The Adomian Decomposition Techniques to Solve The Second Kind Inhomogeneous Fredholm Integral Equation

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الخلاصة

من البديهي القول ان معادلات فريدهولم التكاملية لها تطبيقات كثيرة في مختلف العلوم لذلك أثارت هذه المعادلات انتباه الكثير من العلماء والباحثين في ميدان علم الرياضيات.

في هذا البحث سنقدم طريقة جديدة وموثوقة لحل معادلة فريدهولم التكاملية الغير متجانسة من النوع الثاني وتسمى هذه الطريقة (Adomian Decomposition Method) وسيتم شرح آلية عمل هذه الطريقة بصيغتها الاصلية وصيغتها المعدلة في الحل من خلال مجموعة متنوعة من الامثلة.

الكلمات المفتاحية

معادلات فريدهولم التكاملية، طريقة تحليل أدوميان، طريقة التحليل المعدلة.

Abstract

It is arbitrary that, Fredholm Integral Equations have got many variable applications in different scientific domains. So, such brand of equation has attracted many scientists' and researchers' attention in mathematics science.

Then, in this research, the researcher will produce new authentic method to solve the second kind inhomogeneous Fredholm Integral Equation whose name is Adomian Decomposition Method. The procedural steps of this method will be produced and explained with its original as well as its modified modes in terms of solution process through multiple kinds of examples.

Keywords

Fredholm Integral Equations, Adomian Decomposition Method, Modified Decomposition Method.

1. Introduction

It is evident that, Fredholm Integral Equation exists in many fields of scientific applications. For example, Newton's law, stating that, the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations. These differential equations can be transformed to the equivalent integral equations of Fredholm types.In mathematic field of science, Fredholm came to the prominence as a Swedish scientist who established the brand of (Integral Equation) in the applications mathematical domain.

In this sense, this scientist was enabled to convert the boundary value problems into integral Equations which coined lately by his name. In the last years of previous century, this mode of equation (Fredholm Integral Equations) was employed effectively in variable fields of physical and chemical problems, accodingly. Consequently, the scientific contribution of such Equation engaged many researchers to study it deeply.

The researcher will produce the method to solve these problems which is termed Adomian Decomposition Method. This method was discovered by the scientist Adomian in 1990.

It is well known that, Adomian Decomposition Method has been used by many scientists and engineers in order to solve highly nonlinear integral equations which can not be solved by other methods. But this method has been used globally to solve the second kind from Volterra integral equations and Fredholm integral equations and it is impossible to be used regardimg to solve the first kind from these two equations.

Haifa Ali and Fawzi Abdelwahid have been published a scientific article under a title (Modified Adomian Techniques Applied to Non-Linear Volterra Integral Equations) in (Open Journal of Applied Sciences) in 2013 to explain Adomian Decomposition Method to solve Volterra integral equations. On the other hand, I explained the mechanism and procedures this method regarding to deal with Fredholm integral equations.

2. Basic Definitions

It is understood that, some definitions have been selected in order to help us to understand integral equations and their types generally as well as to understand Fredholm integral equations, specificly.

Definition 1. [1]An integral equation is an equation that involves the unknown function $u(x)$ that appears inside of an integral sign. The most standard type of an integral equation in $u(x)$ is of the form

$$u(x)=f(x)+\lambda\int_{g(x)}^{h(x)}K(x,t)u(t)dt \quad (2.1)$$

Definition 2. [2] If the function $f(x)=0$ in equation (2.1), then equation (2.1) is called homogeneous. Otherwise it is called inhomogeneous.

Definition 3. [3] If the limits of the integral in equation (2.1) are fixed so, is called a Fredholm integral equation.

These types of equations are classified into two types, the general form of Fredholm integral equation of the first kind is

$$f(x)=\lambda\int_{a}^{b}K(x,t)u(t)dt \quad (2.2)$$

where, the unknown function $u(x)$ appears inside the integral sign.

The second kind given by

$$u(x)=f(x)+\lambda\int_{a}^{b}K(x,t)u(t)dt \quad (2.3)$$

where, the unknown function $u(x)$ appears inside and outside the integral sign.

3. The Adomian decomposition method

Polyanin and Manzhirov conclude that, this method arises to work for linear, nonlinear integral equations, differential equations and integro-differential equations.

We shall explain the technique of this method by expressing $u(x)$ in equation (2.3) in the form of a series

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \quad (3.1)$$

or equivalently,

$$u(x)=u_0(x)+u_1(x)+u_2(x)+\dots \quad (3.2)$$

Where the elements of, $u_n(x)$ $n\geq 0$ will be identified redundantly. The mode of the Adomian decomposition method links itself with finding the elements $u_0,u_1,u_2,\dots$. singly. The setting of these elements or components can be solved in a fair easy way through a redundant relation which includes normally simple integrals which in turn can be evaluated, simply [4].

