# The Transitivity of One Disconnecting Arc Spaces

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## الخلاصة

لتكن D فضاء القوس المفصول بنقطة و f:D→D دالة مستمرة في هذا العمل نتوصل الى :

1. اذا كانت f دالة متعدية فان مجموعة النقاط الدورية كثيفة .
2. f تكون متعدية و تمتلك نقطة دورية فردية الدوار دوارها اكبر من الواحد اذا وفقط اذا f^2 تكون متعدية و f تمتلك نقاط دورية كثيفة .
3. $f^2$ تكون متعدية و f تمتلك نقاط دورية كثيفة اذا وفقط اذا f متعددة كليا .
4. تكون متعددة كليا اذا وفقط اذا f تكون تبولوجي خلط .
5. اذا كانت f دالة خطية رتيبة فان f تكون تبولوجي خلط اذا وفقط اذا لكل قوس K⊆D , يوجد n بحيث ان $f^n$ (K)=D .

## الكلمات المفتاحية

(النقاط الدورية، تبولوجي التعدي، تبولوجي الخلط، مجموعة الغاية، قوس مفصول بنقطة).

## Abstract

Let D be a one disconnecting arc space, f:D→D be a continuous map, in this work we get:

1- If f is transitive map. Then the set of all periodic points is dense in D.

2- If f is transitive and has a point of odd period greater than one if and only if $f^2$ is transitive and f has dense periodic points.

3- $f^2$ is transitive and f has dense periodic points if and only if f is totally transitive

4- f is totally transitive if and only if f is topologically mixing.

5- If f is piecewise monotone Then f is topologically mixing if and only if for every arc K⊆D, there is an n such that f^n (K)=D.

## Keywords

Periodic point, topological transitive, topological mixing, limit set, one disconnecting arc.

## 1. Introduction

A several authors have been studied the transitive maps, periodic points and other chaotic properties on one dimensional spaces for examples (the circle, the real interval and graph maps etc). In [1] Alseda, Kolyada, Llibre and Snoha, (1990) studied if X is connected and a compact topological space and if f a transitive map with one disconnecting arc. In [2] Sabbaghana M., Damerchiloob H. , (2011), proved that it is not necessary to assume that X is connected. In this research we will generalize some of the results and theorems on the new space.

## 2. Preliminary

In this paper, we prove that there are relations between the transitivity and periodicity, also the transitivity with topological mixing.

### 2.1. Definition [6]

The orbit of p is the set of points p,f(p),$f^2$(p),…. And is denoted by orb(p)={$f^n$ (p)|n∈$N_\circ$} where $N_\circ$=N∪{0}.

A point p∈X is a periodic point of f if there exists a positive integer n ∈ N such that $f^n$ (p)=p . If p is a periodic point of period n ,then we say that orbit of p is a periodic orbit of period n. It is called the period of f. We denote the set of periodic points by per(f) and the set the period of f is denote by p(f).

### 2.2. Definition

We say that D has one disconnecting arc say J if J is an open subset of D homeomorphic with an open interval of R and when Y is the connected component of D which contains J, then for all x in J the set Y -{x} has exactly two connected components.

### 2.3. Definition

Let a,b∈D, we say that a⊳b if a follows counterclockwise to b .

### 2.4. Definition [3]

We say that the map f is transitive(Do) if ∃x∈X such that the orbit {$f^n$ (x)|n≥0} is dense in X , that is $\overline{(\{f^n (x)|n\geq 0\})}$=X .

### 2.5. Definition [4]

Let f:X→X be a continuous map , if f is transitive , we say that f is totally transitive if f^n is transitive for all integers n>1 .

### 2.6. Definition [3]

Let f:X→X be a continuous map, we say that f is topological mixing if for every pair non-empty open sets U and V in X , there exists a positive integer n such that $f^k$ (U)∩V≠∅ for every k>n.

