Using Simulation to Build a Better Model for Generation Observations of Laplace Distribution and Determine the Optimal Method of Estimation

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الخلاصة

توزيع لإبلاس ينتمي الى عائلة التوزيعات الأسية وله تطبيقات مختلفة منها الاقتصادية، العلوم الهندسية، الطبية، علوم الحياة. وله علاقات مع اختلاف التوزيعات الأسي، الطبيعي، باريتو سواء كان لإبلاس القياسي أو العام، وقد استخدمنا هذه العلاقة لتوليد عينة عشوائية من مشاهدات التوزيع، بالإضافة استخدمنا أساليب مختلفة لتقدير معلمات التوزيع والغرض منه هو تحديد النموذج الأمثل في ظل العديد من المؤثرات (طرق تقدير حجم العينة، والعلاقة بين المعلمات)، بالإضافة إلى تحديد أسلوب التقدير الأمثل تحت ظروف مختلفة (نماذج، حجم العينة، والعلاقة بين المعلمات). وقد تم الاعتماد على تجارب المحاكاة لتحديد الأنموذج الأفضل لتوليد المشاهدات بنسبة أكثر من (77٪) لجميع الحالات، وطريقة (MLE) أفضل بنسبة أكثر من (72٪) مقارنة مع الطرائق الأخرى.

الكلمات المفتاحية

توزيع لإبلاس، التوزيعات الأسية، طريقة (MLE).

Abstract

Laplace distribution is a continuous statistical distribution belong to the exponential family which has multiple applications in many aspects like economic life, engineering, medical ...etc., moreover it has various relationships with other distributions, Exponential, Normal, Pareto, both as standard Lap (0,1) or as general Lap (μ, β), and we used this relationship is to generate random sample of distribution observations. In addition, we used different methods for estimating the distribution parameters in order to determine the optimal model under several effects (methods of appreciation, sample size, and the relationship between the parameters. It has been relying on simulation experiments to determine the best model to generate observations and over (77%) in all cases, the method (MLE) represents over (72%) compared with the other methods.

Keywords

Laplace distribution, Statistical distribution, The method (MLE).

Objective:

The first objective for the paper is the study of the relationship between general Laplace (μ, β) distribution properties and other probability distributions to find out the best relationship to generate random observations of a sample, second objective is to determine the optimal method of finding distribution parameters, taking into account several effects on the two goals, including the effect of the sample size and the relationship of parameters (μ, β) through simulation experiments.

1. Introduction:

The Laplace distribution is belong to exponential family distributions which is similar to the normal and Cauchy distributions as symmetric around mean, but more tail of them, and most of the researches studied the process of obtaining the data distributed of standard Laplace distribution L0,1)) with the standard probability distributions, as N (0,1), Exp1)), ... not Laplace (μ, β) and its relationship with the general probability distributions, such as N (μ, σ), Exp(λ).

This distribution has many applications in real problems such as medical science, communications, environmental science, economics, finance and engineering Aryal (2006) and some researchers were taking the concept and characteristics of the General Laplace distribution in its relationship with other probability distributions (Samuel K., *et al*., 2001).

The probability function for Laplace (μ, β) distribution can be written as following formula:

$$f(x;\mu,\beta)=\frac{1}{2\beta}\,e^{-\frac{|x-\mu|}{\beta}} \quad ;-\infty<x<\infty \quad (1)$$

Where μ is location parameter -∞<μ<∞ and β is scale parameter β>0

Bebasis (2005) used the simulation to compare between the Laplace, Normal as standard distributions and the effect of the sample size on the test power to reject the hypothesis, he noted that the test power increases with raising of the sample size.

In (2006), Aryal applied his thesis on the Laplace distribution in real problems when there are some problems in distribution Skewness and kurtosis. Also, he applied it to real applications in exchange rate finance studies data for six different currencies, Australian, Canadian, and United States Dollar.

Gauss (2011) published article for the Beta Laplace distribution as the case of the expansion of the Laplace using multiple formats to moments generating function, finding moments and MLE for parameters distribution using the real data.