However, Collians maintains that, firstly, we set the value of $u_0(x)$ as the term outside the integral sign of equation (2.3)

$$u_0(x)=f(x)$$

To found the redundant relation, we substitute (3.1) into the Fredholm integral equation (2.3) to obtain [5]:

$$\sum_{n=0}^{\infty}u_n(x)=f(x)+\lambda\int_{a}^{b}K(x,t)\left(\sum_{n=0}^{\infty}u_n(x)\right)dt$$

or equivalently,

$$u_0(x)+u_1(x)+u_2(x)+\dots=f(x)+\lambda\int_{a}^{b}K(x,t)[u_0(t)+u_1(t)+\dots]dt$$

Usman and Zubair state that, we can get the value of the components $u_0(x)$ $u_1(x)$ $u_2(x)$ ..., $u_n(x)$ ... of the unknown function $u(x)$ as follows:

$$u_0(x)=f(x)$$

$$u_1(x)=\lambda\int_{a}^{b}K(x,t)u_0(t)dt$$

$$u_2(x)=\lambda\int_a^b K(x,t)u_1(t)dt$$

$$u_3(x)=\lambda\int_a^b K(x,t)u_2(t)dt$$

$$u_{n+1}(x)=\lambda\int_a^b K(x,t)u_n(t)dt\ ,\quad n\geq 0.\quad (3.3)$$

Then, we can get the solution $u(x)$ by

$$u(x)=u_0(x)+u_1(x)+u_2(x)+u_3(x)+...$$

that converges to a closed form solution [6].

3.1. Some Applications

I have chosen these group of examples which are employed to explain the Adomian decomposition method clearly and comprehensively as compared with other modes of examples.

Example (1):

To Solve the following Fredholm integral equation

$$u(x)=e^x-x+x\int_0^1 t\,u(t)dt\ .\quad (3.4)$$

Here, Wazwaz sees that, the Adomian decomposition method assumes that, the solution $u(x)$ has a series form given in (2.1). Substituting the decomposition series (3.1) into both sides of (3.4) gives [7]

$$\sum_{n=0}^{\infty}u_n(x)=e^x-x+x\int_0^1 t\sum_{n=0}^{\infty}u_n dt\ .\quad (3.5)$$

or equivalently,

$$u_0(x)+u_1(x)+u_2(x)+...=e^x-x+x\int_0^1 t[u_0(t)+u_1(t)+...]dt$$

It is understood that, Hosseini explains that, we identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x)=e^x-x,$$

$$u_{k+1}(x)=x\int_0^1 t\,u_k(t)dt\ ,k\geq 0.$$

Hence,

$$u_0(x)=e^x-x,$$

$$u_1(x)=x\int_0^1 t\,u_0(t)dt\ =x\int_0^1 t(e^t-t)dt\ =\frac{2}{3}x,$$

$$u_2(x)=x\int_0^1 t\,u_1(t)dt\ =x\int_0^1\frac{2}{3}t^2dt\ =\frac{2}{9}x,$$

$$u_3(x)=x\int_0^1 t\,u_2(t)dt\ =x\int_0^1\frac{2}{9}t^2dt\ =\frac{2}{27}x,$$

$$u_4(x)=x\int_0^1 t\,u_3(t)dt\ =x\int_0^1\frac{2}{27}t^2dt\ =\frac{2}{281}x,$$

and so on [8].

Using (3.1) gives the series solution

$$u(x)=e^x-x+\frac{2}{3}x\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....\right).\quad (3.6)$$

It is clearly that, the infinite geometric series at the right side has $a_1=1$, and the ratio $r=\frac{1}{3}$. The sum of the infinite series is therefore given by

$$S=\frac{1}{1-\frac{1}{3}}=\frac{3}{2},\quad (3.7)$$

substituting (3.7) into (3.6) gives the exact solution

$$u(x)=e^x.$$

Example (2):

To Solve the following Fredholm integral equation

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} u(t)dt \,. \quad (3.8)$$

It is knowledgeable that, Porter and Stirling maintain that, by substitute the decomposition series (3.1) into both sides of (3.8) we find

$$\sum_{n=0}^{\infty} u_n(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} u_n dt \,. \quad (3.9)$$

or equivalently,

$$u_0(x) + u_1(x) + u_2(x) + \ldots = \sin x - x + x\int_0^{\frac{\pi}{2}} [u_0(t) + u_1(t) + \ldots]dt$$

Then, we set all terms that are not included under the integral sign as the zeroth component

$$u_0(x) = \sin x - x,$$
$$u_{k+1}(x) = x\int_0^{\frac{\pi}{2}} u_k(t)dt \,, k \geq 0.$$

