Asymptotic Fitting Shadowing Property

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الخلاصة

ليكن(M,d) فضاء متري وليكن ϕ دالة من الفضاء المتري (M,d) الى نفسه وتحقق خاصية مقارب التظليل المناسب (AFSB) (Asymptotic Fitting Shadowing Property) فأن النتائج التالية متحققة: ϕ^m تمتلك خاصيـة مقـارب التظليل المناسب لكل $m\in N$، وخاصية سلسلة متعدية (chain transitive)، وايضاً، اذا كانت ψ تمتلك حالة مقـارب التظليل المنـاسب فأن$\phi\times\psi$ تمتلك خاصية مقـارب التظليل المنـاسب. بالإضافة الى نتائج اخرى حول خاصية مقارب التظليل المناسب.

الكلمات المفتاحية

فضاء متري، خاصية مقارب التظليل المناسب، سلسلة متعدية.

Abstract

Let (M,ď) be a metric space, ϕ be a map from a metric space (M,ď) to itself and satisfy the Asymptotic Fitting Shadowing property(AFSP) then these results are satisfy: For every $m \in N$, ϕ^m has asymptotic fitting shadowing property and ϕ is chain transitive, also, if ψ has the asymptotic fitting shadowing property then $\phi \times \psi$ has the asymptotic fitting shadowing property. In addition to the other results on the asymptotic fitting shadowing property.

Keywords

Metric space, Asymptotic Fitting Shadowing property, chain transitive.

1. Introduction

In dynamical systems the pseudo-orbit tracing property (POTP) is the most main ideas [1]. Blank defined the average shadowing property in studyingchaotic dynamical systems [2]. Yang conversed the relationship between topological ergodicity and the POTP for maps, and proved that the POTP with a chain transitive map is topologically ergodic [3].

Guo and Gu conversed the relationship topological ergodicity with the average shadowing property (ASP) for flows, and proved that the ASP with

Gu introduced the notion of the AASP, which is less power than the asymptotic POTP in the shadowing method, and achieved the relation transitivity with the AASP. He also proved for a compact metric space M, if a map ϕ has the AASP and continues on M, then ϕ is chain transitive[5].

In this paper we try to discuss the concept of the Asymptotic Fitting Shadowing property (AFSP) is topologically ergodic, We get that asymptotic fitting shadowing case with Lyapunov stable map from a compact metric space to itself, is topologically ergodic but not topologically weakly mixing, it is not also topologically mixing.

2. Preliminaries

Let N symbolize the set of natural numbers and Z symbolize the set of integer numbers, Z_+ symbolize the set of nonnegative integer numbers. In this section, we introduce some definitions that we will use in this search, we recall some fundamental definitions. Let (M,ď) be a metric space and ϕ: M→M is a continuous map. For every positive integer m, We define ϕ^m inductively by $\phi^m = \phi \circ \phi^{m-1}$ and $\phi^{-m} = \phi^{-1} \circ \phi^{-m+1}$, If A and B are two nonempty open subsets of M after that we let $N(A \cap B) = \{m \in Z_+ : \phi^m (A) \cap B \neq \emptyset\}$, for $A,B \subset M$, $\omega \in M$, we write $N(A, B)=\{ m \in Z_+ : A \cap \phi^{-m} (B) \neq \emptyset \}$,

$N(\omega,B)=\{n \in Z_+ : \phi^m (\omega) \in B\}$. Symbolize by $N_\varepsilon (B)$ the open ball with center ω and radius ε.