Nikola (2014) studied the basic properties of the estimators of Laplace and Cauchy distributions parameters with three parameters (a, b, r) dependent on the estimation methods MLE, moment method of numerical analysis and comparison was based on simulation experiments with one sample size and fixed repeat, but without providing any generating observations method of a random sample of each above distribution.

Thomas (2014) studied in his paper discussion how Laplace distribution was used in chemical, engineering interest where interpretation of material in terms of the normal distribution is prescribed. Nonetheless, he choosed level of significance in goodness-of-fit tests to determine whether the Laplace distribution is indeed an alternative to the Normal distribu-.tion

In the same year, Song, W. and other studied were the mixture linear regression model when random error term follows the Laplace distribution. They used the proposed method compared with other methods enhanced their studies by simulation experiments in addition to the sensitivity of the estimators in their application to real data.

A recent study in (2015) for each Elsayed & Elamir used Laplace distribution in randomized complete design sectors, using the average of the absolute values deviations analysis they found the limits of mathematical formulas own model (treatments ,blokes, and error) to minimize the sum of the absolute deviations for each term based on simu lation experiments in properties of the model, Furthermore, according to Rahim (2015), he used Laplace distribution as prior information in quintile regression dependent on R package functions to estimate parameters of general Laplace distribution through real data and simulation study.

From above researcher notes that comparison between the generating observations models with the effect of sample size and estimation method.

2.Theoretical Part

2.1. Mathematical formula for distribution

The Laplace distribution belong to exponential family as previously stated, in which several cases according to distribution parameters namely: [1,9]

- When location parameter (0) and scale parameter1)) called standard formula and writes L0,1)) according to the following probability function:

$$f(x;0,1) = \frac{1}{2}\, e^{-|x|} \quad ; \qquad -\infty < x < \infty \qquad (2)$$

- When location parameter0)) and scale parameter(β>0) called classic formula and writes L0) , β) according to the following probability function:

$$f(x;0,1) = \frac{1}{2\beta}\, e^{-\frac{|x|}{\beta}} \quad ; -\infty < x < \infty \; ; \beta > 0 \qquad (3)$$

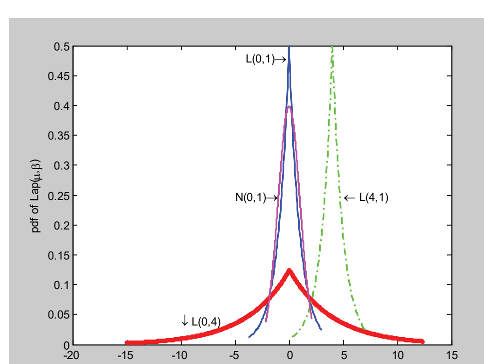
- General formula when μ ≠ 0, and β≠0 writes L (μ, β) as function in formula (1) above.

$$L = \prod_{i=1}^{n} f(x_i, \mu, \beta) = [2\beta]^{-n}.\prod_{i=1}^{n} e^{-\frac{|x-\mu|}{\beta}}$$

$$\therefore \log l = -nlog2 - nlog\,\beta - \frac{\sum_{i=1}^{n}|x-\mu|}{\beta}; \qquad ......(4)$$

In the following figures the [L ,(0,1) L ,(0,4)L ,(4,1) N0,1))] pdfs of distributions as:

Figure)1(:PDF of L (0,1), L (0,4), L (4,1) and N (0,1) distributions.



We noted from above figure that Laplace distribution has an unusual, symmetric shape with a sharp peak and tails.

