AL-Bahir
Journal for Engineering and Pure Sciences
BJEPS

Volume 1 | Issue 2 Article 4

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ORIGINAL STUDY

Functions Approximation by Spectral Graph Wavelets

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Abstract

Approximating function by using Spectral Graph Wavelets is an interesting direction in approximation theory. We essential to well choosing the space of functions that are approximated by Spectral Graph Wavelets. L_p spaces of functions are fantastic choices to study It is more interesting to take the value $0<p<1$. In this paper, new formulas of Spectral Graph Wavelets were constructed and proved to get good rates of approximation. Fundamental properties of L_p Graph Wavelets transform s (L_p GWT) are studied, such as, inversion, scaling Limit and approximation Wavelets. Finally, existence of best approximation can be concluded here for Graph functions in terms of SGWT.

Keywords: Approximation, Spectral Graph, Wavelets, Lp

1. Introduction

In many papers, authors define many formulas of Spectral Graph continuous Wavelet Transform. For examples, continuous wavelet transform has been introduced in $L^2(R)$ by [1]. When a new formula is defined, it should have the appropriate properties to best fit the target function. Note that the graph we use in this paper is the simple undirected graph, which implies a symmetric adjacency matrix. Moreover we consider non-negative weights for the graph we define $L_p, p<1$ wavelet transform, for a given function defined on the vertices of a weighted graph, we need to define $\psi(sx)$. For this purpose, we need to go back to fourier transform to construct a basis for the spectral graph wavelet transform. Now, we begin defining Graph Fourier Transform,

Definition 1.1. Graph Fourier Transform
For a given function $f: V \to R$ on vertices of a graph , we define

$$(\mathcal{L}f)(m) = \sum_{m\sim n} a_{m,n}(f(m) - f(n)) \tag{1}$$

For all adjacent vertices m and n.
Beginning with the graph Laplacian $\mathcal{L} = D - A$ is a real symmetric matrix, has a set of eigenvectors, denote by χ_ℓ for $\ell = 0, \ldots, N-1$, with set of eigenvalues λ_ℓ

$$\mathcal{L}\chi_\ell = \lambda_\ell \chi_\ell \tag{2}$$

For any function $f \in R^N$ defined on the vertices of G, its graph Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\ell) = \sum_{n=1}^{N} \chi_\ell^*(n) f(n), \ell = 0, \ldots, N-1 \tag{3}$$

Where the inverse transform is

$$f(n) = \sum_{\ell=0}^{N-1} \hat{f}(\ell)\chi_\ell(n) \ , n = 1, \ldots, N \tag{4}$$

2. L_p spectral graph wavelet transform

L_p SGWT can be defined in term of the choice of a kernal function $g: R^+ \to R^+$ that satisfies

$$g(0) = 0 \quad and \ \lim_{x\to\infty} g(x) = 0 \ , \tag{5}$$

In particular

$$\widehat{T_g f}(\ell) = g(\lambda_\ell)\hat{f}(\ell) \quad \textbf{where } T_g = g(\ell) \tag{6}$$

With inverse fourier transform,

$$T_g f(n) = \sum_{\ell=0}^{N} g(\lambda_\ell)\hat{f}(\ell)\chi_\ell(n) \tag{7}$$

Wavelet at scale t is

Received 4 September 2022; revised 3 November 2022; accepted 7 November 2022.
Available online 05 December 2022
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https://doi.org/10.55810/2313-0083.1009

$$T_g{}^t = g(t\mathcal{L}) \tag{8}$$

Applying the above operators at each vertex gives

$$\psi_{t,n} = T_g^t\,\delta_n, \tag{9}$$

Where δ_n is the location of ψ w.r.t vertex n

For a given mother function $\psi_{t,n}$, with scale t and location n, we define L_p SGWT in L_p space, with $p<1$, by a kernel g as follow

$$\psi_{t,n}(m) = \sum_{\ell=0}^{N} \left|g(t\lambda_\ell)\chi_\ell^*(n)\chi_\ell\,(m)\,\right|^p \tag{10}$$

where $n, m = 1, \ldots, N$

Also, set

$$w_f(t,n) = \left(T_g^t f\right)(n) = \sum_{\ell=0}^{N} \left|g(\lambda_\ell)\widehat{f}(\ell)\chi_\ell(n)\,\right|^p \tag{11}$$

to be the wavelet coefficient.