2.1. Definition

[6] Let ϕ :M→M be a map and (M,ď) be a compact metric space. A sequence $\{\omega_i\}_{i \in Z}$ is named orbit of ϕ if $\forall i \in Z$ we have $\omega_{i+1}=\phi(\omega_i)$ and we called it a ∝ - pseudo-orbit of ϕ $\forall i \in Z$, we have $ď(\phi(\omega_i),\omega_{i+1}) \leq \propto$. The map ϕ is reminded to have the shadowing property, if ∀ ε>0 , ∃∝ >0, so ∀δ - pseudoorbit $\{\omega_i\}_{i \in Z}$ is ε - shadowed by the orbit $\{\phi^i (\omega), i \in Z\}$,∃ z∈M, it means, ∀ i∈Z we have $ď(\phi^i (z),\omega_i) \leq \varepsilon$.

2.2. Definition

[6] Let $\{\omega_i\} i=0^\infty$ be a sequence in M, if ∃ α>0 and a positive integer N=N(α) such that ∀m∈N and n≥N, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} ď(\phi(\omega_{i+m}), \omega_{i+m+1}) < \propto.$$

then a sequence $\{\omega_i\}_{i=0}^{\infty}$ is named a ∝ - average pseudo orbit of ϕ .

A map ϕ is reminded to have average shadowing property, if ∀ ε>0 , ∃ ∝ >0 , so ∀ ∝ - average pseudo orbit $\{\omega_i\}_{i \in Z}$ is ε - shadowed in average by the point y∈M , it means ,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(y),\omega_i) < \varepsilon.$$

We introduce a new definition as the following:

2.3. Definition

Let $\{\omega_i\}_{i=0}^{\infty}$ be a sequence in M, if ∃ α>0 and a positive integer N=N(α) such that ∀m∈N and n≥N, we have

$$\sum_{i=0}^{n-1} d(\phi(\omega_{i+m}), \omega_{i+m+1}) < \alpha .$$

then $\{\omega_i\}_{i=0}^{\infty}$ is named a α - fitting pseudo-orbit of ϕ.

A map ϕ is reminded to have fitting shadowing property (FSP) if ∀ε>0, ∃ α>0 so ∀ α - fitting pseudo orbit is ε- shadowed in fitting by the point y∈M , it means

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} d(\phi^i(y), \omega_i) < \varepsilon.$$

2.4. Definition

Let $\{\omega_i\}_{i=0}^{\infty}$ be a sequence in M is named the asymptotic

fitting pseudo-orbit of ϕ. if

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} d(\phi (\omega_i), \omega_{i+1}) = 0.$$

A sequence $\{\omega_i\}_{i=0}^{\infty}$ in M is reminded to be asymptotically shadowed in fitting by the point y ∈ M if

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} d (\phi^i(y), \omega_i) = 0$$

We say that ϕ has asymptotic fitting shadowing property (AFSP) if any asymptotic fitting pseudo-orbit of ϕ is asymptotically shadowed in fitting by the point z ∈ M.

2.5. Definition

[7] A map $\phi : Z+ \times (0, \varepsilon_0] \longrightarrow N$ such that $\varepsilon_0>0$, then ϕ is called mistake map if ∀ ε∈ (0, ε_0] and ∀ m∈Z +, we have ϕ (m ,ε) ≤ ϕ (m+1 ,ε) and

$$\lim_{n\to\infty} \frac{\phi (m , \varepsilon)}{m} = 0 .$$

Supposed a mistake map ϕ , if $\varepsilon>\varepsilon_0$, then we know $\phi (m ,\varepsilon)= \phi (m ,\varepsilon_0)$.

2.6. Definition

[7] A continuous map ϕ:M→M has the almost specification property if there is a mistake map $\check{\phi}$ and a function $k_{\check{\phi}}:(0, \infty)\longrightarrow N$ so for any m≥1 , any $\varepsilon_1 ,\ldots,\varepsilon_m >0$ any point $y_1 ,\ldots,y_m$ ∈M, and any integers

$n_1 > k_{\check{\phi}} (\varepsilon_1) ,\ldots,n_m > k_{\check{\phi}} (\varepsilon_m)$ setting $n_0=0$ and $l_j=\sum_{(s=0)}^{(j-1)} n_s$,for j=1,…,m .We can find a point z∈M so for each j=1,…,m, we have $\phi^{\curlyvee(l_j)} (z)\in B_{nj} (\phi : z_j ,\varepsilon_j)$.