2.2. Methods of Estimating parameters

There are several method use to estimate the Laplace distribution parameters (μ, β), and most of these methods are: [1,6]

2.2.1. Method of maximum likelihood M.L.E.

The method maximum likelihood (MLE) depending on the max (maximize of the likely) function of the probability function on the formula (1):

$$L = \prod_{i=1}^{n} f(x_i, \mu, \beta) = [2\beta]^{-n} . \prod_{i=1}^{n} e^{-\frac{|x-\mu|}{\beta}}$$

$$\therefore \log l = -n\log 2 - n\log \beta - \frac{\sum_{i=1}^{n}|x-\mu|}{\beta} ; \quad ......(4)$$

To find the estimator of parameter μ, we find (∂ logL) / ∂μ, through the use of the absolute value is defined as follows:

$$|x_i - \mu| = \begin{cases} x_i - \mu & if\ x_i > \mu \\ -(x_i - \mu) & if\ x_i < \mu \\ 0 & if\ x_i = \mu \end{cases}$$

Taking absolute derivative, we get:

$$\frac{\partial |x_i - \mu|}{\mu} = \begin{cases} 1 & if\ x_i > \mu \\ -1 & if\ x_i < \mu \\ 0 & if\ x_i = \mu \end{cases}$$

So that the researcher takes the values of the set of the values (Median (x_i)). The refore the maximum likelihood estimator for μ is:

$$\hat{\mu}_{ml} = median(x_i) \quad ; \quad ......(5)$$

Also to find estimator of β, we take the derivative (∂ logL) / ∂β as:

$$\frac{\partial \log L}{\partial \beta} = \frac{-n}{\beta} - \frac{-2}{\beta^2} * \sum_{i=1}^{n} |x_i - \mu|$$

$$= \frac{-n}{\beta} + \frac{\sum_{i=1}^{n}|x_i-\mu|}{\beta^2} \quad \text{put } \frac{\partial \log L}{\partial \beta} = 0 \Rightarrow -n\beta + \sum_{i=1}^{n}|x_i - \hat{\mu}| = 0$$

$$\therefore \hat{\beta}_{ml} = \sqrt{\frac{\sum_{i=1}^{n}|x_i - \hat{\mu}_{ml}|}{n}} \quad ; \quad ......(6)$$

2.2.2. Moments Method

This method is based on finding the first and second moment and from relationship with mean and variance distribution using the following relationship:

The relationship of the above is clear to us:

$$\mu'_r = \frac{1}{2} \sum_{k=0}^{r} \binom{r}{k} \beta^k . \mu^{(r-k)} k! \{1 + (-1)^k\} \ ; \ r = 1,2,..n \quad ......(7)$$

If r = 1, we get:

$$\mu'_1 = \mu = E(x) = mean(x) \Rightarrow \hat{\mu}_{mo} = \bar{x} \quad ; \quad ......(8)$$

but if r = 2 we get the following results:

$$\mu'_2 = \frac{1}{2}[2\mu^2 + 4\beta^2] = \mu^2 + 2\beta^2 = E(x^2) \ ; \quad ......(9)$$

In order to simplifying the above equation and finding estimator parameter (β) can be linked equations (8,9) through variation as:

$$var(x) = E(x^2) - (E\,x)^2 = \mu'_2 - \mu'_1$$
$$var(x) = \mu^2 + 2\beta^2 - (\mu)^2 = 2\beta^2$$
$$\therefore var(x) = 2\beta^2 \Rightarrow \hat{\beta}_{mo} = \sqrt{\frac{var(x)}{2}} \quad ; \quad ......(10)$$

2.2.3. The proposed method

For the purpose of balancing between the two methods maximum likelihood and moments, we proposed in this paragraph using weighted estimator, or what it is known as shrinkage estimator according to the following formula:

$$\hat{\theta}_{sh} = \lambda\, \hat{\theta}_{ml} + (1-\lambda)\hat{\theta}_{mo} ; 0 \le \lambda \le 1 \quad ; \quad ......(11)$$

Whereas:

$\hat{\theta}_{sh}$ Vector weighted estimators (contraction) of the Laplace distribution parameters (μ, β).