Hence, we obtain that,

$$u_0(x) = \sin x - x,$$
$$u_1(x) = x\int_0^{\frac{\pi}{2}} u_0(t)dt = x\int_0^{\frac{\pi}{2}} (\sin x - x)dt = x - \frac{\pi^2}{8}x,$$
$$u_2(x) = x\int_0^{\frac{\pi}{2}} u_1(t)dt = x\int_0^{\frac{\pi}{2}} (x - \frac{\pi^2}{8}x)dt = \frac{\pi^2}{8}x - \frac{\pi^4}{64}x,$$
$$u_3(x) = x\int_0^{\frac{\pi}{2}} u_2(t)dt = x\int_0^{\frac{\pi}{2}} (\frac{\pi^2}{8}x - \frac{\pi^4}{64}x)dt = \frac{\pi^4}{64}x - \frac{\pi^6}{512}x,$$
$$u_4(x) = x\int_0^{\frac{\pi}{2}} u_3(t)dt = x\int_0^{\frac{\pi}{2}} (\frac{\pi^4}{64}x - \frac{\pi^6}{512}x)dt = \frac{\pi^6}{512}x - \frac{\pi^8}{4096}x,$$

and so on [9].

Hence, by using (3.1) gives the series solution

$$u(x) = \sin x - x + \left(1 - \frac{\pi^2}{8}\right)x + \left(\frac{\pi^2}{8} - \frac{\pi^4}{64}\right)x$$
$$+ \left(\frac{\pi^4}{64} - \frac{\pi^6}{512}\right)x + \left(\frac{\pi^6}{512} - \frac{\pi^8}{4096}\right)x + \ldots \quad (3.10)$$

By canceling the identical terms with opposite signs in (3.10), we will obtain the exact solution

$$u(x) = \sin x.$$

Example (3):

To Solve the following Fredholm integral equation

$$u(x) = x + e^x - \frac{4}{3} + \int_0^1 t\,u(t)dt \,. \quad (3.11)$$

Malrknejad and Mahmoudi state that, substituting (3.1) into both sides of (3.11) gives

$$\sum_{n=0}^{\infty} u_n(x) = x + e^x - \frac{4}{3} + \int_0^1 t\sum_{n=0}^{\infty} u_n dt \,.$$

or equivalent,

$$u_0(x) + u_1(x) + u_2(x) + \ldots = x + e^x - \frac{4}{3} + \int_0^1 t[u_0(t) + u_1(t) + \ldots]dt$$

Then, we set

$$u_0(x) = x + e^x - \frac{4}{3},$$
$$u_{k+1}(x) = x\int_0^1 t\,u_k(t)dt \,, k \geq 0.$$

Hence,

$$u_0(x) = x + e^x - \frac{4}{3},$$
$$u_1(x) = \int_0^1 t\,u_0(t)dt = \int_0^1 t(x + e^x - \frac{4}{3})dt = \frac{2}{3},$$
$$u_2(x) = \int_0^1 t\,u_1(t)dt = \int_0^1 t(\frac{2}{3})dt = \frac{1}{3},$$

$$u_3(x)=\int_0^1 u_2(t)dt=\int_0^1 t(\frac{1}{3})dt=\frac{1}{6},$$

$$u_4(x)=\int_0^1 u_3(t)dt=\int_0^1 t(\frac{1}{6})dt=\frac{1}{12},$$

and so on [10].

By using (3.1) we obtain,

$$u(x)=x+e^x-\frac{4}{3}+\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....\right).$$

(3.12)

It is clear that, the infinite geometric series has $a_1=1$, and the ratio $r=\frac{1}{2}$. Therefore, the sum of the infinite series is given by

$$S=\frac{1}{1-\frac{1}{2}}=2.$$

The series solution (3.12) converges to the closed form solution

$$u(x)=x+e^x$$

4. The modified decomposition method

If the function $f(x)$ consists of a mixture of two or more of trigonometric functions, hyperbolic functions, polynomials, and others, the evaluation of the components $u_j, j\geq 0$ requires long time and difficult work.

Henceforth, Ali and Abdelwahid accept that, we can set the function $f(x)$ as the sum of two partial functions, such as $f_1(x)$ and $f_2(x)$ In other words, we can set,

$$f(x)=f_1(x)+f_2(x)$$

we identify the component $u_0(x)$ by one part of $f(x)$ to minimize the size of calculations. We will use the other part of $f(x)$ to find the value of the component $u_1(x)$ In other words, the modified decomposition method introduces the modified recurrence relation

$$u_0(x)=f_1(x)$$

$$u_1(x)=f_2+\lambda\int_a^b K(x,t)u_0(t)dt$$

$$u_{k+1}(x)=\lambda\int_a^b K(x,t)u_k(t)dt\,. \quad (4.1)$$

We can get the exact solution $u(x)$ by correct selection of the functions $f_1(x)$ and $f_2(x)$ and by using very few iterations, and sometimes by evaluating only two or three components. The success of this method depends only on the correct choice of $f_1(x)$ and $f_2(x)$ and this can be made through experience only. A rule that may help for the correct choice of $f_1(x)$ and $f_2(x)$ could not be found until now.