2.7. Definition

[6] Let $\{\omega_i\}_{i=0}^{\infty}$ be a sequence in M is named asymptotic average pseudo-orbit of ϕ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\phi (\omega_i), \omega_{i+1}) = 0.$$

A sequence $\{\omega_i\}_{i=0}^{\infty}$ in M is called an asymptotically shadowed in average by the point y ∈ M if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(y), \omega_i) = 0$$

We say that ϕ has asymptotic average shadowing case if any asymptotic average pseudo-orbit of ϕ is asymptotically shadowed in average by the point $y \in M$.

2.8. Definition

[8] A map ϕ is called topologically transitive if for A, B⊂ M where A, B≠∅, N(A∩B)≠∅. y∈M is named a transitive point if orbit y is dense in M.

2.9. Definition

[9] Let ϕ:M→M be a continuous map, (M,d) be a compact metric space. For ∝ ∈R, so ∝ >0, a sequence $\{y_i\}_{i=0}^{\infty}$ in M is named an δ - chain if $d(\phi(y_i), y_{i+1}) \leq \propto$, ∀i=0,1,2,…, the map ϕ is said to be chain transitive if ∀ z,y∈M and each ∝ >0 , there is a finite ∝-chain $\{y_0,y_1,\dots,y_n\}$ such that $y_0=z$ and $y_n=y$.

2.10. Definition

[8] A map ϕ is called topologically mixing A, B, ∃m∈Z+, N(A∩ B) ⊃ {m, m+1, …}A map is named topologically weak mixing if ϕ×ϕ topologically transitive.

2.11. Definition

[8] A map ϕ is named chain mixing if for all z,y∈M and every δ>0 , there exists N∈N such that for any n≥N there is δ - chain from x to y with length n.

2.12. Definition

[10] Let ϕ:M→M be a continuous map and Let (M,ɗ) be a compact metric space with metric ɗ . We say that ϕ be topologically ergodic if for any A, B⊂M and A, B are open sets, A, B≠∅, then N(A ,B) has positive upper density, that is,

$$\bar{D}\big(N(A,B)\big) = \limsup_{m\to\infty} \frac{\text{Card}\,(N(A,B)\cap\{0,1,\dots,m-1\})}{m} > 0\,.$$

$q \subset Z_+$ is called syndetic if ∃N∈N, where [m,m+N]∩q≠∅ for every n∈N. ϕ is called strongly ergodic if for any pair of nonempty open subset A, B⊂M, N(A ,B) is syndetic set. If ∀k∈N, ϕ^k is strongly ergodic, we call ϕ totally strongly ergodic. We observe that

totally strongly ergodic ⟹ strongly ergodic ⟹ topologically ergodic ⟹

topologically transitivity.

If each y∈M is transitive point, then we say ϕ is minimal. y∈M is said to be minimal, if for every neighborhood B of y, N(y,B) is syndetic. Let AP(ϕ) symbolize of the set minimal point of ϕ.

2.13. Definition

[10] Let (M,ɗ) be a compact metric space with metric ɗ and ϕ:M→M be a map, a point y∈M is named stable point of ϕ if ∀ ε>0,∃ γ>0, where $ɗ(\phi^n(y), \phi^n(\omega)) < \varepsilon$ for all ω∈M with ɗ(y,ω) < γ and ∀ $n \in Z_+$. A map ϕ is named Lyapunov stable if each point of M is stable point of ϕ.

3.Main result and proof.

Let ϕ:M→M be a map and (M,ɗ) be a metric space. we prove some results of the asymptotic fitting shadowing property, also we show that if ϕ has the asymptotic fitting shadowing property then ϕ^m has the asymptotic fitting shadowing property for every positive integer m.