$\hat{\theta}_{ml}$ Vector estimators of the maximum likelihood method $\hat{\theta}_{ml} = (\hat{\mu}_{ml}, \hat{\beta}_{ml})$

$\hat{\theta}_{mo}$ Vector estimators of the moment's method $\hat{\theta}_{mo} = (\hat{\mu}_{mo}, \hat{\beta}_{mo})$

The method of determining the weight λ is done by choice A set of values within the range [0,1] or more iteration until the convergence between the MSE of the parameters is happening in the iteration (k) compared with (k-1) in this case we stop, this has been applied in the search where the relationship of weight according to the following equation:

$$\lambda_k = \lambda_{k-1} + \delta \quad ; k = 1,2,\dots . \ \& \ (\lambda_0, \delta > 0) \quad ; \quad \dots\dots(12)$$

2.3. Comparison Measurements

Using comparison measures is to determine the best estimators results from the generation model of Laplace distribution data, as well as results from the method of computing estimators and then finding the success rate of each case and the method gives the highest success rate model which is considered the best and depending on the model of user, based on three scales to get an odd number so it can discriminate success rate and these measures are: [3]

$$RMSE(\hat{\theta}_j) = \sqrt{\frac{1}{rep}\sum_{i=1}^{rep}(\hat{\theta}_{ij} - \bar{\theta}_j)^2} \ ; \forall j = 1,2 \quad ; \quad \dots\dots(13)$$

RMSE: Root mean square error of parameters is calculated from the following relationship:

whereas:

$\hat{\theta}_{ij} = [\hat{\mu}, \hat{\beta}]$:Estimated values for the parameter j in the iteration i of random experiment.

$\bar{\theta}_j$: Estimators above average during each simulation experiments (rep).

$(Bias)^2$: Bias Square through the following formula:

$$Bias\ Squar = (\hat{\theta}_j - \theta_{oj})^2 \ ; \forall j = 1,2 \quad ; \dots\dots.(14)$$

Where $\hat{\theta}_j = [\hat{\mu}, \hat{\beta}], \theta_{oj} = [\mu_o, \beta_o]$ and θ_{oj} are initial values of parameters.

MSE of parameters: Average joint error squares of parameters could be measured through determining of covariance matrix, according to the following formula:

$$\text{MSE paramter} = \det\left[(\hat{\theta} - \theta_o)(\hat{\theta} - \theta_o)'\right] \quad ; \dots(15)$$

And the above formula codes are defined in the previous formula14)).

3. Experimental Part

3.1. Simulation Concept

Simulation models are mathematical models that represent and reflect all the properties and behavior of the real system, used to study and analyze the behavior of a particular issue is difficult to study the real model for several reasons, which are important because they allow for changes in the simulation model components quickly and according to cases that are interest in them.

Many experiments tests can be high speed, accuracy and choose the closest to the real model of them, a task in statistics, mathematics, and engineering sciences when you cannot compare the best model from several models or best solution for a particular model from several solutions theoretically, so they are relying on simulation models through workout these models or methods under several conditions of the experiment to determine which gives the best statistical measure.

3.2. Generating observations technique

Laplace distribution is one of the most distributions which no generating observations implicit function in the application of Matlab, so they are relied on the definition software function through relationship of this distribution with other probability distributions that their functions implied in Matlab, but much of these relationships give random observations of Lap (0,1) distribution [9], and thus getting observations of Lap (μ,β) distribution a few, that it was not interested when it was used in comparison between more than one technique (model) to generate data and determine the best of it. Therefore, this research was done relying on the following models as:

1. Dependence on the relationship between standard Normal N (0,1), standard Exponential Exp. (1) distributions, assuming that Z = N (0,1), and V = Exp (1) where they can get a variable distributed Lap (μ, β) according to the following:

$$x = \mu + \beta * \sqrt{2\,V}\ Z \qquad ; \qquad ...(16)$$

Where observations can be got in a random sample size for above distributions depending on the Matlab application as following code formula:

$$Z = randn(n,1)\ ,\quad V = exprnd(1,n,1)\ ; \qquad ......\ (17)$$

Or possible relationship Z =N0,1)), and V= Exp2)) and the form of the relationship (16) are as follows:

$$x = \mu + \beta * \sqrt{V}\ Z \quad \&\ V = exprnd(2,n,1)\ ;\ ......(18)$$

and it can be shown to reduce $y = \sqrt{V}\,Z$ formulas (16,18) to prove the mathematical model of the L (0,1) distribution either through moments generating function as follows:

$$M_x(t) = E[e^{itx}] = E[e^{it(\mu+\beta y)}] = [e^{it\mu} * \psi(\beta y)] = \frac{e^{it\mu}}{1-\beta^2 t^2}$$

Or by relying on transformation method $y = \sqrt{V}\,Z$ as follows:

$$f(x) = |J| * f_1\left(\frac{y}{\sqrt{v}}\right) * f_2\left(\frac{y^2}{z^2}\right)$$

And then it added effect of location parameter μ and scale parameters β as in the formulas (16, 18).

