On Bi-domination in Graphs

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الخلاصة

المجموعة المهيمنة هي المجموعة ذات الهيمنة المزدوجة في الرسم البياني ، إذا كان كل قمة في هذه المجموعة، تسيطر تمامًا على رأسين لا ينتميان إلى المجموعة المهيمنة. في هذا العمل ، تم تقديم خمسة تعريفات جديدة للهيمنة، وهي عبارة عن صيغ معدلة للهيمنة المزدوجة: «هيمنة ثنائية مستقلة ؛ الهيمنة الثنائية الكاملة ، الهيمنة الثنائية المتصلة، الهيمنة المقيدة ، شجرة تكميلية ثنائية الهيمنة ». يتم تحديد الحدود العليا والدنيا لحجم الرسوم البيانية التي تحتوي على هذه المعلمات.

الكلمات المفتاحية

ثنائية الهيمنة ، مجموعة ثنائية الهيمنة مستقلة ، مجموعة ثنائية الهيمنة الكلي ، مجموعة ثنائية الهيمنة المتصلة ، شجرة تكميلية ثنائية الهيمنة، هيمنة ثنائية مقيدة.

Abstract

A dominating set is a bi-dominating set in a graph, if every vertex in this set, dominates exactly two vertices that, do not belong to the dominating set. In this work, five new definitions of domination have been presented, which they are a modified versions of bi-domination: “independent bi-domination; total bi-domination, connected bi-domination, restrained domination, and complementary tree bi-domination”. Upper and lower bounds are determined for the size of graphs having these parameters.

Keywords

bi-domination, independent bi-dominating set, total bi-dominating set, connected bi-dominating set, complementary tree bi-domination, restrained bi-domination.

Mathematical subject classification: (05C69)

1. Introduction

Let G=(E,V) be a graph with a set V(G) of vertices of order n and a set E(G) of edges of size m. The "degree of a vertex v∈ V(G)", of any graph G is defined as the number of edges incident on v. It is denoted by deg(v). The "minimum and maximum degrees of vertices in G denoted by δ(G) and Δ(G) respectively ". The "open neighborhood "N(v) is the set of vertices adjacent to v, and the "closed neighborhood" N[v]=N(v)∪{v}." The subgraph of G induced by the vertices in D is denoted by G[D],[2].

O. Ore is the first one who introduced the term "domination number" and "dominating set "[5]. A set D "⊆" V is a "dominating set " in G, if every vertex in V – D is adjacent to a vertex in D, that is N[D]=V. The minimum cardinality over all dominating set in G is the "domination number γ(G)"[7]. In recent years, many types of domination-related parameters have been studied. For a historical reference in this regard see books written by Haynes et al [3, 4, 6].

A dominating set D is an "independence dominating set" in G if [D] ,doesn't has edges, and where a set D is a "total dominating set" in G if G[D] , has no isolated vertex and, if G[D] is connected, then this is known as a "connected dominating set" in G, if G[D] is connected, also set D known as a "restrained dominating set" where, "every vertex doesn't belongs to dominating set D, is adjacent to a vertex in D and, to another vertex not in D"[3]. A set D is a "complementary tree dominating set" if, G[V-G] is a tree [8]. Finally, the new definition is the "bi-domination" where a dominating set D is a "bi-dominating set" in G, if every vertex in set D dominates exactly two vertices in V-D [1].

Here, the definitions of the parameters of some types of bi-domination like "independent bi-dominating set, total bi-dominating set, connected bi-dominating set, complementary tree bi-dominating set and restrained bi-dominating set" are introduced, this study includes the bounds of the size of any graph has these types of domination.

1.1. Definition [1]

"For any graph G(V,E) which is finite and simple undirected graph without isolated vertex, a subset D ⊂ V(G) is a bi-dominating set if every vertex in D dominates exactly two vertices in V-D, such that |N(v) ∩ V-D| = 2. The minimum cardinality of bi- dominating set is denoted by γ_{bi} (G). The domination number of G, denoted γ_{bi} (G) is a minimum cardinality over all bi-dominating set in G".

1.2. Observation [1]

For any finite simple graph G(n,m) with a bi- dominating set D and bi-domination number γbi(G). We have

The order of G is n ≥ 3.

δ(G) ≥1 , Δ(G) ≥2.

Every v ∈ D, deg(v)≥ 2.

Every support vertex v,v∈ D.

$\gamma(G) \leq \gamma_{bi}(G)$.