### 2.7. Definition [5]

Let X be a metric space, and let f:X→X be a continuous map. The ω -limit set of x∈X , denoted by ω(x,f), is the set of cluster points of the forward orbit $\{f^n (x)\}_{n\in N}$ of the iterated functionf. Hence, y∈ω(x,f) if and only if there is a strictly increasing sequence of natural numbers $\{n_k\}_{k\in N}$ such that $f^{n_k}(x)\to y$ ask→∞. Another way to express this is ω(x,f)=$\cap_{n\in N}\overline{\{f^k}$

(x)|k>n} where ¯A denotes the closure of the set A.

### 2.8. Proposition [6]

A continuous map $f:X\to X$ of a compact metric space into itself is transitive if and only if there exists a point $x\in X$ such that $\omega(x,f)=X$.

### 2.9. Lemma [2]

Let X be a topological space and let $f: X \to X$ be a transitive map. If X has a connected component with nonempty interior, then X has only a finite number of connected components and they form a regular periodic decomposition. That is, $\bigcup_{i=0}^{n-1} D_i$ where for each $0< i < n-1$, $D_i$ is the closure of an open set and for every $0 \le i < j \le n-1$, $D_i \cap D_j$ is nowhere dense and for each positive integer k, $f^k(D_i) \subseteq D_{i+k(mod n)}$.

The set $D = \{D_0, D_1,..., D_{n-1}\}$ is called a regular periodic decomposition for f on X.

### 2.10. Lemma [6]

If $f:X\to X$ is a continuous map, then the following statements are equivalent:

1) f is transitive.

2)for every non-empty open set W in X $,\bigcup_{n=1}^{\infty} f^n(W)$ is dense in X.

3) for every pair of non-empty sets U and V in X , there is a positive integer k such that $f^{-k}(U)\cap V\neq\emptyset$.

4) for every non-empty open set W in X $,\bigcup_{n=1}^{\infty} f^{-n}(W)$ is dense in X.

5)for every proper closed invariant subset of X has empty interior.

### 2.11. Theorem [2]

Let X be a compact space and let X have a disconnecting interval . Let also $f:X \to X$ be a transitive map. Then the set of all periodic points of f is dense in X.

## 3.Main Result:

Some theorems and lemmas are proved, we generalize the results of the spaces from R to R to maps from the one disconnecting arc space to itself :

### 3.1. Proposition

Let $f:D\to D$ be transitive map and the set of all limit points of f of a point x in D is D .Then exactly one of the following conditions holds:

1) the set of all limit points of f^s of a point x in D is D, for every positive integer s.

2) there exist non-degenerate closed arcs J and K with $J\cup K=D$ and $J\cap K=\{y\}$ where y is a fixed point of f such that $f(J)=K$ and $f(K)=J$.

Proof:

Let r be an arbitrary integer and $B_r=\omega(f^k(x), f^r)$ and for $0\le k<r$. Since $B_0\cup B_1\cup\ldots\cup B_{r-1}=D$ at least of $B_k$ has nonempty interior. Moreover since the orbit of x cannot folding the arc to the a point. Since $f(B_k)=B_{k+1}$ for $0\le k<r$ and $f(B_{k-1})=B_0$ , it follows that each $B_i$. We claim that if the interior of $B_k$ and $B_j$ intersect then $B_k=B_j$. To see that suppose that $B^{\circ}_k\cap B^{\circ}_j\neq\emptyset$. Then for some positive integer n $f^{rn+i}(x)\in B^{\circ}_k\cap B^{\circ}_j$

It follows that $B_k \subseteq B_j$. Since $B_j$ is $f^r$-invariant and $B_k=\omega(f^{kn+i}(x),f^r)$ . Since k,j can be interchanged.

Let A denote the collection of subsets of D which are component of int($B_k$) for some k∈{0,…,r-1}. Then by Lemma 2.9 D has n connected components. Since A is finite. We may assume that has one connected component say $E_1$. Then $F_1=\overline{(E_1)}$ .So we have $B_k$=D for k=0,…,r-1 . Since f is transitive, then by proposition 2.8 ω(x,f)=D.