We also prove if (M,d) be a compact and a map ϕ has the asymptotic fitting shadowing property on M, then ϕ is chain transitive. We get that a Lyapunov stable map with the AFSP from a compact metric space onto itself is topologically ergodic.

3.1. Lemma

[7] If $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence of non-negative real numbers, then

the following conditions are equivalent:

(1) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_n = 0$,

(2) There is a subset $J \subset N$ of density zero, that is,

$$\lim_{n\to\infty} \frac{\mathrm{Card}(J \cap \{0,1,2,\ldots,n-1\})}{n} = 0 \text{ such that } \lim_{n\notin J} a_n = 0.$$

3.2. Theorem

Let ϕ:M→M be a map and (M,d) be a compact metric space. If $\{y_k\}_{k=0}^{\infty}$ is an asymptotic fitting pseudo-orbit of ϕ and ϕ is chain mixing, then there is an asymptotic pseudoorbit $\{z_k\}_{k=0}^{\infty}$ of ϕ so the set $\{k:y_k\neq z_k\}$

has asymptotic density zero.

proof: By chain mixing of ϕ:M→M, for every j there exists an integer L_j so that for any points y,z∈M there exists a $1/2^j$ - chain of length L_j+1 from y to z. we may suppose that L_j+1 is a multiple of L_j for every j. Assume that $\{y_k\}_{k=0}^{\infty}$ is an asymptotic fitting pseudo-orbit. By Lemma 3.1 there exists a set k so d(k)=0 and (lim)┬(k∉K) $d(\phi(y_i),y_{i+1})=0$. (3.1)

Since d(k)=0, there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that for every j we have L_j+1 divides n_j and $L_j \cdot \mathrm{card}(k\mid n)<1/2^j$, for every $n>n_k$. By (3.1) we may assume that if k∉k and $k\geq n_j$ then $d(\phi(y_i), y_{i+1})<1/2^j$. Now we define a set k' and sequence $\{z_k\}\sim_{(k\notin k')}$ in the following way: for every j and for every t such that $[sL_j, (t+1) L_j) \subset [n_j, n_{j+1})$ if $k \cap [sL_j, (t+1) L_j) \neq \emptyset$ then we include the set $[tL_j,(t+1) L_j)\cap N$ in k'. We define $\{z_k\}_{(k=tLj)}^{((t+1) L\_j)}$ as a $1/2^j$ - chain from to $y_{((t+1) Lj)}$ of length L_j+1. Note that z_(tL_j)= $y_{(tLj)}$ and $z_{((t+1) Lj)} = y_{((t+1) Lj)}$. Let k' and $\{z_k\}_{(k\in k')}$ be obtained by the above procedure . For k∉k we put $y_k= x_k$. First note that for $k\geq n_j$ we have $(d(\phi(z_k), z_{k+1})<1/2^k$ and hence $\{z_k\}_{k=0}^{\infty}$ is an asymptotic pseudo- orbit. Furthermore, if we fix any $n> n_1$ then there is $n_j<n \leq n_{j+1}$ so j>0 and

$\mathrm{card}(k'\mid n)=\mathrm{card}(k'\mid n_1)+\sum_{(t=1)}^{j}\mathrm{card}((k'\cap[n_t, n_{t+1})\mid n)$

$\leq \mathrm{card}(k'\mid n_1)+\sum_{s=1}^{k} \mathrm{card}\ L_t((k\cap[n_t, n_{t+1})\mid n)$

$\leq \mathrm{card}(k'\mid n1)+L_j\ \mathrm{card}(k\mid n) < \mathrm{card}(k'\mid n_1) + 1/2^j$

This shows that d(k')=0.

The proof is completed by noting that $\{k:y_k\neq z_{(k)}\}\subset k'$. ■

3.3. Lemma

[7] Let ϕ:M⟶M be continuous map and (M,d) be a compact metric space, If ϕ has the almost specification property and surjective, then ϕ is chain mixing.