2. By depending on CDF function for distribution as following:

$$F(x) = \frac{1}{2} + \frac{1}{2} sign(x-\mu)\left(1 - \exp\left(-\frac{|x-\mu|}{\beta}\right)\right);\ ......\ (19)$$

After using the inverse of CDF for any distribution in simulation technique we can obtain on a variable for that distribution, in above Laplace function were obtained the following formula:

$$x = \mu - \beta * sign(U) * Ln(1 - 2|U|)\ ; \qquad ......(20)$$

μ،β : Initial values of hypothetical parameters for Laplace distribution and U follows a continuous uniform distribution within the period (-1/2, 1/2] and his random observations can be gotten by Matlab application according to general formula:

$$U = unifrnd(a,b,n,m)\ ; \qquad ......(21)$$

whereas:

a and b are the parameters distribution within the period (-1/2, 1/2], n sample size, m dimension, m ≥1.

3. Dependence on generating observations of Double Exponential with scale parameter through function $y_o = exprnd(\beta, n, 1)$ and then generate a vector of random numbers by function r = r and (n,1) <0.5 and the latter has access to observations of the Lap(μ,β) distribution be in accordance to relationship:

$$r = rand(n, 1) < 0.5$$

It was tested efficiency of the models and methods solution through the different samples sizes as (25,50,100,200) and three cases of the relationship between the initial values for the parameters distribution (<,>, =), with the use of three methods to estimate the distribution parameters and thus was total executing tests (108) experiment and each experiment repeated (5000) times all these are run by the program described in the Appendix.

4. Discuss the results

After running of simulation software under all cases of experiments was shown the mean of parameters distribution. Also the mean error squares of parameters and model according to the following tables:

Table (1): The average estimation values of the parameters distribution simulation experiments

	Sample size	Models	Methods					
			MLE		Moments		Shrinkage	
			μ̂	β̂	μ̂	β̂	μ̂	β̂
(μ<β) (2,4)	25	Model_1	2.007	3.910	1.985	3.910	2.004	3.900
		Model_2	2.040	3.759	2.201	3.692	2.059	3.752
		Model_3	1.978	3.918	1.976	3.913	1.979	3.917
	50	Model_1	2.003	3.971	2.018	3.965	2.004	3.970
		Model_2	2.047	3.806	2.209	3.729	2.068	3.797
		Model_3	1.999	3.949	2.033	3.941	1.999	3.948
	100	Model_1	2.002	3.981	2.008	3.980	2.004	3.981
		Model_2	2.038	3.817	2.202	3.736	2.060	3.807
		Model_3	1.995	3.973	2.003	3.968	1.997	3.972
	200	Model_1	2.008	3.990	2.012	3.986	2.009	3.990
		Model_2	2.046	3.828	2.204	3.743	2.068	3.817
		Model_3	1.996	3.988	1.990	3.986	1.997	3.988