3. Properties of L_p SGWT

In this section, some properties are studied for L_p SGWT. We first estimate an inverse formula for the transform under admissible condition.

Theorem 3.1.
Suppose that L_p SGWT with kernel g satisfies the admissibility condition

$$\int_0^\infty \frac{|g(x)|^{2p}}{x}dx = C_g < \infty$$

and $g(0)=0$, then

$$\frac{1}{C_g(p)}\sum_{n=1}^{N-1}\int_0^\infty w_f(t,n)\psi_{t,n}(m)\,\frac{dt}{t} \le f(m) - \widehat{f}(0)\chi_0(m)$$

Proof:
By (10) , (11) and admissibility condition, we have

$$\frac{1}{C_g}\sum_{n=1}^{N-1}\int_0^\infty w_f(t,n)\psi_{t,n}(m)\,\frac{dt}{t}$$

$$= \frac{1}{C_g}\int_0^\infty \frac{1}{t}\sum_n$$

$$\left(\sum_\ell |g(t\lambda_\ell)\widehat{f}(\ell)\chi_\ell(n)|^p \sum_{\ell'} |g(t\lambda_{\ell'})\chi_{\ell'}^*(n)\chi_{\ell'}\,(m)|^p\right)dt$$

$$\le \frac{C(p)}{C_g}\int_0^\infty \frac{1}{t}$$

$$\left(\sum_{\ell,\ell'=0} |g(t\lambda_{\ell'})g(t\lambda_\ell)\widehat{f}(\ell)\chi_{\ell'}(m)|^p \sum_n \chi_{\ell'}^*(n)\chi_\ell\,(n)\right)dt$$

Since of orthonormality of the χ_ℓ, we get

$$\frac{1}{C_g}\sum_{n=1}^{N-1}\int_0^\infty w_f(t,n)\psi_{t,n}(m)\,\frac{dt}{t}$$

$$\le \frac{C(p)}{C_g}$$

$$\left\{\sum_{\ell=1}^{N-1}\left(\int_0^\infty \frac{|g^2(t\lambda_\ell)|^p}{t}dt\right)\widehat{f}(\ell)\chi_\ell(m) + \widehat{f}(0)\chi_0(m)\right\}$$

$$= f(m) - \widehat{f}(0)\chi_0(m)$$

Since $g(0)=0$, and admissibility condition on $(t\lambda_\ell)$.

Lemma 3.2.
Let G be a weighted graph, $\mathcal{L}$ the graph Laplacian (normalized or non-normalized) and $s>0$ an integer. For any two vertices m and n, if $d_G(m,n)>s$ then $(\mathcal{L}^s)_{m,n} = 0$.
If two kernels g and $\tilde{g}$ are close to each other in some sense, then the resulting wavelets should be close to each other. More precisely, we have the following theorem

Theorem 3.3.
Let $\psi_{t,n} = T_g^t\delta_n$ and $\tilde{\psi}_{t,n} = T_{\tilde{g}}{}^t\delta_n$ be the wavelets at scale t generated by the kernels g and $\tilde{g}$. If $g(t\lambda) - \tilde{g}\,(t\lambda)_p \le C(t)$ for all $\lambda\in[0,\lambda_{N-1}]$, **then** $\psi_{t,n}(m) - \tilde{\psi}_{t,n}(m)_p \le C(t,p)$ for each vertex m.
Proof: First recall (10)

$$\psi_{t,n}(m) = \sum_\ell \left|\chi_\ell\,(m)g(t\lambda_\ell)\chi_\ell^*(n)\,\right|^p$$

