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IRAQI JOURNAL FOR COMPUTER SCIENCE AND MATHEMATICS 2025;6:203–209

ORIGINAL STUDY

A New Class of Endo-R.B Module and Its Relationship with Modules

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ABSTRACT

This paper gives a definition of a new class of T-module and T-submodule called an Endo-Restricted Bounded module (submodule) written shortly by Endo-R.B. module (submodule) and present some different approaches to connect this class of module with other types of modules such as: compressible modules, monoform modules, critically compressible modules, retractable modules, and quasi-Dedekind modules. One of the main purpose of this work is to introduce a few new conditions and reveal some properties and corollaries. This paper considered to be another solution or answer for Zelmanowitz's problem. In fact, an Endo-R.B. T-module plays an important role to this problem and we say under what condition compressible T-module would be a critically compressible T-module so we present a positive solution depend on an Endo-R.B module. The T-homomorphism of compressible and monoform modules join in a nice way with an endomorphism of a T-module Ω that we need it for the definition of an Endo-R.B module. Moreover, an Endo-R.B module gives us directly three different modules and these modules are bounded module, finitely annihilated module, and retractable module. Finally, polyform and fully polyform modules are involved in this article.

Keywords: Endo-Restricted Bounded module, Compressible module, Monoform module, Retractable module, Quasi-Dedekind module

1. Introduction

The ring in this paper is commutative with identity denoted by T and Ω is a unitary left-T-module. Motivated by the notion of bounded module that was introduced by Carl Faith, where a T-module Ω is called bounded if there exists $x \in \Omega$ such that $ann_T(x) = ann_T(\Omega)$ [1]. Also, the concept of bounded module studied and expanded in some details by Al-Ani where a bounded submodule also defined in the same context so a submodule A of Ω is said to be bounded if these exists $x \in A$ such that $ann_T(x) = \ ann_T(A)$ [2]. Moreover, prime modules and scalar modules both are involved in many properties as condition to connect an Endo-R.B module with other modules where a T-module Ω is called prime module if for every submodule A of Ω once we have $ann_T(\Omega) = ann_T(A)$ [3]. A T-module Ω is called compressible if it can be embedded in any non-zero submodule A of Ω [3] and every compressible is a prime module [3]. Note that if Ω is a finitely generated T-module. Then Ω is compressible module if and only if it is uniform and prime module [4]. A T-module Ω is called retractable T-module if for every non-zero submodule A of Ω we have $Hom(\Omega.A) \neq 0$. Equivalently, Ω is called retractable T-module if there exists a non-zero endomorphism $\varphi \in End(\Omega)$ such that $Im\varphi \subseteq A$ for every non-zero submodule A of Ω [4]. A retractable T-module Ω is called a critically compressible T-module if every non-zero partial endomorphism of Ω is a monomorphism [4]. In addition, a T-module Ω is said to be multiplication if submodule A of Ω there exists an ideal I of T such that $I\Omega = A$ [5]. Also, there are some properties of T-module that we need them to show more results such as torsionless and self-generator modules along

Received 15 September 2024; revised 30 April 2025; accepted 4 May 2025.
Available online 24 May 2025

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https://doi.org/10.52866/2788-7421.1256

with the trace of T-module Ω. Ω is called torsionless T-module if $f \in Hom(\Omega, T)$ such that $\cap_f \ker f = 0$ [6]. Also, a T-module Ω is said to be self-generator if $A = \sum_{\varphi} Im\ \varphi$, for every submodule A of Ω where $\varphi \in Hom(\Omega, A)$ [6] and the trace of T-module Ω is denoted by $T_r(\Omega) = \sum_{\varphi} \varphi(\Omega), \varphi \in Hom(\Omega, T)$.

In this paper, we introduce a new class of module called an *Endo-R.B* and this class of module properly contain the concept of bounded module in sense of Carl Faith's definition, which means that every Endo-R.B module is bounded. Also, we came up with some certain analogies between an Endo-R.B module and other types of modules such as compressible, monoform, retractable, and more. In this works, we give several properties and provide many conditions that play a crucial role to concrete these relationships. The paper is organized as follows: In Section 2, some related notions are reviewed and an Endo-R.B module is presented with some properties in Section 3. In Section 4, some significant relationships and compartion with other modules are given while in Section 5 we prove some further properties that are related with different prospective.

2. Preliminaries

Definition 2.1 ([6]): A T-module Ω is called scalar if for every $\varphi \in End(\Omega)$ there exists $r \in T$ such that $\varphi(x) = rx, \ \forall x \in \Omega$.

Corollary 2.2 ([6]): *If Ω is finitely generated multiplication T-module, then Ω is a scalar T-module.*

Proposition 2.3 ([6]): Let Ω be a torsion-free scalar T-module where (T is an integral domain). Then every element $\varphi \in \mathrm{End}(\Omega)$ is a T-monomorphism.