(μ>β) (4,2)	25	Model_1	4.004	1.955	3.992	1.955	4.003	1.954
		Model_2	4.020	1.879	4.101	1.846	4.031	1.876
		Model_3	3.989	1.959	3.988	1.956	3.989	1.958
	50	Model_1	4.001	1.985	4.009	1.982	4.002	1.985
		Model_2	4.024	1.903	4.104	1.864	4.036	1.898
		Model_3	3.999	1.975	4.002	1.970	3.999	1.974
	100	Model_1	4.001	1.991	4.004	1.990	4.001	1.991
		Model_2	4.019	1.909	4.101	1.868	4.033	1.903
		Model_3	3.998	1.987	4.001	1.984	3.999	1.986
	200	Model_1	4.004	1.995	4.006	1.993	4.004	1.995
		Model_2	4.023	1.914	4.102	1.871	4.037	1.908
		Model_3	3.998	1.994	3.995	1.993	3.998	1.994
(μ=β) (6,6)	25	Model_1	6.011	5.864	5.977	5.866	6.005	5.864
		Model_2	6.060	5.638	6.302	5.538	6.086	5.628
		Model_3	5.967	5.877	5.964	5.869	5.968	5.876
	50	Model_1	6.004	5.956	6.027	5.947	6.006	5.955
		Model_2	6.071	5.709	6.313	5.593	6.098	5.696
		Model_3	5.998	5.924	6.005	5.911	5.999	5.922
	100	Model_1	6.004	5.972	6.012	5.971	6.006	5.972
		Model_2	6.057	5.726	6.302	5.605	6.088	5.712
		Model_3	5.993	5.960	6.004	5.952	5.994	5.958
	200	Model_1	6.013	5.985	6.018	5.979	6.014	5.985
		Model_2	6.069	5.742	6.306	5.614	6.099	5.727
		Model_3	5.994	5.982	5.985	5.979	5.995	5.982

Table (2): Average MS to parameters distribution model by simulation experiments

	Sample size	Models	Methods								
			MLE			Moments			Shrinkage		
			RMSμ̂	RMSβ̂	Mse _model	RMSμ̂	RMSβ̂	Mse model	RMSμ̂	RMSβ̂	Mse model
(μ<β) (2,4)	25	Model_1	0.823	0.625	104.23	1.244	0.754	189.69	0.815	0.627	103.53
		Model_2	0.836	0.514	95.72	1.130	0.54	145.62	0.836	0.510	95.70
		Model_3	0.848	0.638	109.27	1.295	0.744	194.78	0.844	0.638	108.89
	50	Model_1	0.388	0.319	12.43	0.635	0.388	24.70	0.389	0.321	12.52
		Model_2	0.378	0.262	11.41	0.555	0.280	21.11	0.377	0.260	11.48
		Model_3	0.396	0.307	12.29	0.640	0.374	24.18	0.398	0.308	12.36
	100	Model_1	0.187	0.155	1.45	0.328	0.193	3.17	0.192	0.157	1.50
		Model_2	0.185	0.136	1.58	0.278	0.146	3.43	0.186	0.134	1.62
		Model_3	0.192	0.154	1.49	0.33	0.188	3.12	0.195	0.155	1.52
	200	Model_1	0.089	0.079	0.17	0.155	0.097	0.38	0.091	0.08	0.181
		Model_2	0.089	0.069	0.224	0.139	0.075	0.62	0.09	0.068	0.24
		Model_3	0.091	0.077	0.176	0.163	0.098	0.40	0.093	0.078	0.182
(μ>β) (4,2)	25	Model_1	0.206	0.156	6.52	0.311	0.189	11.86	0.206	0.157	6.54
		Model_2	0.213	0.128	6.11	0.282	0.134	9.10	0.211	0.127	6.07
		Model_3	0.212	0.159	6.83	0.324	0.186	12.17	0.212	0.160	6.84
	50	Model_1	0.097	0.080	0.78	0.159	0.097	1.54	0.099	0.080	0.80
		Model_2	0.094	0.066	0.71	0.139	0.07	1.32	0.095	0.065	0.73
		Model_3	0.10	0.077	0.77	0.160	0.094	1.51	0.100	0.077	0.78
	100	Model_1	0.047	0.039	0.09	0.082	0.048	0.20	0.049	0.039	0.10
		Model_2	0.046	0.034	0.10	0.070	0.037	0.21	0.047	0.034	0.10
		Model_3	0.048	0.039	0.09	0.082	0.047	0.19	0.050	0.039	0.09
	200	Model_1	0.022	0.02	0.01	0.039	0.024	0.023	0.023	0.020	0.012
		Model_2	0.022	0.017	0.01	0.035	0.019	0.038	0.023	0.017	0.02
		Model_3	0.023	0.019	0.01	0.041	0.024	0.03	0.024	0.020	0.01
(μ=β) (6,6)	25	Model_1	1.852	1.407	527.68	2.80	1.698	960.3	1.83	1.412	524.13
		Model_2	1.920	1.155	494.71	2.54	1.206	737.22	1.877	1.143	482.85
		Model_3	1.907	1.434	553.18	2.913	1.675	986.10	1.894	1.435	549.75
	50	Model_1	0.873	0.720	62.95	1.428	0.872	125.03	0.873	0.721	63.10
		Model_2	0.850	0.590	57.76	1.250	0.630	106.89	0.842	0.584	57.64
		Model_3	0.892	0.692	62.20	1.440	0.842	122.39	0.890	0.692	62.14
	100	Model_1	0.421	0.350	7.37	0.738	0.434	16.04	0.428	0.352	7.54
		Model_2	0.417	0.305	7.99	0.626	0.329	17.36	0.415	0.302	8.14
		Model_3	0.432	0.347	7.55	0.741	0.423	15.77	0.437	0.349	7.66
	200	Model_1	0.200	0.177	0.889	0.348	0.218	1.90	0.202	0.179	0.91
		Model_2	0.201	0.155	1.14	0.313	0.168	3.12	0.202	0.153	1.20
		Model_3	0.204	0.174	0.889	0.366	0.220	2.02	0.207	0.176	0.91