Now suppose that A have two connected components $E_1$ and $E_2$ such that $F_1=\overline{(E_1)}$ and $F_2=\overline{(E_2)}$. Let y be a fixed point. If $y \in E_1,E_2$. Then $f(F_1)=F_1$ and $f(F_2)=F_2$ which is impossible. If y cannot to be endpoint of arc in D. The only possibility y is that y is a common endpoint of $F_1$ and $F_2$. Then $f(F_1)=F_2$ and $f(F_2)=F_1$. Thus we have (2) satisfy and hence $f^2$ is not transitive. Since r is arbitrary, this proves that (1) satisfied if $f^2$ is transitive and (2) satisfied if is not.

## 3.2. Lemma

Let f:D→D be transitive.Then of the following holds:

1) f^2 is transitive in which f^(r+l) is transitive ∀r≥1,l≥0.

2) $f^2$ is not transitive in which case $D=\overline{(\{f^2n\ (x)|n\in N_\circ\})} \cup \overline{(\{f^{2n+1}(x)|n\in N_\circ\})}$ and $\overline{(\{f2n\ (x)|n\in N_\circ\})} \cap \overline{(\{f^{2n+1}\ (x)|n\in N\_°\})} \neq \emptyset$ and $f(\overline{(\{f^2n\ (x)|n\in N_\circ\})})=\overline{(\{f^{2n+1}\ (x)|n\in N_\circ\})}$ and $f(\overline{(\{f^{2n+1}\ (x)\,|\,n\in N_\circ\})})=\overline{(\{f^2n\ (x)|n\in N_\circ\})}$. Moreover ∀l≥1, $\{f^2ln\ (x)|n\in N_\circ\}$ is dense in $\overline{(\{f^2n\ (x)|n\in N_\circ\})}$ and $\{f^{2ln+1}\ (x)|n\in N_\circ\}$ is dense in $\overline{(\{f^{2n+1}\ (x)|n\in N_\circ\})}$.

Proof:

Let r be an integer, r≥1and for each integer s ,0≤s≤r-1, let $B_s=\overline{(\{f^{rn+s}\ (x)|n\in N_\circ\})}$. Then since $\cup_{s=0}^{r-1}\{f^{rn+s}\ (x)|n\in N_\circ\}=\{f^n\ (x)|n\in N_\circ\}$, this implies that $\cup_{s=0}^{r-1} B_s=D$. Then there is an s $0\le s\le r-1$, thus $B_s^\circ\neq\emptyset$ . Since f is transitive, so by Lemma 2.9 $D=\cup_{s=0}^{r-1} B_s$, for each $0\le s\le r-1$ and ∀ $0\le i,j\le r-1$ , $B_i\cap B_j$ is nowhere dense and for each positive integer k, $f^k(B_r)\subseteq B_{r+k \bmod n}$.

Now we must prove that if $B_i^\circ\cap B_j^\circ\neq\emptyset$, then $B_i^\circ= B_j^\circ$.

If $B_i^\circ\cap B_j^\circ\neq\emptyset$ , then there is a positive integer n , so that $f^{2n+i}(x)\in B_i^\circ\cap B_j^\circ$, and there is a sequence $n_1,n_2,n_3,\ldots$of positive integer such that $f^{n_k r+j}(x)\to f^{rn+i}(x)$. Then for every integer m>0 we have $f^{r(n_k+m+j)}(x)\to f^{r(n+m)+i}(x)$.

This implies that $\overline{(\{f^{rn+i}(x), f^{r(n+m)+i}(x),.\})}\subset B_j$ and hence that $B_i\subset B_j\cup\{f^{i}(x), f^{r+i}(x),\ldots, f^{(n-1)r+i}(x)\}$.Then $B_i^\circ\subset B_j^\circ$. In the same way we can prove that $B_j^\circ\subset B_i^\circ$, and so $B_i^\circ=B_j^\circ$.

Let h be a component of $B_s^\circ$ such that H={h|for some s,0≤s≤r-1} . Since D has n of components. Then $H=\{h_1, h_2, \ldots, h_n\}$. Let $F_i=\overline{(h_i)}$. Since $h_i$ is finite, and so $F_i$ is finite and hence the set $\{F_1, F_2, \ldots, F_n\}$ is finite . By transitivity , we get the set $\{F_1, F_2, \ldots, F_n\}$ is permuted by f. We next prove that n≤2. Let p be a fixed point of f. If $p\in nF_i^\circ$, then $f(F_i)=F_i$ which is impossible unless n=1 . In the same way, if p is an endpoint of D, then n=1. If p is a common endpoint of $F_i$ and $F_j$ , then $f(F_i)= F_j$ and $f(F_j)= F_i$ which is satisfied only if n=2 . Notice that the integer n depend on s. It follows that we will refer n as n(r).