Remark 2.4 ([6]): If Ω is an injective scalar T-module, then A is a scalar submodule of Ω.

Definition 2.5 ([7]): A T-module Ω is called finitely annihilated T-module if there exists a finitely generated submodule A of Ω such that $ann_T(\Omega) = \mathrm{ann}_T(A)$.

It is clear that every an Endo-R.B module is a finitely annihilated since every bounded module is finitely annihilated.

Proposition 2.6 ([7]): If Ω is multiplication T-module, then Ω is finitely annihilated module if and only if Ω is finitely generated.

Definition 2.7 ([8]): A T-module Ω is called monoform if every non-zero submodule A of Ω is dense and a submodule A of Ω is said to be dense if for any $x, y \in \Omega$ with $x \neq 0$ there exists $r \in T$ such that $ry \in A$ and $rx \neq 0$.

Equivalently, a T-module Ω is called monoform if for every non-zero submodule A of Ω and for every non-zero homomorphism $\varphi \in H(A, \Omega)$ implies that φ is monomorphism [9]. Note that if Ω is monoform module, it is uniform and prime module [9]. Also, from monoform module we infer that $ann_T(\Omega)$ is prime ideal of T.

Definition 2.8 ([10]): A T-module Ω is called quasi-Dedekind if every non-zero submodule A is quasi-invertible where a non zero-submodule A is said to be quasi-invertible if $Hom(\frac{\Omega}{A}, \Omega) = 0$. Equivalently, a T-module Ω is called quasi-Dedekind if for each non-zero endomorphism of Ω is a T-monomorphism.

Proposition 2.9 ([11]): Let Ω be a retractable T-module. If every non-zero $\varphi \in End(\Omega)$ is a monomorphism, then Ω is a compressible module.

Proposition 2.10 ([11]): Let Ω be a retractable T-module. Then Ω is critically compressible module if and only if every non-zero partial endomorphism of Ω is a monomorphism.

Definition 2.11 ([11]): A T-module Ω is called fully retractable if for every non-zero submodule A of and each non-zero homomorphism $h \in Hom(A, \Omega)$ implies that $Hom(A, \Omega)h \neq 0$.

Proposition 2.12 ([11]): Let Ω be a fully retractable T-module such that End(Ω) is a domain. Then Ω is polyform.

Proposition 2.13 ([11]): Let Ω be a retractable T-module such that End(Ω) is a domain. Then Ω is critically compressible module if and only if it is polyform.

3. Endo-R.B modules

A new class of a T-module will be established in this section with some examples and remarks. In order to do that we need firstly give a definition of an Endo-R.B submodule. Throughout the paper, we will use the symbol $End(\Omega)$ as the set of all endomorphism of T-module Ω and "$<$ or $\leq$" for proper submodule of Ω.

Definition 3.1: A proper T-submodule A of Ω is said to be Endo-R.B if there exists an endomorphism φ of Ω and $\varphi(x) \in A$ for some $x \in \Omega$ such that $ann_T(\varphi(x)) = ann_T(A)$.

Example 3.2:

1) *Suppose that $\Omega = Z_3 \oplus Z$, $T = Z$ and let $A = \langle\bar{0}\rangle \oplus 3Z$. Then we can find $\varphi :\ Z_3 \oplus Z \to Z_3 \oplus Z$ defined by $\varphi(\bar{a}, b) = (\bar{0}, b)$. It is clear that φ is an endomorphism. Now, let $x = (\bar{1}, 3) \in \Omega$ Then $\varphi(\bar{1}, 3) = (\bar{0}, 3) \in A$, and hence we see that $ann_T(\bar{0}, 3) = ann_T(A) = \langle\bar{0}\rangle$. We conclude that a submodule A is an Endo-R.B submodule of Ω.*
2) *Assume $\Omega = Z_4$ as Z-module and $A = \langle\bar{2}\rangle$. Define $\varphi : Z_4 \to Z_4$ as $\varphi(\bar{a}) = \bar{0}$, $\forall\, \bar{a} \in Z_4$ and φ is endomorphism. Then $\varphi(\bar{a}) \in A$, but $ann_Z(A) = ann_Z\langle\bar{2}\rangle = 2Z$ is not equal to $ann_Z(\varphi(\bar{3})) = ann_Z(\bar{0}) = Z$. Therefore, A is not Endo-R.B submodule.*

For more information about an Endo-R.B T-submodule and its properties see [12].

Now, we are ready to give the definition of Endo-R.B module with some examples that explain the structure of the definition.

Definition 3.3: A T-module Ω is called Endo-R.B module if every proper submodule of Ω is Endo-R.B submodule.