We can not use this method if $f(x)$ consists of one term only, in this case the standard decomposition method can be used [11].

4.1. Some Applications

Example (1):

To Solve the Fredholm integral equation by using the modified decomposition method:

$$u(x)=e^x-1+\int_0^1 u(t)dt\,. \quad (4.2)$$

Firstly, we will solve this equation by the “standard decomposition method” as compared with modified decomposition method.

Substituting the decomposition series (3.1) into both sides of (4.2) gives

$$\sum_{n=0}^{\infty} u_n(x) = e^x - 1 + \int_0^1 t \sum_{n=0}^{\infty} u_n dt \; . \quad (4.3)$$

or equivalently,

$$u_0(x) + u_1(x) + u_2(x) + \ldots = e^x - 1 + \int_0^1 t[u_0(t) + u_1(t) + \ldots]dt$$

we identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x) = e^x - 1,\; u_{k+1}(x) = \int_0^1 t u_k(t) dt \;, k \ge 0.$$

Hence,

$$u_0(x) = e^x - 1,$$

$$u_1(x) = \int_0^1 t u_0(t) dt = \int_0^1 t(e^t - 1) dt$$

$$u_1(x) = \left[e^t(t-1) - \frac{t^2}{2} \right]_0^1$$

$$u_1(x) = \left[0 - \frac{1}{2} \right] - [-1] = \frac{1}{2}$$

$$u_2(x) = \int_0^1 t u_1(t) dt = \int_0^1 \frac{1}{2} t dt$$

$$u_2(x) = \left[\frac{t^2}{4} \right]_0^1 = \frac{1}{4}$$

$$u_3(x) = \int_0^1 t u_2(t) dt = \int_0^1 \frac{1}{4} t dt$$

$$u_3(x) = \left[\frac{t^2}{8} \right]_0^1 = \frac{1}{8}$$

$$u_4(x) = \int_0^1 t u_3(t) dt = \int_0^1 \frac{1}{8} t dt$$

$$u_4(x) = \left[\frac{t^2}{8} \right]_0^1 = \frac{1}{8}$$

and so on [10].

By using (3.1) we obtain,

$$u(x) = e^x - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right). \quad (4.4)$$

It is clear that, the infinite geometric series has $a_1 = 1$, and the ratio $r = \frac{1}{2}$. Therefore, the sum of the infinite series is given by

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

The series solution (4.4) converges to the closed form solution

$$u(x) = e^x$$

To solve equation (4.2) by using the modified decomposition method, Wazwaz sees that, first, we set

$$f(x) = e^x - 1$$

hence,

$$f_1(x) = e^x \;, \quad f_2 = -1$$

By using (4.1) we obtain,

$$u_0 = f_1(x) = e^x$$

$$u_1(x) = -1 + \int_0^1 t u_0(t) dt \;,$$

$$u_1(x) = -1 + \int_0^1 t e^t dt \;,$$

$$u_1(x) = -1 + \left[e^t(t-1) \right]_0^1$$

$$u_1(x) = -1 + \left[e^1(0) - e^0(0-1) \right]$$

$$u_1(x) = -1 + [0 - (-1)]$$

$$u_1(x) = -1 + 1$$

$$u_1(x) = 0$$

It is clearly that, each component of

$u_k, k \geq 1$

is zero. So the exact solution given by [12]

$u(x) = e^x$.

Obviously, we got the solution for the equation (4.2) in a standard Adomian method after we got the value of four components. While, we were able to get the solution after we got the value of two components when we used the modified Adomian decomposition method. This mode of solution " modified Adomian decomposition method" has been employed to reduce the mathematical calculations as compared with the standard technique.

Example (2):

To Solve the Fredholm integral equation by using the modified decomposition method:

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^x) + \int_0^1 t\,u(t)\,dt .$$

Almazmumy and Hendi conclude that, first, we set,

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^x)$$

hence,

$$f_1(x) = 3x + e^{4x}, \quad f_2 = -\frac{1}{16}(17 + 3e^x)$$

By using (4.1) we obtain,

$$u_0 = f_1(x) = 3x + e^{4x},$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4t}) + \int_0^1 t\,u_0(t)\,dt ,$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4t}) + \int_0^1 t\left(3t + e^{4t}\right)dt ,$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4t}) + \int_0^1 \left(3t^2 + t e^{4t}\right)dt ,$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4t}) + \left[\frac{3t^3}{3} + \frac{e^{4t}}{16}(4t-1)\right]_0^1$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4}) + \left[1 + \frac{e^4}{16}(3)\right] - \left[-\frac{1}{16}\right]$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^{4}) + \frac{3e^4}{16} + \frac{17}{16}$$

$$u_1(x) = 0$$

It is clearly that, each component of

$u_k, k \geq 1$

is zero. So the exact solution given by [13]

$u(x) = 3x + e^{4x}$.