Now we satisfy the conclusion of the Lemma, so we first suppose that $f^2$ is transitive. Let r be an integer, r≥0 and suppose that n(r)=2. Then there are closed arcs $F_1$and $F_2$with $F_1 \cup F_2=D$, $F_1 \cap F_2=\{pt\}$, $f(F_1)= F_2$ and $f(F_2)=F_1$.

Suppose that $x \in F_2$, we see that, for each m, $f^2m\ (x) \in F_2$ and hence $\{f^{2n}\ (x)|n \in N_\circ\} \cap F_1^\circ \neq \emptyset$. Which is contradiction with the fact that f^2is transitive, and hence n(r)=1 . Then $\forall s, 0 \leq s \leq r-1$, $B_s$= D. Then $\overline{\ }(\{f^{rn+s}(x) \mid n \in N_\circ\})$= D and so $f^{r+s}$is transitive. This implies that for any integer l≥0, $f^{r+l}$ is transitive.

Next suppose that $f^2$ is not transitive. Let r=2. Since $f^2$ is not transitive $B_\circ \neq D$ and so n(2)=2 .Now let j be an integer, j≥1.Then for each integer k, $\{f^{2jn+k}\ (x) \mid n \in N_\circ\} \subset \{f^{2n+k}(x) \mid n \in N_\circ\}$, and since $\overline{\ }(\{f^{\,2}n\ (x) \mid n \in N_\circ\}) \neq D$, we have n(2j)=2.

Since the common endpoints of $F_1$ and $F_2$ is the only fixed point for the map f.Then the arcs $F_1$ and $F_2$ which we construct for r=2j are independent of j. Since by assumption $x \in F_2$, we have $F_2=\overline{\ }(\{f^2n\ (x) \mid n \in N_\circ\})$, $F_1\overline{\ }(\{f^{(2n+1)}\ (x) \mid n \in N_\circ\})$ for each integer l≥1, $\overline{\ }(\{f^2 ln\ (x) \mid n \in N_\circ\})= F_2$ and $\overline{\ }(\{f^{\,2} ln\ (x) \mid n \in N_\circ\}) =F_1$. This establishes the Lemma.

In the same way the proof of Theorem 2.11, we can prove that the next Theorem:

## 3.3. Theorem

Let $f:D \to D$ be transitive map. Then the set of all periodic point is dense in D.

## 3.4. Theorem

f2 is transitive and f has dense periodic points if and only if for each arc K in Dand each pair a, $b \in D^\circ$, there is an integer M such that n>M then $[a,b] \subset f^{\,n}\ (K)$.

**Proof:**

Suppose that $f^{\,2}$ is transitive and f has a dense periodic points. Let K be an arc in D. Since the periodic points of fare dense, there is a periodic point $q \in K^\circ$.Suppose that q has a periodic point of period j. Let $g=f^{\,j}$. Let $E=\overline{\ }(\bigcup_{n=0}^{\infty} g^n\ (K))$ .Then E is a closed arc. Let $z \in K^\circ$such that $\{f^{\,2}n\ (z) \mid n \in N_\circ\}$ is dense D. Then $f^{\,2}$ is transitive .Thus from Lemma 3.2, we get $\{g^n\ (z) \mid n \in N_\circ\}$ is dense in D. This implies that g is transitive. Hence E=D.

We will prove that if q is aperiodic point such that $orb(q) \subset D^\circ$, then there is an integer t such that $orb(q) \subset f(K)^t$.

To see that, suppose that q is a periodic point with period l and $orb(q) \subset D^\circ$. Let $p_1 \in orb(q)$ such that $p_1$ is the nearest point of begin point of D and let $p_2 \in orb(q)$ such that $p_2$ is the furthest point of begin point of D.

