right)^{\alpha_2+\alpha_3} + \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_2+\alpha_3} \quad (23)$$

for $y_1 = y_2 = y$. And using $\exp\left[-\lambda\left(\frac{y}{p}\right)^{\theta}\right] = e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}$

Proof.

The reliability function of (Y_1,Y_2) can be obtained by:

$$R_{Y_1,Y_2}(y_1,y_2) = 1 - [F_1(y_1) + F_2(y_2) - F_{Y_1,Y_2}(y_1,y_2)] \quad (24)$$

$$R_{Y_1,Y_2}(y_1,y_2) = 1 - \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_3} - \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^{\theta}}\right)^{\alpha_2+\alpha_3} + \left(1 - e^{-\lambda\left(\frac{u}{p}\right)^{\theta}}\right)^{\alpha_1+\alpha_2+\alpha_3}$$

where u=min(y_1,y_2)

that is if y1= y2= y, we use the following formula

$$R_{Y_1,Y_2}(y_1,y_2) = R_3(y,y) = 1 - F_1(y_1) - F_2(y_2) + F_3(y,y) \quad (25)$$

while we use the following formula when y1 < y2,

$$R_{Y_1,Y_2}(y_1,y_2) = R_1(y_1,y_2) = 1 - F_1(y_1) - F_2(y_2) + F_1(y_1,y_2) \quad (26)$$

and if $y_2 < y_1$, we use the following formula

and if y2 < y1, we use the following formula

$$R_{Y_1,Y_2}(y_1,y_2) = R_2(y_1,y_2) = 1 - F_1(y_1) - F_2(y_2) + F_2(y_1,y_2) \quad (27)$$

4.2. Hazard Function

Let (Y_1,Y_2) be bivariate random variable with joint pdf $f_{Y1,Y2}(y_1,y_2)$. We define hazard function $h_{Y1,Y2}(y_1,y_2)$ as

hazard function $h_{Y_1,Y_2}(y_1,y_2)$ as

Then, $$h_{Y_1,Y_2}(y_1,y_2) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{R_{Y_1,Y_2}(y_1,y_2)} \quad (28)$$

if $y_1 < y_2$

$$h_1(y_1,y_2) = \frac{f_1(y_1,y_2)}{R_1(y_1,y_2)} \quad (29)$$

where f_1 (y_1,y_2) from equation(5) and R_1 (y_1,y_2) from equation (23), if $y_2 < y_1$

$$h_2(y_1, y_2) = \frac{f_2(y_1,y_2)}{R_2(y_1,y_2)} \quad (30)$$

where $f_2(y_1,y_2)$ from equation(6) and $R_2(y_1,y_2)$ from equation(24)

if $y_1 = y_2 = y$

$$h_3(y, y) = \frac{f_3(y,y)}{R_3(y,y)} \quad (31)$$

where $f_3(y,y)$ from equation(7) and $R_3(y,y)$ from equation(25)

4.2.1. Reversed Hazard Function and Its Gradient Vector

4.2.2. Reversed Hazard Function

The reversed hazard function by:

$$r_{Y_1,Y_2}(y_1, y_2) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{F_{Y_1,Y_2}(y_1,y_2)} \quad (32)$$

Now, we find the reversed hazard function of BEEPD as

$$r_{Y_1,Y_2}(y_1, y_2) = \begin{cases} r_1(y_1, y_2) & \text{for } y_1 < y_2 \\ r_2(y_1, y_2) & \text{for } y_1 > y_2 \\ r_3(y, y) & \text{for } y_1 = y_2 = y \end{cases}$$

1) $r_1(y_1, y_2) = (\alpha_1 + \alpha_3)\,\lambda\,\theta\,p^{-\theta} y_1^{\theta-1} e^{-\lambda\left(\frac{y_1}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^\theta}\right)^{-1}$

$$\times\, \alpha_2\,\lambda\,\theta\,p^{-\theta} y_2^{\theta-1} e^{-\lambda\left(\frac{y_2}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^\theta}\right)^{-1} \quad (33)$$

for $y_1 < y_2$

2) $r_2(y_1, y_2) = \alpha_1\,\lambda\,\theta\,p^{-\theta} y_1^{\theta-1} e^{-\lambda\left(\frac{y_1}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y_1}{p}\right)^\theta}\right)^{-1}$

$$\times\, (\alpha_2 + \alpha_3)\,\lambda\,\theta\,p^{-\theta} y_2^{\theta-1} e^{-\lambda\left(\frac{y_2}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y_2}{p}\right)^\theta}\right)^{-1} \quad (34)$$

for $y_2 < y_1$

3) $r_3(y, y) = \alpha_3\,\lambda\,\theta\,p^{-\theta} y^{\theta-1} e^{-\lambda\left(\frac{y}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y}{p}\right)^\theta}\right)^{-1} \quad (35)$

for $y_1 = y_2 = y$

4.2.3. Gradient Vector

The gradient vector of bivariate reversed hazard function by:

$r_{Y_1,Y_2}(y_1, y_2) = (r_1(y_1), r_2(y_2))$, where

$r_j(y_j) = \frac{f_{Y_j}(y_j)}{F_{Y_j}(y_j)} = \frac{\partial}{\partial y_j} \log F_{Y_j}(y_j)$; $j = 1,2$, then (36)

$$r_j(y_j) = (\alpha_j + \alpha_3)\,\lambda\,\theta\,p^{-\theta} y_j^{\theta-1} e^{-\lambda\left(\frac{y_j}{p}\right)^\theta} \left(1 - e^{-\lambda\left(\frac{y_j}{p}\right)^\theta}\right)^{-1}, \quad j = 1,2 \quad (36)$$

5. Maximum Likelihood Estimation

In this section we can estimate the unknown parameters of the BEEP distribution, by using the method of maximum likelihood (Kundu and Gupta [3]).

Suppose $((Y_{11},Y_{21}),(Y_{12},Y_{22}),\ldots,(Y_1n,Y_{2n}))$ is a random sample from BEEP distribution where

$$n_1 = (j; Y_{1j} < Y_{2j}),\, n_2 = (j; Y_{1j} > Y_{2j}),\, n_3 = (j; Y_{1j} = Y_{2j} = Y_j),\, n = \sum_{j=1}^{3} n_j \quad (37)$$

We find that the likelihood of the sample is given by

$l(\alpha_1,\alpha_2,\alpha_3,p,\lambda,\theta) = \prod_{j=1}^{n1} f_1(y_{1j},y_{2j}) \prod_{j=1}^{n2} f_2(y_{1j},y_{2j}) \prod_{j=1}^{n3} f_3(y_j,y_j)$

The log-likelihood function becomes:

$L(\alpha_1,\alpha_2,\alpha_3,p,\lambda,\theta) = n_1 \ln(\alpha_1+\alpha_3) + + n_1 \ln(\alpha_2) + 2n_1 \ln(\lambda)$

$$+2n_1 \ln(\theta) - 2n_1\,\theta \ln(p) + (\theta - 1)\sum_{j=1}^{n_1} \ln(y_{1j}) - \lambda \sum_{j=1}^{n_1} \left(\frac{y_{1j}}{p}\right)^\theta$$

$$+(\alpha_1 + \alpha_3 - 1)\sum_{j=1}^{n_1} \ln\left(1 - e^{-\lambda\left(\frac{y_{1j}}{p}\right)^\theta}\right) + (\theta - 1)\sum_{j=1}^{n_1} \ln(y_{2j})$$

$$-\lambda \sum_{j=0}^{n_1} \left(\frac{y_{2j}}{p}\right)^\theta + (\alpha_2 - 1)\sum_{j=1}^{n_1} \ln\left(1 - e^{-\lambda\left(\frac{y_{2j}}{p}\right)^\theta}\right) + n_2 \ln(\alpha_1)$$

$$+n_2 \ln(\alpha_2 + \alpha_3) + 2n_2 \ln(\lambda) + 2n_2 \ln(\theta) - 2n_2\,\theta \ln(p)$$

$$+(\theta - 1)\sum_{j=1}^{n_2} \ln(y_{1j}) - \lambda \sum_{j=1}^{n_2} \left(\frac{y_{1j}}{p}\right)^\theta + (\alpha_1 - 1)\sum_{j=1}^{n_2} \ln\left(1 - e^{-\lambda\left(\frac{y_{1j}}{p}\right)^\theta}\right)$$

$$+(\theta - 1)\sum_{j=1}^{n_2} \ln(y_{2j}) - \lambda \sum_{j=1}^{n_2} \left(\frac{y_{2j}}{p}\right)^\theta + (\alpha_2 + \alpha_3 - 1)$$

$$\times \sum_{j=1}^{n_2} \ln\left(1 - e^{-\lambda\left(\frac{y_{2j}}{p}\right)^\theta}\right) + \ln(\alpha_3) + n_3 \ln(\lambda) + n_3 \ln(\theta) - n_3\,\theta \ln(p)$$

$$+(\theta - 1)\sum_{j=1}^{n_3} \ln(y_j) - \lambda \sum_{j=1}^{n_3} \left(\frac{y_j}{p}\right)^\theta + (\alpha_1 + \alpha_2 + \alpha_3 - 1)$$

$$\times \sum_{j=1}^{n_3} \ln\left(1 - e^{-\lambda\left(\frac{y_j}{p}\right)^\theta}\right) \quad (38)$$