Example (3):

To Solve the Fredholm integral equation by using the modified decomposition method:

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} t\,u(t)\,dt .$$

Davies states that, first, we set

$f(x) = \sin x - x$

hence,

$f_1(x) = \sin x, \quad f_2 = -x$

By using (4.1) we obtain,

$$u_0 = f_1(x) = \sin x$$

$$u_1(x) = -t + t\int_0^1 t\,u_0(t)\,dt .$$

$$u_1(x) = -t + t\int_0^1 t \sin t\,dt ,$$

$$u_1(x) = -t + t\left[t \sin t - t \cos t\right]_0^{\frac{\pi}{2}}$$

$$u_1(x) = -t + t\left[1 - \frac{\pi}{2}(0)\right] - [0 - 0(1)]$$

$$u_1(x) = -t + t[1 - 0]$$

$$u_1(x) = -t + t$$

$$u_1(x) = 0$$

It is clearly that, each component of $u_k, k \geq 1$ is zero. So the exact solution given by [14] $u(x) = \sin x$.

5. Conclusion

It is clear that, the integral Equations have got many applications in the fields of sciences and to help us understanding the natural phenomena in terms of design for instance. In these papers, we produce and explain new authentic and dependable method which is called (Adomian Decomposition Method) in order to solve (Fredholm integral Equations).

The researcher produced and explained accordingly, the procedural mechanism of this method in terms of its original mode as well as its modified one through variable sets of examples.

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Convergence in Possibility Measure

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الخلاصة

قياس الامكانية هي دالة مجموعة معرفة من الحقل المتكامل $\mathfrak{I}$ الى الفترة المغلقة $[0,1]$ ، والمتغيرات لقياس الامكانية هي دالة قابلة للقياس من فضاء قياس الامكانية الى مجموعة الاعداد الحقيقية . في هذا البحث نسمي قياس الامكانية بقياس ρ ومتغيرات الامكانية بمتغيرات ρ .

نناقش في هذا البحث انواع من التقارب في هذا النوع من القياس وندرس العلاقة بينهم ونبرهن بعض المبرهنات المهمة.

الكلمات المفتاحية

الحقل المتكامل σ،الفضاء القابل للقياس، قياس ρ ، التقارب، متغيرات ρ .

Abstract

possibility measure as a set function $\rho : \Im \to [0,1]$ where $\Im$ is σ -field. Possibility variable is measurable function of possibility space to set of real number, in this paper we called possibility measure as ρ-measure and possibility variable is ρ-variable.

In this paper we discuss the kind of convergence in ρ-measure and study relations between them by prove some important theorem.

Keywords

σ-field, measurable space, ρ-measure, convergence , ρ-variable.

1-Introduction

Possibility theory is mathematical theory with certain kind of uncertainty and is substitutional to probability theory [1], possibility measures were introduced by Zadeh [2] in 1978.

In this paper we study the kinds of converge in ρ-measure first almost everywhere. We called almost everywhere ρ-almost everywhere . After that define almost uniformly, uniform convergence is a kind of convergence stronger than point wise convergence. It is clear from these definitions that uniform convergence imply pointwise convergence for every, and define converge in

ρ- measure many author discuss converge in measure (see [3,4,5]) in this paper we discuss converge in another measure is called ρ - measure .

In [6] discussion convergence in another measure.

After that discussion of kinds of converge in possibility variable as converge almost surely, converge in mean and converges in distribution, and prove some properties theory.

2.Preliminaries

2.1. Definition [1]

A family $\Im$ of subsets of a set X is called a σ-field on a set X if

(1) $X \in \Im$

(2) If $A \in \Im$, then $A^c \in \Im$

(3) If $A_n \in \Im, n=1,2,...$ then $\bigcup_{n=1}^{\infty} A_n \in \Im$

$(X,\Im)$ is a measurable space, where X is a set and $\Im$ is σ-field on X

a subset A of X is called measurable to the σ-field $\Im$, for all $A \in \Im$ is called a measurable set.