2. Independent bi-dominating set

2.1. Definition

"A subset $D\subset V(G)$ is an independent bi-dominating set in graph G if, D is a bi-dominating set such that, the induced subgraph <D> has no edges, while the minimum cardinality of an independent bi-dominating set is denoted by $\gamma_{bi}^{i}(G)$".

2.2. Theorem

If graph G(n,m) has an independent bi-domination number $\gamma_{bi}^{i}(G)$ then

$$2\,\gamma_{bi}^{i}(G) \le m \le \frac{n^2 - n + (\gamma_{bi}^{i}(G))^2 - 2n\,\gamma_{bi}^{i}(G) + 5\,\gamma_{bi}^{i}(G)}{2}$$

Proof.

The proof requires two cases and as follows: let D be a γ_{bi}^{i} - set of G.

Case 1. To prove the lower bound $2\,\gamma_{bi}^{i}(G)\le m$:

By the definition of independent bi-dominating set, G[D] is a null graph, and there exist exactly two edges incident to, every vertex in D. Let G[V-D] be a null graph. So G contains as few edges as possible and G[V-D] will not violate the definition of independent bi-domination. Therefore, the number of edges is $2\,|D|=2\,\gamma_{bi}^{i}(G)$. Thus, in general $m\ge 2\,\gamma_{bi}^{i}(G)$.

Case 2. To prove the upper bound:

Since set D is independent then, all vertices of G[D] are isolated and since, the number of edges of G[V-D] does not affect vertices of set D then, let G[V-D] be a complete induced subgraph. Let m_1 be the number of edges of G[V- D] Therefore,

$$m_1 = \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{bi}^{i}(G))(n-\gamma_{bi}^{i}(G)-1)}{2}$$

Since, D is a bi-dominating set, so let $m_2=2|D|=2\,\gamma_{bi}^{i}(G)$,so $m=m_1+m_2$.

Hence, $m = 2|D| + \frac{|V-D||V-D-1|}{2} = 2\,\gamma_{bi}^{i}(G) + \frac{(n-\gamma_{bi}^{i}(G))(n-\gamma_{bi}^{i}(G)-1)}{2}$.

3. Total bi-dominating set

3.1. Definition

"A subset $D\subset V(G)$ is, a total bi-dominating set in G if set D is a bi-dominating such that, <D> has no isolated vertex, $\gamma_{bi}^{t}(G)$ is the minimum cardinality of a total bi-dominating set in G".

3.2. Theorem

Let G(n,m) be a graph having a total bi-domination number $\gamma_{bi}^{t}(G)$ then

$$3\,\gamma_{bi}^{t}(G) \le m \le \frac{n^2-n}{2} + (\gamma_{bi}^{t}(G))^2 - n\,\gamma_{bi}^{t}(G) + 2\,\gamma_{bi}^{t}(G)$$

Proof.

Let set D be a γ_{bi}^{t}-set of G.

Case 1. To prove the lower bound $3\,\gamma_{bi}^{t}(G)\le m$:

This case occurs when the induced subgraph G[V-D] is a null graph, and since D is a total dominating set, so every vertex in D has at least a degree equal to 3. Therefore, the number of edges is m=3 |D| so $3\,\gamma_{bi}^{t}(G)\le m$.

Case 2. To prove the upper bound, this case occurs where, the two induced subgraphs G[D] and G[V-D] are complete, so let m_1 and m_2 be the number of edges of G[D] and G[V- D] respectively. Therefore,

$$m_1 = \frac{|D|\,|D-1|}{2} = \frac{\gamma_{bi}^{t}(G)(\gamma_{bi}^{t}(G)-1)}{2}$$

$$m_1 = \frac{|D|\,|D-1|}{2} = \frac{\gamma_{bi}^t(G)(\gamma_{bi}^t(G)-1)}{2}$$

And according to the definition of bi-domination set we have m_3 edges between G[D] and G[V-D] where, $m_3=2|D|=2\gamma_{bi}^t(G)$, so in this case $m=m_1+m_2+m_3$.

Hence,

$$m = 2|D| + \frac{|D|\,|D-1||}{2} + \frac{|V-D||V-D-1|}{2} = 2\gamma_{bi} + \frac{\gamma_{bi}^t(G)(\gamma_{bi}^t(G)-1)}{2} + \frac{(n-\gamma_{bi}^t(G))(n-\gamma_{bi}^t(G)-1)}{2}.$$

In general, $m \le m_1+m_2+m_3$.