3.4. Theorem

[7] Let ϕ:M⟶M be surjective map and (M,d) be a compact metric space, if ϕ has almost specification property, then ϕ has the asymptotically average shadowing property.

3.5. Remark

The Theorem 3.4 does not give the asymptotic fitting shadowing property ϕ because: let $\{y_i\}_{i=0}^{\infty}$ be an asymptotic fitting pseudo-orbit of ϕ. By Lemma 3.3 is chain mixing, and therefore we may use Theorem 3.2 to obtain an asymptotic pseudo-orbit of ϕ, denoted $\{z_i\}_{i=0}^{\infty}$,such that d({i:$y_i \neq z_i$})=0.

So we cannot to show that $\{z_i\}_{i=0}^{\infty}$ can be asymptotically shadowed on fitting by point ω ∈Y, then ϕ does not satisfy the asymptotically fitting shadowing property. ■

3.6. Proposition

Let ϕ:M⟶M be continuous map and (M,d) be a compact metric space, if for every positive integer m, ϕ has the asymptotic fitting shadowing property then ϕ^m has the asymptotic fitting shadowing property .

Proof:

Suppose ϕ has the asymptotic fitting shadowing property and m is positive integer. Let $\{y_i\}_{i=0}^{\infty}$ be an asymptotic fitting pseudo- orbit of ϕ^m, that is,

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi^k(y_i), y_{i+1}) = 0. \qquad (3.2)$$

Let $z_{tm+k} = \phi^m(y_i)$, for all $0 \le k < m$ and all $t \ge 0$.

Since $$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi(z_i), z_{i+1}) \le \frac{n}{tm+k}\lim_{n\to\infty}\sum_{i=0}^{t} d(\phi^m(y_i), y_{i+1}),$$

we get from (3.2)that $$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi(z_i), z_{i+1}) = 0$$

That is , the sequence {z_i }_(i=0)^∞ is an asymptotic fitting pseudo orbit of ϕ. So , there is point ω∈Ysuch that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi^i(\omega), z_i) = 0, \qquad (3.3)$$

note that $$\lim_{n\to\infty}\sum_{i=0}^{t-1} d(\phi^{mi}(\omega), y_i) \le \lim_{n\to\infty}\sum_{s=0}^{t-1}\sum_{k=0}^{m-1} d(\phi^{sm+k}(\omega), z_{sm+k})$$

$$= \lim_{n\to\infty}\sum_{i=0}^{tm-1} d(\phi^i(\omega), z_i).$$

From (3.3)that $$\lim_{n\to\infty}\sum_{i=0}^{t-1} d(\phi^{mi}(\omega), y_i) = 0.$$

Thus ϕ^m has the asymptotic fitting shadowing property . ■

3.7. Remark

There are difference between the asymptotic average shadowing and the asymptotic fitting shadowing case for example : if ϕ^m has the asymptotic average shadowing for some m∈Z_+, then so does ϕ, but if ϕ^m has the asymptotic fitting shadowing property for some m∈Z_+, then does not ϕ.

3.8. Theorem

Let ϕ:M⟶M be continuous map and (M,d) be a compact metric space, if ϕ has the asymptotic fitting shadowing property then ϕ is chain transitive.

Proof:

Assume that y, z∈M such that y ≠z and γ>0. It is enough to prove that exists a γ- chain from y to z.

we defined a sequence $\{U_0\}_{i=0}^{\infty}$ as follows.

Let U_0= y, z= U_1

U_2=y, z= U_3

U_4=y, ϕ (y), z-1, z= U_7

……….

U_2^m =y, ϕ (y),…,$\phi^{2^{m-1}-1}$ (y), z-2^{m-1}+1,…,z-1, z= $U_{2^{m+1}-1}$

………..

where ϕ (z-k)=z-k+1 for each k>0 and z_0=z. It is clear that for 2^m≤n<2^{m-1}, and

$$\sum_{i=0}^{n-1} d(\phi(\mathcal{U}_i), \mathcal{U}_{i+1}) < \frac{2(m+1)\times D}{2^m} \quad ,$$

where D is the diameter of M , that is , D= max {d(y ,z) :y,z∈Y} .