Suppose that $p_1 \neq q$ . Since $\bigcup_{n=0}^{\infty} g^n\ (K)=D^\circ$, there is an integer r such that $[p_1,q]=g^s\ (K)$. Let $h=g^s$ and observe that $h^l\ (p_1)= p_1$, $h^l\ (q)=q$. By Lemma 3.2, there is a point $y \in [p_1, q]^\circ$ such that $\{h^l n\ (y)|n \in N_\circ\}$ is dense in D, and hence $h^l$ is transitive.

Therefore there is an integer r such that $p_2 \rhd h^l r\ (y)$. Then we have $h^l\ r\ (q)=q$, $h^l\ r\ (p_1)= p_1$, and $p_2 \rhd h^l\ r\ (y)$. This implies that $orb(q) \subset [p_1, p_2] \subset f^{\,l}\ r\ (K)$. Thus for r=l.r.s.j, we have $orb(q)=f^t\ (K)$. Then if r≥t, $orb(q) \subset f^{\,r}\ (K)$.

Now suppose that $a,b\in D^{\circ}$ and that $a\rhd b$. Let c and d be periodic points such that $[a,b]\subset[c,d]$ and $orb(c)\cup orb(d)\subset D^{\circ}$. Then there are a positive integer $r_1$ and $r_2$ such that $orb(c)\subset f^{r_1}(K)$ and $orb(d)\subset f^{r_2}=(K)$ .Let $N=\max\{r_1, r_2\}$. Then if $n>N$, $[a,b]\subset[c,d]\subset f^{n}(K)$.

Suppose that for each arc K in D and each pair $a,b\in D^{\circ}$, there is a positive integer N such that if $n>N$, then $[a,b]\subset f^{n}(K)$. We must prove that f is transitive. Let U be an open arc in D, and let $x\in U$. We will prove that $\cup_{n=0}^{\infty} f^{-n}(x)$ is dense in D . If not, there is a closed arc K such that $K\cap\cup_{n=0}^{\infty} f^{-n}(x)=\emptyset$. But by the condition, there is a positive integer k such that $x\in f^{k}(K)$. Hence, there is a point $y\in K$ such that $f^{k}(y)=x$ . Then $y\in K\cap\cup$_(n=0)$^{\infty}$ $f^{-n}(x)$. Thus $\cup_{n=0}^{\infty}f^{-n}(x)$ is dense in D. Since $x\in U$, and so $\cup_{n=0}^{\infty} f^{-n}(U)$ is dense in D. Then by Lemma 2.10 f is transitive. Since f is transitive . Then there is a point $y\in D$ such that $\{f^{n}(y)|n\in N_{\circ}\}$ is dense in D.

If $f^{2}$ is not transitive. Hence $\{f^{2}n(y)|n\in N_{\circ}\}$ is not dense in D, then this implies that there are closed arcs $F_1$ and $F_2$ in D such that $F_1\cup F_2=D$, $F_1\cap F_2=\{pt\}$, $f(F_1)=F_2$ and $f(F_2)=F_1$.

Let $a\in F_1^{\circ}$, $b\in F_2^{\circ}$ and let K be $F_1$. Then for each positive integer $[a,b]\not\subset f^{n}(K)$.This is contradiction the assumption that $[a,b]\subset f^{n}(K)$. Since f is transitive, then by Theorem 3.3 f has dense periodic points and hence $f^{2}$ has a dense orbit .This implies that $f^{2}$ is transitive and this is establishes the theorem.

## 3.5. Theorem

If $f^{2}$ is transitive and f has a dense periodic points then f has a point of odd period.

**Proof:**

Let K and L be arcs in D such that $K\cap L=\emptyset$. Then by Theorem 3.3, there is a positive integer N such that if $n>N$, then $(K\cup L)\subset f^{n}(K)\cap f^{n}(L)$. Let r be a positive integer which is prime and larger than $2N+2$. Let $i=((r-1))/2, j=(r+1)/2$.Then $i>N, j>N$, and $i+j=r$.