2.2. Definition [5]

Let $(X,\Im)$ is a measurable space, the set function $\rho : \Im \to [0,1]$ is called ρ- measure if it is holds the following axioms:

(1) $\rho(X)=1$, $\rho(\phi)=0$

(2) For every sequence $\{A_n\}$ in F, we have

$$\rho(\bigcup_{n=1}^{\infty} A_n) = \max_{1\le n\le\infty}\{\rho(A_n)\}$$

a space is a tripe $(X,\Im,\rho)$ where X is a set, $\Im$ is σ- field, ρ- measure on $\Im$.

2.3. Definition [3]

Let $(X,\Im,\rho)$ is a ρ- measure. A function $\xi : X \to R$ is said

a ρ-variable if X is a Borel measurable . i.e.

$\xi^{-1}(A)=\{X \in A \}=(\{x \in X : \xi(x)\in A \}\in$

Let ξ be a ρ-variable on $(X,\Im,\rho)$. The ρ-distribution ϑ defined as $\vartheta_x : R \to [0,1]$ of any ρ-variable $\vartheta_x(x)=\rho\{w\in X:\xi(w)\le x\}$ for any $x \in R$.

2.4. Definition [3]

Let X is a ρ- variable on $(X,\Im,\rho)$, then the expected value of X is defined as

$$E(X)=\int_0^{+\infty}\rho\{X\ge r\}dr - \int_{-\infty}^{0}\rho\{X\le r\}dr$$

2.5. Theorem

Let $(X,\Im,\rho)$ be ρ-measure space, then

(1) $0 \le \rho(A) \le 1$.

(2) $\rho(A_1 \cup A_2) \le \rho(A_1) + \rho(A_2)$

(3) If $A_1, A_2 \in F$ and $A_1 \subset A_2$ then $\rho(A_1) \le \rho(A_2)$.

(4) If $A_1, A_2 \ldots, A_n \in F$, then $\rho(\bigcup_{k=1}^{n} A_k) \le \max_{1\le k\le n}\{\rho(A_k)\}$.

(5) $\rho(\bigcup_{k=1}^{\infty} A_k) \le \sum_{i=1}^{\infty} \rho(A_k)$

(6) $\rho(A_1 \cap A_2) \le \min\{\rho(A_1)\ \rho(A_2)\}$

(7) $(2_A \cap 1_A)\rho \ge 1-(2_A)\rho+(1_A)\rho$

(8) If $A_1, A_2 \in F$, then . $\rho(A_1\text{-}A_2) \ge \rho(A_1)\text{-}\rho(A_2)$

In the following definition we define the kind of converge by ρ- measure by using the definition in [1]and [6] that was use converge another measure

2.6. Definition

Let $(X,\Im,\rho)$ ρ-measure space, sequence (f_n) of real–valued measurable function on X said be

1- converge ρ-almost everywhere to *a.e* real –valued measurable function f denoted by $f_n \xrightarrow{a.e} f$ if for each $\varepsilon > 0$ and $x \in X$ exist a set $E \in \Im$ and a natural number N such that $\rho(E) < \varepsilon$ and $|f_n(x) - f(x)| < \varepsilon$, $x \in E^c$ and each $n \ge N$.

2- converge ρ-almost uniformly to an a.e real –valued measurable function f denoted by $f_n \xrightarrow{a.u} f$ if for each $\varepsilon > 0$ there is a set $E \in \Im$ and a natural number N such that $\rho(E) < \varepsilon$ and $\|f_n - f\|_\infty = \sup_{x\in E^c}|f_n(x) - f(x)| < \varepsilon$, $n \ge N$.

3- converge in ρ-measure to an a.e real –valued measurable function f denoted by $f_n \xrightarrow{\rho} f$ if for each $\varepsilon > 0$ $\lim_{n\to\infty} \rho(x \in X : |f_n(x) - f(x)| \ge \varepsilon) = 0$.

2.7. Definition [7]

Let $(X,\Im,\rho)$ ρ-measure space is called complete if $\Im$ contains all subsets of measure zero. That is, if $\mathrm{H} \in \Im, \rho(\mathrm{H}) = 0$ and $A \subset \mathrm{H}$ then $A \in \Im$.

2.8. Proposition

Let $(X,\Im,\rho)$ complete ρ-measure space and $f = g$ a.e if f is measurable on $\mathrm{H} \in \Im$ as well g.

Proof

Let $\psi \in R$ and $N = \{x \in H : g(x) \ne f(x)\}$ then $N \in \Im$ and $\rho(N) = 0$

Now

$$\{x \in \mathrm{H} : g(x) > \psi\} = \{x \in \mathrm{H} \,|\, N : g(x) > \psi\} \cup \{x \in N : g(x) > \psi\}$$

$$= \{x \in \mathrm{H} \,|\, N : f(x) > \psi\} \cup \{x \in N : g(x) > \psi\}$$

2.9. Proposition

Let $(X,\Im,\rho)$ complete ρ- measure space , f_n is sequence of measurable function on $\mathrm{H} \in \Im$ is converges to f a.e , then f is measurable on H.