The first partial derivatives of (38) are

$$\frac{\partial L}{\partial \alpha_1} = \frac{n_1}{\alpha_1+\alpha_3} + \frac{n_2}{\alpha_1} + (\alpha_3 - 1)\sum_{j=1}^{n_1} \ln\left(1 - e^{-\lambda\left(\frac{y_{1j}}{p}\right)^\theta}\right) - \sum_{j=1}^{n_2} \ln\left(1 - e^{-\lambda\left(\frac{y_{1j}}{p}\right)^\theta}\right)$$

$$+(\alpha_2+\alpha_3-1)\sum_{j=1}^{n_3} \ln\left(1-e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\right) \quad (39)$$

$$\frac{\partial L}{\partial \alpha_2}=\frac{n_2}{\alpha_2+\alpha_3}+\frac{n_2}{\alpha_2}-\sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)+(\alpha_3-1)\sum_{j=1}^{n_2} \ln\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)$$

$$+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_3} \ln\left(1-e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\right) \quad (40)$$

$$\frac{\partial L}{\partial \alpha_3}=\frac{n_1}{\alpha_1+\alpha_3}+\frac{n_2}{\alpha_1+\alpha_3}+\frac{n_3}{\alpha_3}+(\alpha_1-1)\sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)$$

$$+(\alpha_2-1)\sum_{j=1}^{n_2} \ln\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)+(\alpha_1+\alpha_2-1)\sum_{j=1}^{n_3} \ln\left(1-e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\right) \quad (41)$$

$$\frac{\partial L}{\partial \lambda}=\frac{2n_1}{\lambda}-\sum_{j=1}^{n_1}\left(\frac{y_{1j}}{p}\right)^{\theta}+(\alpha_1+\alpha_3-1)\sum_{j=1}^{n_1}\frac{\left(\frac{y_{1j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)}-\sum_{j=1}^{n_1}\left(\frac{y_{2j}}{p}\right)^{\theta}$$

$$+(\alpha_2-1)\sum_{j=1}^{n_1}\frac{\left(\frac{y_{2j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}+\frac{2n_2}{\lambda}-\sum_{j=1}^{n_2}\left(\frac{y_{1j}}{p}\right)^{\theta}+(\alpha_1-1)\sum_{j=1}^{n_2}\frac{\left(\frac{y_{1j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)}$$

$$-\sum_{j=1}^{n_2}\left(\frac{y_{2j}}{p}\right)^{\theta}+(\alpha_2+\alpha_3-1)\sum_{j=1}^{n_2}\frac{\left(\frac{y_{2j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}+\frac{n_3}{\lambda}-\sum_{j=1}^{n_3}\left(\frac{y_j}{p}\right)^{\theta}$$

$$+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{j=1}^{n_3}\frac{\left(\frac{y_j}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\right)} \quad (42)$$

$$\frac{\partial L}{\partial p}=\frac{-2n_1\theta}{p}-\frac{\lambda\theta}{p^{\theta+1}}\sum_{j=1}^{n_1}y_{1j}^{\theta}-(\alpha_1+\alpha_3-1)\sum_{j=1}^{n_1}\frac{\lambda\theta y_{1j}^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{p^{\theta+1}\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)} \quad \frac{\lambda\theta}{p^{\theta+1}}\sum_{j=1}^{n_1}y_{2j}^{\theta}$$

$$+(\alpha_2-1)\sum_{j=1}^{n_1}\frac{\lambda\theta y_{2j}^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{p^{\theta+1}\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}-\frac{2n_2\theta}{p}-\frac{\lambda\theta}{p^{\theta+1}}\sum_{j=1}^{n_2}y_{1j}^{\theta}+(\alpha_1-1)$$

$$\times\sum_{j=1}^{n_2}\frac{\lambda\theta y_{1j}^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{p^{\theta+1}\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)}-\frac{\lambda\theta}{p^{\theta+1}}\sum_{j=1}^{n_2}y_{2j}^{\theta}+(\alpha_2+\alpha_3-1)\sum_{j=1}^{n_2}\frac{\lambda\theta y_{2j}^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{p^{\theta+1}\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}$$