To give a clear picture to the reader was preparing a summary of the results Table (1) to represent the number of cases and higher than of each method according to each data generation model and for all the experiments, according to the following table.

Table (3): Summary of results represents the number of cases exceeds estimation method according to data generation model.

Methods	Models			total of method	Ratio%
	Model 1	Model 2	Model 3		
MLE	31	16	31	78	72%
Moments	0	0	0	0	0
Shrinkage	5	20	5	30	28%

From the above table, it is clear that MLE method that is the best by more than 72%, while the Shrinkage method came in second by (28%) and its superiority in the second model, but Moments method did not achieve any result.

Table (4): Summary of results number of exceeds data generation model, according to the estimation method.

Model	Method			total of model	Ratio%
	MLE	Moments	Shrinkage		
1	10	4	7	21	19%
2	26	29	28	83	77%
3	0	3	1	4	4%

To clarify of comparison between models depending on the estimation methods, it has been found the average of the measurements by comparing data generation models as in the table following:

Table (5): Average statistical measures for each model at the level of all method.

Model	RMSE	RMSE	MSE	%Ratio
1	0.519	0.368	77.25	0
2	0.490	0.289	66.37	100%
3	0.533	0.366	79.69	0

From the above table, it is clear that the second model is the best model to generate the distribution of Laplace data because it gave the best statistical measures with a rapprochement between the two models (1,3), while according to estimation methods shown in the :following table

Table (6): Average statistical measures for each method estimation on the level of all models.

	RMSE	RMSE	MSE	%Ratio
MLE	0.435	0.322	59.91	67%
Moments	0.670	0.377	103.99	0
Shrinkage	0.437	0.324	59.38	33%

From the above table, it is clear that MLE method is the best to find Laplace distribution parameters across all models because it gave the best statistical measures by more than (67%) overall and these results are compatible with the results in tables (3, 4).

5. Conclusions and Recommendations

Through the practical side of the search may be mentioned the most important the following conclusions:

1. The second model is the best model to generate random observations follow Laplace distribution, this result obtained from equation (20) belonged from CDF distribution, causing the results to be better than the other models, of method of estimation.

2. The best method is to appreciate the Laplace distribution parameters MLE then followed Shrinkage method while Moments method did not record any result excellence.

3. The estimated values of the parameters is to be less biased and c onsistent with increasing sample size can be seen in Table (2), this is consistent with statistical theory, which emphasizes the sample size its impact on the accuracy and reliability of the estimators.

6. Recommendations

1. We recommend the adoption of the model described in equation (20) in the generation of track the distribution of public Laplace and use MLE method of finding estimators.
2. Use Robust methods for the statistical inference (estimation and testing) for the purpose of knowing the vulnerability of generating observations model outlier values or containment.
3. Study the Laplace Multivariate distribution.

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