2.10 .Theorem

Let $(X,\Im,\rho)$ is ρ-measure space and f_n is sequence of real-valued measurable functions on X, if a sequence f_n converge ρ- almost uniformly , then it is converge in the ρ-measure to f.

Proof

$\because$ the sequence f_n converges ρ-almost uniformly to f,

there exist measurable set H and a natural number N such that

$\rho(E) < \varepsilon$ and , $\varepsilon > 0$

and

$|f_n(x) - f(x)| < \varepsilon \quad , \quad \forall \quad x \in X \mid H$

and $n \geq N$.

now , for all $n \geq N$.

$\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} \subset H$

Then $\forall \quad n \in N$

$\rho(\{ x \in X : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon$.

2.11. Theorem

Let $(X, \Im, \rho)$ ρ-measure space , f_n is a sequence of a.e real-valued measurable function on X , if a sequence f_n converges ρ -almost uniformly, then it is converges ρ-almost everywhere.

Proof

Let f_n converges ρ- almost uniformly to f, then for each $n \in \mathrm{N}$ there is

a measurable set A_n for $\rho(A_n) < \frac{1}{n}$ such that $f_n \to f$ uniformly of

$X \mid A_n$, Let $A = \bigcup_{n=1}^{\infty}(X \mid A_n)$ then

$$\rho(X \mid A) = \rho(\bigcap_{n=1}^{\infty} A_n) \leq \rho(A_n) = \frac{1}{n} \to 0$$

That is $\rho(X \mid A) = 0$

Then , for each $x \in A, f_n(x) \to f(x)$ as $n \to \infty$ we gate $f_n \overset{a.e}{\to} f$

2.12. Theorem

Let $(X, \Im, \rho)$ is a finite ρ-measure space and f_n is a sequence of a.e real-valued measurable function on X. If the sequence f_n converges

ρ-almost everywhere to f, then it is converges ρ-almost uniformly to f.

2.13. Theorem

Let $(X, \Im, \rho)$ is a ρ-measure space and f_n is a sequence of a.e

real-valued measurable function on X, if a sequence f_n converge in

ρ-measure to f, then there exist a subsequence f_k to f_n is converges ρ-almost everywhere to f.

Proof

Let $n_1 \in N$ such that for all $n \geq n_1$

$$\rho(\{ x \in X : |f_n(x) - f(x)| \geq 1\} < \frac{1}{2}$$

Now , choose $n_2 \in N$ such that $n_2 \geq n_1$ and for all $n \geq n_2$

$$\rho(\{ x \in X : |f_n(x) - f(x)| \geq \frac{1}{2}\} < \frac{1}{2^2}$$

Next , choose $n_3 \in N$ such that $n_3 \geq n_2$ and for all $n \geq n_3$

$$\rho(\{ x \in X : |f_n(x) - f(x)| \geq \frac{1}{3}\} < \frac{1}{2^3}$$

Continue this process obtaining an increasing sequence (n_k) of natural numbers

$$\rho(\{ x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\} < \frac{1}{2^k}$$

For each $k \in N$, let

$$A_k = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$$

$\because$ for all $k \in \mathrm{N}$, $\rho(A_k) < \frac{1}{2^k}$ have

$\sum_{k=1}^{\infty} \rho(A_k)$ converges

Then have $\rho(\limsup A_k) = 0$

$$A = \limsup A_k = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j$$

Choose $x \in X \mid A$, then $\exists j_x \in N$, $x \in X \mid A_{j_x}$ then

$$X \mid A_{j_x} = X \mid \bigcup_{k=j_x}^{\infty} \{x \in X : |f_k(x) - f(x)| \geq \frac{1}{k}\}$$
$$\bigcap_{k=j_x}^{\infty} [X \mid x \in X : |f_k(x) - f(x)| \geq \frac{1}{k}\}$$
$$\bigcap_{k=j_x}^{\infty} [x \in X : |f_k(x) - f(x)| \geq \frac{1}{k}\}$$

Then for $k \geq j_x$, $|f_k(x) - f(x)| < \frac{1}{k}$

Let $\varepsilon > 0$, $k_o \geq j_x$ such that $\frac{1}{k_o} < \varepsilon$ then for all $k \geq k_o$

$$|f_k(x) - f(x)| < \frac{1}{k} < \varepsilon$$

Therefore, for all $x \in X \mid A$, $f_k(x) \to f(x)$

Then $f_k \to f$ almost everywhere.