$$-\frac{n_3\theta}{p}-\frac{\lambda\theta}{p^{\theta+1}}\sum_{i=1}^{n_3}y_j^{\theta}+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{j=1}^{n_3}\frac{\lambda\theta y_{1j}^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{p^{\theta+1}\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)} \quad (43)$$

$$\frac{\partial L}{\partial \theta}=-2n_1\ln p+\sum_{j=0}^{n_1}y_{1j}-\lambda\sum_{j=1}^{n_1}\ln\left(\frac{y_{1j}}{p}\right)\left(\frac{y_{1j}}{p}\right)^{\theta}+(\alpha_1+\alpha_3-1)$$

$$\times\sum_{j=1}^{n_1}\frac{\lambda\ln\left(\frac{y_{1j}}{p}\right)\left(\frac{y_{1j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)}+\sum_{j=1}^{n_1}y_{2j}-\lambda\sum_{j=1}^{n_1}\ln\left(\frac{y_{2j}}{p}\right)\left(\frac{y_{2j}}{p}\right)^{\theta}+(\alpha_2-1)$$

$$\times\sum_{j=1}^{n_1}\frac{\lambda\ln\left(\frac{y_{2j}}{p}\right)\left(\frac{y_{2j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}-2n_2\ln p+\sum_{j=1}^{n_2}y_{1j}-\lambda\sum_{j=1}^{n_2}\ln\left(\frac{y_{1j}}{p}\right)\left(\frac{y_{1j}}{p}\right)^{\theta}$$

$$+(\alpha_1-1)\sum_{j=1}^{n_2}\frac{\lambda\ln\left(\frac{y_{1j}}{p}\right)\left(\frac{y_{1j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{1j}}{p}\right)^{\theta}}\right)}+\sum_{j=1}^{n_2}y_{2j}-\lambda\sum_{j=1}^{n_2}\ln\left(\frac{y_{2j}}{p}\right)\left(\frac{y_{2j}}{p}\right)^{\theta}$$

$$+(\alpha_2+\alpha_3-1)\sum_{j=1}^{n_2}\frac{\lambda\ln\left(\frac{y_{2j}}{p}\right)\left(\frac{y_{2j}}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_{2j}}{p}\right)^{\theta}}\right)}-n_3\ln p+\sum_{j=1}^{n_3}y_j-\lambda\sum_{j=1}^{n_3}\ln\left(\frac{y_j}{p}\right)$$

$$\times\left(\frac{y_j}{p}\right)^{\theta}+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{j=1}^{n_3}\frac{\lambda\ln\left(\frac{y_j}{p}\right)\left(\frac{y_j}{p}\right)^{\theta}e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}}{\left(1-e^{-\lambda\left(\frac{y_j}{p}\right)^{\theta}}\right)} \quad (44)$$

Setting each of these first partial derivatives to be equal to zero. These equations cannot be solved analytically but numerically by using the statistical software, to get the ML of the unknown parameters.

6. Conclusions

In this research we presented a bivariate Exponentiated Exponential Pareto distribution whose marginal are Exponentiated Exponential Pareto distribution. We discussed some statistical properties of the new bivariate model. they observed that the MLE of the unknown parameters can be obtained numerically.

References

[1] Al-Kadim, K. and Boshi, M., Exponential Pareto distribution, Mathematical Theory and Modeling, Vol.3, No.5,135-146, (2013 a).

[2] Johnson, R. A., Wichern, D. W., Applied Multivariate Statistical Analysis, Pearson Prentice Hall. New Jersey, (2007).

[3] Sherpieny, E., Ibrahim, S., Bedar, R, A New Bivariate Distribution with Generalized Gompertz Marginals, Asian Journal of Applied Sciences,Vol. 01, Issue 04,141-150, (2013b).

[4] Kundu, D., and Gupta, R, bivariate generalized exponential distribution, Journal of Multivariate Analysis, 100, 581-593. (2009).

[5] Sarhan, A. M., Balakrishnan, N., A new class of bivariate distributions and its mixture, Journal of Multivariate Analysis ,98, 1508 – 1527, (2007).

[6] Block, H. W., Langberg, and N.A., Stoffer, D., bivariate exponential and geometric autoregressive and autoregressive moving average models, Applied Probability Trust, 20, 798-821, S (1988).

[7] Al-Khedhairi, A., El-Gohary, A., a new class of bivariate gompertz distributions and its mixture Int. Journal of Math. Analysis, Vol. 2, no. 5, 235 – 253, (2008).

[8] Kundu, D., bivariate geometric (maximum) generalized exponential distribution, Journal of Data Science, 13, 693-712, (2015).