Thus $$\lim_{n\to\infty} \sum_{i=0}^{n-1} d(\phi(\mathcal{U}_i), \mathcal{U}_{i+1}) = 0 .$$

It means $\{y_i\}_{i=0}^{\infty}$ is an asymptotic fitting pseudo-orbit of ϕ. Since ϕ has the asymptotic fitting shadowing property , there is a point ω in M so,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} d\big(\phi^i(\omega), \mathcal{U}_i\big) = 0 \qquad (3.4)$$

Since γ>0 and ϕ is continuous map, ∃ μ∈(0,γ) such that d (s,t)<μ

implies d(ϕ (s), ϕ(t))<γ for all s,t∈M .

Claim. (1) There is no limits of integers number k>0 , so

U_{nk}∈{y, ϕ(y),…,$\phi^{2^{k-1}-1}$ (y)} and d(ϕ^{nk} (ω), U_{nk}<μ .

(2) There is no limits integers number r>0 , so

U_{nr}∈{$z_{-2^{r+1}}$, …, z-1, z} and d(ϕ^{nr} (ω), U_{nr})<μ .

Proof (1) Assume on conversely that ∃N∈Z_+, so ∀m∈Z, m>N, so that U_i∈{y, ϕ(y),…,$\phi^{2^{m-1}}$ (y)}, it is got that d(ϕ^i (ω), U_i)>μ. Then it would be got that

$$\lim_{n\to\infty} \inf \sum_{i=0}^{n-1} d(\phi^i(\omega), \mathcal{U}_i) \geq \frac{\mu}{2}$$

This contradiction with (3 .4) and the proof (1) is complete.

Proof (2) is similar to the Proof (1).

From this Claim , we pick 〚 k〛_0 〚 ,r〛_0 are positive integers and so

$n_{k_0} > n_{r_0}$ and

$U_{n_{k_0}}$∈{y, ϕ(y),…,$\phi^{2^{k_0-1}}$ (y)} and d($\phi^{n_{k_0}}$ (ω), $U_{n_{k_0}}$ <μ .

$U_{n_{r_0}}$ ∈{$z_{-2^{r_0+1}}$,…, z-1, z } and d($\phi^{n_{r_0}}$ (ω),$U_{n_{r_0}}$ <μ

It may be assumed $U_{n_{k_0}}$ = $g^{k_1)}$ (x) for some k_1>0 ;

$U_{n_{r_0}}$ = y-r_1 for some r_1>0 .

That is, γ- chain from y to z:

y ,ϕ(y),…,ϕ^{k_1} (y) = $U_{n_{k_0}}$, $\phi^{n_{k_0}+1)}$ (ω), $\phi^{n_{k_0}+2}$(ω),…,$\phi^{n_{r_0}-1}$(ω),

$U_{n_{r_0}}$ = z- r_1, z- r_1+1, …, z .

Thus, ϕ is chain transitive. ■

3.9. Proposition

Let ϕ:M⟶M and ψ:T⟶T be maps ,(M,d) and (T,d ') be metric spaces,. Then, ϕ× ψ has the asymptotic fitting shadowing property if and only if ϕ and ψ have the asymptotic fitting shadowing property.