Thus,

$$\|\psi_{t,n}(m) - \tilde{\psi}_{t,n}\,(m)\|_p^p$$

$$= \left\|\sum_\ell |\chi_\ell(m)g(t\lambda_\ell)\chi_\ell^*(n)|^p - \sum_\ell |\chi_\ell(m)\tilde{g}(t\lambda_l)\chi_l^*(n)|^p\right\|_p^p$$

$$= \sum_{\ell} \left| \left| \sum_{\ell} \chi_{\ell}(m)(g(t\lambda_{\ell}) - \tilde{g}(t\lambda))\chi_{\ell}^{*}(n) \right|^{p} \right|^{p}$$

$$\leq C(t,p) \sum_{\ell} |\chi_{\ell}(m)\chi_{\ell}(n)^{*}|^{p^2}$$

$$\leq C(t,p) \sum_{\ell} |\chi_{\ell}(m)\chi_{\ell}(n)^{*}|^{p}$$

$$\leq C(t,p)$$

The following theorem we show that any two kernels of conditions of Theorem 3.3, such kernels can be approximated by a single monomial for small scales.

Theorem 3.4.
Let $g \in L_p^{K+1}$, g be $K+1$ times continuously differentiable, satisfying

$$g(0) = 0 \ , g^{(r)}(0) = 0 \, for\ all \quad r < K \, , and\, g^{(K)}(0) = C \neq 0.$$

Assume that there is some $t_1 > 0$ such that $g(t\lambda)_p \leq B$ for all $\lambda \in [0,\ t_1\lambda_{N-1}]$. Then, for $\tilde{g}(t\lambda) = (C/K!)(t\lambda)^K$ we have

$$C(t) = g(t\lambda) - \tilde{g}(t\lambda)_p \leq B\frac{t^{K+1}\lambda_{N-1}^{K+1}}{(K+1)!}$$

for all $t < t_1$.
Proof:
By hypothesis of first $K-1$ derivatives of g we conclude Taylor expansion at $t\lambda$ as follow,

$$g(t\lambda) = C\frac{(t\lambda)^K}{K!} + g^{(K+1)}(x^*)\frac{(t\lambda)^{K+1}}{(K+1)!}$$

For some $x^* \in [0, t\lambda]$. Now fix $t < t_1$. For any $\lambda \in [0, \lambda_{N-1}]$, we have $t\lambda < t_1\lambda_{N-1}$, and so the corresponding $x^* \in [0, t_1\lambda_{N-1}]$, we get

$$g(t\lambda) - \tilde{g}(t\lambda)_p \leq C\frac{(t\lambda)^K}{K!} + g^{(K+1)}(x^*)\frac{(t\lambda)^{K+1}}{(K+1)!} - \frac{C}{K!}(t\lambda)^K(t\lambda)_p$$

$$B\frac{t^{K+1}\lambda^{K+1}}{(K+1)!} \leq B\frac{t^{K+1}\lambda_{N-1}^{K+1}}{(K+1)!}$$

For all $\lambda \in [0, \lambda_{N-1}]$
For any kernel g satisfies theorem 3.3 , we get the following main result about localization,

Theorem 3.5.
Let G be a weighted graph with Laplacian $\mathcal{L}$. Let m and n be vertices of G such that $d_G(m,n) > K$. Then

$$\frac{\psi_{t,n}(m)}{\|\psi_{t,n}\|_p} \leq Dt$$

For some t, D s.t.$t < \min(t_1, t_2)$.
Proof:
Set $\tilde{g}(\lambda) = \frac{g^{(K)}(0)}{K!}\lambda^K$ and $\tilde{\psi}_{t,n} = T_{\tilde{g}}{}^{t}\delta_n$. We have

$$\tilde{\psi}_{t,n}(m) = \frac{g^{(K)}(0)}{K!}t^K \sum |\tilde{g}(t\lambda_{\ell})\chi_{\ell}^{*}(n)\chi_{\ell}(m)|^p = 0$$

by Lemma (3.2), as $d_G(m,\ n) > K$. By Theorems (3.3).and (3.4).we have

$$\psi_{t,n}(m) - \tilde{\psi}_{t,n}(m) = \psi_{t,n}(m) \leq t^{K+1}\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B$$

Also, we have

$$\|\tilde{\psi}_{t,n}\|_p = t^K\frac{g^{(K)}(0)}{K!}\|L^K\delta_n\|_p$$

and we have from Theorem (3.4)

$$\|\psi_{t,n} - \tilde{\psi}_{t,n}\|_p \leq t^{K+1}\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B$$