4. Connected bi-dominating set

4.1. Definition

"Let D be a subset of V(G) then D is a connected bi-dominating set in G if, D is a bi-dominating set, such that, <D> is a connected induced subgraph. $\gamma_{bi}^c(G)$ is the minimum cardinality of a connected bi-dominating set"

4.2. Theorem

The size of graph G(n,m) has connected bi-domination number $\gamma_{bi}^c(G)$ is

$$3\gamma_{bi}^c(G) - 1 \le m \le \frac{n^2-n}{2} + \gamma_{bi}^c(G)^2 - n\,\gamma_{bi}^c(G) + 2\gamma_{bi}^c(G)$$

Proof.

We prove the required by two cases that depend on the bounds as follows: Let set D be a γ_{bi} - set of G

Case 1. To prove the lower bound $3\gamma_{bi}-1 \le m$.

Based on a Theorem 2.3 the number of edges is $m_1= 2|D|=2\gamma_{bi}$ since D is a bi-dominating set. Moreover, the induced subgraphs G[D] should contain as few edges as possible to be a connected. Thus, the minimum number of edges is $m_2=(D-1)=(\gamma_{bi}^c(G)-1)$ thus, $m=m_1+m_2=2\gamma_{bi}^c(G)+(\gamma_{bi}^c(G)-1)=3\gamma_{bi}^c(G)-1$. In general, $m \ge m_1+m_2$.

Case 2. The same proof in Theorem 3.2 (case 2).

5. Restrained bi-dominating set

5.1. Definition

"A bi-dominating set is a restrained bi-dominating set D in a graph G, such that G[V-D] has no isolated vertices. The minimum cardinality of a restrained bi-dominating set, is the restrained bi-domination number of G, is denoted by $\gamma_{bi}^r(G)$."

5.2. Theorem

The size of graph G(n,m) has restrained bi-dominating set $\gamma_{bi}^r(G)$ is

$$2\gamma_{bi}^r(G) + \left\lceil\frac{n-\gamma_{bi}^r(G)}{2}\right\rceil \le m \le \frac{n^2-n}{2} + \gamma_{bi}^r(G)^2 - n\,\gamma_{bi}^r(G) + 2\gamma_{bi}^r(G)$$

Proof.

Let D be a γ_{bi}^r - set of G, so the number of edges is calculated from the two following cases.

Case 1. To prove the lower bound $2\gamma_{bi}^r(G) + \left\lceil\frac{n-\gamma_{bi}^r(G)}{2}\right\rceil \le m$:

This case occurs where the induced subgraph G[D] is a null graph, and since D is a bi-dominating set then the number of edges is $m_1= 2|D|=2\gamma_{bi}^r(G)$. Additionally the graph G[V-D] should be containing as few edges as possible to be a graph with isolated vertices. Thus, the number of edges is $m_2 = \left\lceil\frac{|V-D|}{2}\right\rceil = \left\lceil\frac{n-\gamma_{bi}^r(G)}{2}\right\rceil$, so in this case $m=m_1+m_2$. Hence, $m = 2\gamma_{bi}^r(G) + \left\lceil\frac{n-\gamma_{bi}^r(G)}{2}\right\rceil$ .In general, $m \ge m_1+m_2$.

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Case 2. The same proof in Theorem 3.2 (case 2).

6. Complementary tree bi-dominating set

6.1. Definition

“The complementary tree bi-dominating set D is, a bi-dominating set such that, the induced subgraph <V-D> is a tree. The minimum cardinality of a complementary tree bi-dominating set, is denoted by γ_{bi}^{ct} (G) “.

6.2.Theorem

Let G(n,m) has complementary tree bi-dominating set γ_{bi}^{ct} (G), then

$$\gamma_{bi}^{ct}(G)+n-1\le m\le \frac{n^2-n}{2}$$
$$+\,(\gamma_{bi}^{ct}(G))^2-n\,\gamma_{bi}^{ct}(G)+2\,\gamma_{bi}^{ct}(G)$$

Proof.

Let D be a γ_{bi}^{ct}- set of G, so we prove the required by two cases that depend on the boundaries as follows:

Case 1. To prove the lower bound γ_{bi}^{ct} (G)+n-1≤m.

Depending on a definition of bi-dominating set the number of edges is m_1= 2 |D|= γ_{bi}^{ct}(G). Moreover, the induced subgraph G[V-D] should contain as few edges to be a tree graph so the number of these edges is m_2=|V-D|-1=(n- γ_{bi}^{ct}-1). So, in this case, m=m_1+m_2=2 γ_{bi}^{ct}+(n- γ_{bi}^{ct}-1). Hence,m≥ γ_{bi}^{ct}+n-1.

Case 2. The same proof in Theorem 3.2 (case 2).

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