Proof: ⟸)

we choose the metric d '' on ϕ× ψ as following : For m=(m_1,t_1), t= (m_2, t_2) ∈ ϕ×ψ, d '' (m,t)= max {d(m_1, m_2), d ' (t_1, t_2)}, Assume that ϕ and ψ have the asymptotic fitting shadowing property and $\{(m_i, t_i)\}_{i=0}^{\infty}$ be asymptotic fitting pseudo-orbit of ϕ× ψ , that is ,

$\lim_{n\to\infty} \sum_{i=0}^{n-1} d''((\phi\times\psi)(m_i,t_i),(m_{i+1},t_{i+1})) = 0$.(3.5)

It means

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi(m_i), m_{i+1}) = 0 \qquad (3.6)$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\psi(t_i), t_{i+1}) = 0,$$

such that $\{m_i\}_{i=0}^{\infty}$ and $\{t_i\}_{i=0}^{\infty}$ are asymptotic fitting pseudo orbit of ϕ and ψ ,respectively, thus there is $r_1, r_2 \in M$ such that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\phi^i(r_1), m_i) = 0, \qquad (3.7)$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d(\psi^i(r_2), t_i) = 0. \qquad (3.8)$$

By Lemma (3.1) and (3. 7), $\exists J_0 \subset Z_+$ of zero density so

$$\lim_{j\to\infty} d(\phi^i(r_1), m_j) = 0 \qquad (3.9)$$

so $j \notin J_0$.Similarly, $\exists J_1 \subset Z_+$ of zero density so

$$\lim_{j\to\infty} d(\psi^i(r_2), t_j) = 0 \qquad (3.10)$$

so $j \notin J_1$. Let $J = J_0 \cap J_1$. Then, $J \subset Z_+$ subset of zero density and $\lim_{j\to\infty} d''((\phi\times\psi)^i(r_1,r_2),(m_j,t_j)) = 0$ (3.11)

so $j \notin J$. so that, by Lemma 3 .1 we have

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} d''\left((\phi\times\psi)^i(r_1,r_2),(m_j,t_j)\right) = 0. \qquad (3.12)$$

Thus $\phi\times\psi$ has the asymptotic fitting shadowing property.

$\Longrightarrow$) Similarly ,we can show that if $\phi\times\psi$ has the asymptotic fitting shadowing property then ϕ and ψ have the asymptotic fitting shadowing property. ■

3.10. Theorem

Let $\phi: M\to M$ be a Lyapunov stable map and (M,d) be a compact metric space, If ϕ has the asymptotic fitting shadowing case, then ϕ is topologically ergodic.

Proof:

Let $S_1, S_2 \subset M$ and $S_1, S_2 \neq \emptyset$. We pick $m\in S_1$,$z\in S_2$ and $\varepsilon>0$ such that $B(m,\varepsilon)\subset S_1$ and $B(z,\varepsilon)\subset S_2$. Since ϕ is Lyapunov stable and M is compact, $\exists \propto>0$ so for any s, $t\in M, d(s,t)<\propto$ implies that $d(\phi^n(s),\phi^n(t))<\varepsilon, \forall\, n\geq 0$.

Construct the sequence $\{u_i\}_{i=0}^{\infty}$ as shown:

$u_0 = m$, $z = u_1$

$u_2 = m$, $z = u_3$

$u_4 = m-1$,y ,z-1, $z = u_7$

$u_{2k} = m_{-2^{r-1}+1}, \dots, m-2, m, z_{-2^{r-1}+1}, \dots, z-2, z-1, z = u_{2^{r+1}-1}$

where $\phi(m_{-j}) = m_{-j+1}, \forall\, j>0$, $z_0 = z$ and $\phi(z_l) = z_{-l+1} \forall\, l>0$,

$z_0 = z$. It is clear for $2^r \leq n < 2^{r+1}$,

$$\sum_{i=0}^{n-1} d(\phi(u_i), u_{i+1}) < \frac{2(r+1)\times D}{2^r},$$

where $D = \max\{d(m,z): m,z \in M\}$ is the diameter of M. Hence

$\lim_{(n\to+\infty)}\sum_{(i=0)}^{(n-1)} d(\phi(u_i),u_{(i+1)})=0.$

Hence $\{u_i\}_{i=0}^{\infty}$ is an asymptotic fitting pseudo-orbit of ϕ. Since ϕ has the asymptotic fitting shadowing case, $\exists\, u\in M$, so