By quasi-triangle inequality we get

$$\|\psi_{t,n}\|_p \geq C\left(\|\psi_{t,n}\|_p - \|\psi_{t,n} - \tilde{\psi}_{t,n}\|_p\right)$$

$$\geq t^K\left(\frac{g^{(K)}(0)}{K!}\|L^K\delta_n\|_p - t\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B\right)$$

On the other hand,

$$\|\psi_{t,n}\|_p \geq C\|\psi_{t,n} - \tilde{\psi}_{t,n}\|_p$$

Hence,

$$\frac{\psi_{t,n}(m)}{\|\psi_{t,n}\|_p} \leq C\frac{t\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B}{\frac{g^{(K)}(0)}{K!}\|L^K\delta_n\|_p - t\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B}$$

An elementary calculation shows

$$\frac{\frac{\lambda_{N-1}^{K+1}}{(K+1)!}t}{\frac{g^{(K)}(0)}{K!}\|L^K\delta_n\|_p - t\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B} \leq \frac{2\frac{\lambda_{N-1}^{K+1}}{(K+1)!}}{\frac{g^{(K)}(0)}{K!}\|L^K\delta_n\|_p}t$$

If

$$t \le C\frac{\frac{g^{(K)}(0)}{K!}\left\|L^K\delta_n\right\|_p}{2\frac{\lambda_{N-1}^{K+1}}{(K+1)!}B}$$

The proof completes with

$$D = C\frac{2\frac{\lambda_{N-1}^{K+1}}{(K+1)!}BK!}{g^{(K)}(0)\left\|L^K\delta_n\right\|_p} \text{ and } t_2 = \frac{g^{(K)}(0)\left\|L^K\delta_n\right\|_p(K+1)}{2\lambda_{N-1}^{K+1}B}$$

$$D = C\frac{2\frac{\lambda_{N-1}^{K+1}}{(K+1)!}BK!}{g^{(K)}(0)\left\|\mathcal{L}^K\delta_n\right\|_p} \text{ and } t_2 = \frac{g^{(K)}(0)\left\|\mathcal{L}^K\delta_n\right\|_p(K+1)}{2\lambda_{N-1}^{K+1}B}$$

Now, we study frames of wavelete (11) with $S_f(n)$, with kernel h and bounds A, B

Theorem 3.6.
Given a set of scales $\{t_j\}_{j=1}^J$, the set $F = \{\phi_n\}_{n=1}^N \cup \{\psi_{t\,j\,,n}\}_{j=1}^J{}_{n=1}^N$ forms a frame

$$A\|f\|_p \le \left\|W_f(t,n)\right\|_p\left\|S_f(n)\right\|_p \le B\|f\|_p$$

for any scaling function coefficients $S_f(n)$, with kernel h and bounds A, B given by

$$A = \min_{\lambda\in[0,\lambda_{N-1}]} G(\lambda), \text{and}$$

$$B = \max_{\lambda\in[0,\lambda_{N-1}]} G(\lambda),$$

where

$$G(\lambda) = |h(\lambda_\ell)|^p + \sum_j\left|g(t_j\,\lambda)\right|^p$$

Proof:
Let $f\in L_p(R^N)$ be any vertex function then by (11)

$$\left\|W_f(t,n)\right\|_p^p = \sum_{n=1}^N\left|W_f(t,n)\right|^p$$

$$= \sum_{n=1}^N\sum_{\ell=0}^{N-1}\left|g(t\lambda_\ell)\chi_\ell(n)\hat{f}(\ell)\right|^p$$

$$\le C\sum_{\ell=0}^{N-1}|g(t\lambda_\ell)|^p\left|\hat{f}(\ell)\right|^p$$

Similarly, for any scaling function coefficients S_f, generated by h we have

$$\left\|S_f(n)\right\|_p^p = \sum_{n=1}^N\left|S_f(t,n)\right|^p$$

$$= \sum_{n=1}^N\left|h(\lambda_\ell)\chi_\ell(n)\hat{f}(\ell)\right|^p$$

$$\le C\sum_{\ell=0}^{N-1}|h(\lambda_\ell)|^p\left|\hat{f}(\ell)\right|^p$$

Combining terms together, we get

$$A\sum_{\ell=0}^{N-1}\left|\hat{f}(\ell)\right|^p \le C\sum_{\ell=0}^{N-1}\left(|h(\lambda_\ell)|^p + \sum_{j=1}^J\left|g(t_j\lambda_\ell)\right|^p\right)$$