3. Relation between converge concept:

3.1 .Definition

Let $(X, \Im, \rho)$ is ρ-measure space

1- a sequence ζ_n is said to be convergent ρ-almost surely (a.s) to ρ- variable ζ if exists an event β with $\rho\{\beta\} = 1$ such that $\lim_{n\to\infty} |\zeta_n(x) - \zeta(x)| = 0$, for all $x \in \beta$.

2- a sequence ζ_n is said to be convergent in ρ-measure to a ρ-variable ζ if $\lim_{n\to\infty} \rho\{|\zeta_n(x) - \zeta(x)| \geq \varepsilon\} = 0$, for all $\varepsilon > 0$.

3- a sequence ζ_n is called convergent in mean to ζ

if $\lim_{n\to\infty} E[|\zeta_n(x) - \zeta(x)|] = 0$, such that $\zeta, \zeta_1, \zeta_2, \ldots$ be ρ- variables with finite expected values.

4- a sequence ζ_n is said to be convergent in distribution to ζ if $\varphi_n \to \varphi$, at any point φ such that $\varphi, \varphi_1, \varphi_2, \ldots$ be ρ- distribution of ρ-variable $\zeta, \zeta_1, \zeta_2, \ldots$, respectively.

3.2. Proposition

Let $\zeta, \zeta_1, \zeta_2, \ldots$ be ρ- variable then ζ_n converges a.e to ζ if and only if for any $\varepsilon > 0$ we have $\rho\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in X \| \zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$.

3.3. Definition

Let $\zeta_n, \zeta, n \geq 1$ be ρ- variable defined in the possibility measure space $(X, \Im, \rho)$. The sequence ζ_n is said converges ρ-uniformly a.s to ζ if exist $\mathrm{A}_k, \rho(\mathrm{A}_k) \to 0$ such that ζ_n converges ρ- uniformly to ζ in $\Im - \mathrm{A}_k$ for any k.

3.4. Proposition

Let $\zeta, \zeta_1, \zeta_2, \ldots$ be ρ-variable then ζ_n converges ρ- uniformly to ζ if and only if

$$\lim_{n\to\infty} \rho\left(\bigcup_{n=m}^{\infty} (x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

Proof

Let ζ_n be converges ρ- uniformly a.s to ζ then for $\gamma > 0$ there exist β such that $\rho\{\beta\} < \gamma$ and

ζ_n converges uniformly to ζ on $\Im \mid \beta$.

Then for any $\varepsilon > 0$ there exists $m > 0$ s.t $|\zeta_n(x) - \zeta(x)| < \varepsilon$ where $n \geq m$ and $x \in \Im \mid \beta$ that $\bigcup_{n=m}^{\infty}\{(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\} \subset \beta$

Then

$$\rho(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon) \leq \rho(\beta) < \gamma$$

Thus

$$\lim_{n\to\infty} \rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

In the second hand

Let

$$\lim_{n\to\infty} \rho\left(\bigcup_{n=m}^{\infty}(x \in \mathrm{X} \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

For any $\varepsilon > 0$ then for any $\gamma > 0$, $k \geq 1$ there exists m_k

$$\rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \frac{1}{k}\right) < \frac{\gamma}{2^k}$$

Let $\bigcup_{k=1}^{\infty}\bigcup_{n=m_k}^{\infty}\{x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \frac{1}{k}\}$

Then $\rho\{\beta\} < \gamma$ we have $\sup_{x\in X|\beta} |\zeta_n(x) - \zeta(x)| < \frac{1}{k}$ for any k=1,2,…, and $n > m_k$.

3.5. Theorem

Let $\zeta, \zeta_1, \zeta_2, \ldots$ be ρ-variable if ζ_n converges ρ- uniformly a.s to ζ then ζ_n converges a.s to ζ.

Proof

Since ζ_n converges ρ- uniformly a.s to ζ then

$$\lim_{n\to\infty} \rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

From proposition 3.3.

$$\rho\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x \in X \| \zeta_n(x) - \zeta(x)| \geq \varepsilon\right)$$

$$\leq \rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right)$$

Taking the limit of $m \to \infty$

$$\rho\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x \in X \| \zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

Then ζ_n converges a.s to ζ.

3.6 .Theorem

Let $\zeta, \zeta_1, \zeta_2, \ldots$ be ρ-variable if ζ_n converges ρ- uniformly a.s to ζ then ζ_n converges in ρ- measure to ζ.

Proof

Since ζ_n converges ρ- uniformly a.s to ζ then

$$\lim_{n\to\infty} \rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right) = 0$$

From proposition 3.3.

$$\rho\{x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\}$$

$$\leq \rho\left(\bigcup_{n=m}^{\infty}(x \in X \mid |\zeta_n(x) - \zeta(x)| \geq \varepsilon\right)$$

then ζ_n converges in ρ-measure to ζ.

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