$$\lim_{n\to+\infty}\sum_{i=0}^{n-1} d(\phi^i(u), u_i) = 0. \qquad (3.13)$$

Let $J_m = \{i: u_i \in \{m_{-2^{i-1}+1}, m_{-2^{i-1}}, \dots, m_{-1}, m\}\}$,

$J_z = \{i: u_i \in \{z_{-2^{i-1}+1}, z_{-2^{i-1}}, \dots, z_{-1}, z\}\}$ and $d(\phi^i(u), u_i) < \propto$.

Claim: J_m and J_z have positive upper density, that is , $\overline{D}(J_m)>0$ and $\overline{D}(J_z)>0$.

proof of Claim. Now, we will prove $\overline{D}(J_y)\geq 0$. assume conversely, $\overline{D}(J_m) = 0$ then we have

$$\lim_{n\to+\infty}\frac{\mathrm{Card}(J_m \cap \{0,1,\dots,n-1\})}{n} = 0.$$

Let $J'_m = \{i: u_i \in \{m_{-2^{i-1}+1}, m_{-2^{i-1}}, \dots, m_{-1}, m\}$

and $d(\phi^i(u), u_i) < \propto\}$.

Then $\lim_{n\to+\infty} \frac{\mathrm{Card}(J'_m \cap \{0,1,\dots,n-1\})}{n} = \frac{1}{2}$.

so ∀ μ∈(0,1/2\), ∃ N>0 so

so $\forall\, \mu \in \left(0, \frac{1}{2}\right), \exists\, N > 0$ so

$$\frac{\mathrm{Card}(J'_m \cap \{0, 1, \dots, n-1\})}{n} > \frac{1}{2} - \mu, \quad \forall\ n \geq N.$$

$$\text{So,} \limsup_{n\to+\infty} \sum_{i=0}^{n-1} d(\phi^i(u), u_i) \geq \limsup_{n\to+\infty} \sum_{i\in J'_m \cap \{0,1,\dots,n-1\}} d(\phi^i(u), u_i)$$

$$\geq \propto \limsup_{n\to+\infty} \frac{\mathrm{Card}(J'_m \cap \{0, 1, \dots, n-1\})}{n}$$

$$\geq \propto \left(\frac{1}{2} - \mu\right)$$

Since μ is arbitrary, we have

$$\limsup_{n\to+\infty} \sum_{i=0}^{n-1} d(\phi^i(u), u_i) \geq$$

This is a contradiction with (3.13). Hence $\overline{D}(J_m)>0$.

Proof that $\overline{D}(J_z)>0$ in the same way.

Now let $J_t(z)=\{l\in J_z : u_l = z_{-t}\}$, $\forall t\geq 0$. Then, by Claim, $\exists\, t_0 \geq 0$, so $\overline{D}(J_{t_0}(z))>0$.

We pick $l_0 \geq 0$ and $0 \leq k_0 \leq 2^{l_0-1)}-1$, so $\phi^{l_0}(u)\in B(m_{-k_0}, \propto)$.

$\forall\ J\in J_{t_0}(z)$ with $J\geq l_0+k_0$, we have $\phi^j(u)\in B(z_{-t_0}, \propto)$. Because ϕ Lyapunov stable , we have $\phi^{l_0+k_0}(u)\in B(m,\varepsilon)$ and $\phi^{j+t_0}(u)\in B(z,\varepsilon)$.

Let $n_j=(j+t_0)-(l_0+k_0)$. We have $\phi^{n_j}(B(m,\varepsilon)\cap B(z,\varepsilon)) \neq \emptyset$. So, $\phi^{n_j}(S_1) \cap S_2 \neq \emptyset$. Thus $\overline{D}(K(S_1),S_2) \geq \overline{D}(J_{t_0}(z))>0$. Hence ϕ is topologically ergodic. ■

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