$$\left|\hat{f}(\ell)\right|^p = C\sum_{\ell=0}^{N-1}G(\lambda_\ell)\left|\hat{f}(\lambda_\ell)\right|^p \le B\sum_{\ell=0}^{N-1}\left|\hat{f}(\ell)\right|^p$$

4. Approximation by L_p SGWT, application in approximation theory

In this section, we benefit from the construction of L_p SGWT to find a best approximation of vertex functions out of space of L_p SGWT. As the approximation began with polynomials, we approximate the kernel g with a polynomial p in a way that leads to the approximation of their corresponding wavelet operators in L_p spaces. The following theorem shows that the degree of approximation between the two wavelets is at most the degree of approximation between their generators.

Theorem 4.1. Polynomial Approximation:
For any upper bound of $\mathcal{L}$, that is , $\lambda_{max} \ge \lambda_{N-1}$ and fixed $t > 0$, *let* P_n be a polynomial best approximant of $g(tx)$ with degree of approximation $E_n(g)_p = B$ for the space of polynomials of degree at most n. Then the approximate wavelet coefficients $\tilde{w}_f(t,n) = (P(L)f)_n$ satisfies

$$\left\|w_f(t,n) - \tilde{w}_f(t,n)\right\|_p \le C\|f\|_p$$

Proof.
By using (11) we get

$$\left\|w_f(t,n) - \tilde{w}_f(t,n)\right\|_p^p = \left\|\sum_l\left|g(t\lambda_l)\hat{f}(l)\chi_l(n)\right|^p - \sum_l\left|P(t\lambda_l)\hat{f}(l)\chi_l(n)\right|^p\right\|_p^p$$

$$= \left\|\sum_l\left|(g(t\lambda_l) - P(\lambda_l))\hat{f}(l)\chi_l(n)\right|^p\right\|_p^p$$

$$\leq C(p)\sum \left\| \left| (g(t\lambda_l) - P(\lambda_l))\hat{f}(l)\chi_l(n) \right|^p \right\|_p^p$$

$$\leq C(p)\mathbf{B}^p \|f\|_p^p$$

The last step follows from the orthonormality of the χ_ℓ

5. Existence of best approximation

In order to approximate vertex functions by wavelets of spectral graph type, here is a theorem that confirm the possibility of existence of best approximation of vertex functions out of the space of L_p SGWT, namely $\mathbf{\Omega}$.

Theorem 5.1.
For any $f \in L_p(R^N)$, then there exist $w_f \in \mathbf{\Omega}$ of the from (11) that is generated by a graph G and a kernel g s.t

$$\|f - w_f(t,n)\|_p \leq \varepsilon$$

Proof:
Set $\tilde{w}_f \in \mathbf{\Omega}$, with a polynomial P, that satisfies Theorem (4.1), so that

$$\|w_f - \tilde{w}_f\|_p < C(p)\mathbf{B}\|f\|_p$$

Also, set

$$\|f - \tilde{w}_f(t,n)\|_p \leq \frac{\varepsilon}{2}$$

which is true by WEIERSTRASS Theorem, then by Quasi-Triangle Inequality, we get the desired result

$$\|f - w_f(t,n)\|_p \leq C\left[\|f - \tilde{w}_f(t,n)\|_p + \|w_f - \tilde{w}_f\|_p\right]$$

$$\leq C\left[\frac{\varepsilon}{2} + B\|f\|_p\right] \leq \varepsilon$$

By choosing the constant B, that satisfies

$$B \leq \frac{\varepsilon}{2\|f\|} < \varepsilon$$

Uncited references

[2]; [3]; [4]; [5]; [6]; [7]; [8]; [9].

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