Now, since $K\subset f^{i}(L)$, there is an arc $L_1$ of L such that $f^{i}(L_1)=K$. Then $L_1\subset f^{j}(K)=f^{i+j}(L_1)=f^{r}(L_1)$. Then there is a point $y\in L_1$ such that $f^{r}(y)=y$. Since $y\in L_1$, $f^{j}(y)\in K$,and $K\cup L=\emptyset$., this implies that $f(y)\neq y$.Since r is prime we have the period of y is r. This establishes Theorem.

In [7] coven proved that the following theorem for maps from R to itself. We will generalize the same theorem to maps from D to itself.

## 3.6. Proposition

Let $f:D\to D$ be a continuous map .Then f is transitive and has a point of odd period greater than one if and only if $f^{2}$ is transitive and f has dense periodic points .

Proof:

Suppose that f is transitive and has a point of odd period greater than one. Since f is transitive. Then by Proposition 3.2 , $f^{2}$ is transitive and by Theorem 3.5 f has a dense periodic point.

Now Suppose that $f^{2}$ is transitive and f has a dense periodic point. Since $f^{2}$ is transitive .

Then by Definition 2.4, f is transitive .Since $f^2$ is transitive and f has a dense periodic point. Then by Theorem 3.4 f has a point of odd period.

### 3.7. Proposition

Let f:D→D be a continuous map. Then $f^2$ is transitive and f has dense periodic points if and only if f is totally transitive map.

**Proof:**

Suppose that $f^2$ is transitive and f has a dense periodic point. Then by Proposition 3.2, $f^n$ is transitive for every n>0 and hence f is totally transitive.

Now Suppose that f is totally transitive. Then by Definition 2.4 $f^2$ is transitive and by Theorem 3.5f has dense periodic points.

### 3.8. Proposition

Let f:D→D be a continuous map, f is totally transitive if and only if f is topologically mixing.

**Proof:**

Case (1): To prove the topological mixing map imply the totally transitive

Assume that f is topologically mixing. Then By Definition 2.4, for every pair of non-empty open sets U and V in X there exists a positive integer n such that $f^n$ (U)∩V≠∅ and Definition 2.8 implies, there exists a positive integer n such that $f^k$ (U)∩V≠∅ for every k>n . Hence $f^n$ is transitive for every n>0 and so f is totally transitive.

Case two (2): suppose that $f^n$ is transitive for every n>0 imply the topological mixing map . Since $f^n$ is transitive for every n>0 .Then $f^2$ and f are transitive .So by Theorem 3.5 f has a dense periodic point. Thus by proof of Theorem 3.3 f is topologically mixing.

### 3.9. Proposition

Let f:D→D be a continuous map and f is piecewise monotone, f is totally transitive if and only if for every arc K⊆D, there is an n such that f^n (K)=D.

**Proof:**

Assume that f is totally transitive. To prove that for every arc K⊆D, there is an n such that f^n (K)=D. We have two cases:

Case one: there is an arc L⊆D^°such that f^2 (L)=D . If K is an arc such that K⊆D . Then by Proposition3.7 , f^2 is transitive and f has dense periodic points and by Theorem 3.3 , there is an n such that L⊆f^n (K). Thus f^(n+2) (K)=D.

Case two :assume that $f^2$ (L)≠D for every arc L⊆D^°. Let $e_1$ be start point in D and $e_2$ be endpoint in D. Since f is onto, either f^(-1) $(e_1)\subseteq\{e_1,e_2\}$ or $f^1$ $(e_2)\subseteq\{e_1,e_2\}$ . This implies that either $f^{-2}$ $(e_1)=\{e_1\}$ or $f^{-2}$ $(e_2)=\{e_2\}$. We assume the former. Let q be the smallest turning point of $f^2$. Then $f^2$ has no fixed points in $(e_1,q)$ and y⊳$f^2$ (y) for every y∈$(e_1,q)$. (If not, then y⊳$f^2$ (y) for every y∈$(e_1,q)$, and hence $[e_1,q]$ is $f^2$-invariant). But $f^2$ $[q,e_2]\subseteq[m,e_2]$ for some m> 0 . Thusf^2n $[q,e_2]\subseteq[m,e_2]$ for every n > 0, contradicting the assumption that $f^2$ is transitive.

Now suppose that for every arc K⊆D, there is an n such that $f^n$ (K)=D. Then immediately f is totally transitive.

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