Example 3.4:

i) *Z_P as Z-module is Endo-R.B where P is prime number since $(\bar{0})$ is the only proper submodule of Z_P and to show that take an endomorphism $\varphi \in End(\Omega)$ as $\varphi : Z_3 \to Z_3$ such that $\varphi(\bar{a}) = 3\bar{a}$, $\forall\, \bar{a} \in Z_3$ then $\varphi(\bar{a}) \in (\bar{0})$ and $ann_T(\langle\bar{0}\rangle) = ann_T(\varphi(\bar{a})) = Z$.*
ii) *Let $\Omega = Z_4 \oplus Z_2$ as Z-module. Define $\varphi :\ \Omega \to \Omega$ as $\varphi(\bar{a}, \bar{b}) = (\bar{a}, 0)$, $\forall(\bar{a}, \bar{b}) \in \Omega$. Then if we take $A = Z_4 \oplus (\bar{0})$ we have that $\varphi(\bar{a}, \bar{b}) \in A$. Thus we conclude that Ω is not Endo-R.B Z-module since if we let $x = (\bar{2}, \bar{0})$, then $4Z = ann_Z(A) \neq ann_Z(\bar{2}, \bar{0}) = 2Z$.*

Remark 3.5: (1) Every Endo-R.B.T-module is bounded but the converse is not true and to show that.

Let $\Omega = Z_2 \oplus Z_2$ as Z-module and let Let $A = Z_2 \oplus (\bar{0})$. Define $\psi : \Omega \to\ \Omega$ by $\psi(\bar{a}, \bar{b}) = (\bar{0}, \bar{0})$.

Since $\psi(\bar{a}, \bar{b}) = (\bar{0}, \bar{0}) \in A$ but $2Z = ann_Z(A) \neq ann_Z(\bar{0}, \bar{0}) = Z$. Therefore, Ω is not Endo-R.B while Ω is bounded T-module since there exists an element $x = (\bar{0}, \bar{1}) \in Z_2 \oplus Z_2$ such that $2Z = ann_Z(\Omega) = ann_Z(\bar{0}, \bar{1}) = 2Z$.

(2) Every proper submodule of Endo-R.B module is also an Endo-R.B.

4. Endo-R.B modules and related modules

In this section, compressible and monoform modules play a major role in order to obtain an Endo-R.B module. Also, a retractable module and quasi-Dedekind module both are related to an Endo-R.B module and we shall show this relationship.

Proposition 4.1: Let Ω be a compressible T-module, then Ω is an Endo-R.B. T-module.

Proof: Let $\varphi \in End(\Omega)$ and define $\varphi : \Omega \to \Omega$ as $\varphi(x) = rx$, $x \in \Omega$. Since Ω is a compressible module, then for each non-zero submodule N of Ω there exists a monomorphism $\psi : \Omega\ \to N$. Put $\varphi = i \circ\ \psi$ where i is the inclusion map.

Thus $\varphi(x) = (i\psi)(x) = i(\psi(x)) = \psi(x) \in N$. Now, let $t \in ann_T(\varphi(x))$ implies that $t\varphi(x) = 0$, $\forall x \in \Omega$. Then $t\psi(x) = 0$, $\forall x \in \Omega$ so that $\psi(tx) = 0$, $\forall x \in \Omega$. Hence, $\psi(tx) = \psi(0)$ implies that $tx = 0$, $\forall\ x \in \Omega$. Therefore, $t \in ann_T(\Omega) = ann_T(N)$ since Ω is prime module. Hence, $ann_T(\varphi(x)) = ann_T(N)$.

Corollary 4.2: *Let Ω be a finitely generated T-module, then every uniform prime T-module is an Endo-R.B module.*

Proof: Since Ω is finitely generated T-module, then Ω is compressible module. Thus by previous proposition the result we get the result.

Proposition 4.3: Let Ω be a multiplication injective T-module. Then Ω is an Endo-R.B module if and only if Ω is a compressible module.

Proof: Suppose that Ω is an Endo-R.B module, then Ω is finitely annihilated module and Proposition 2.6 Ω is finitely generated. Thus, by Corollary 2.2, we conclude that Ω is scalar module. Now, Let $N < \Omega$, then N is scalar T-submodule and thus there exists $\varphi \in End(\Omega)$ such that $\varphi : \Omega \to \Omega$ define as follow $\varphi(x) = rx$, $\forall\ x \in N$. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\varphi"'] & \Omega \arrow[dl, dashed, "g"] \\ & N & \end{tikzcd}$$

Fig. 1. $\Omega - Injective$.

Then there exists a T-homomorphism g such that $g \circ i = \varphi$. See Fig. 1. It remains to prove that g is monomorphism.

$(g \circ i)(x) = g(x) = \varphi(x)$. Let $g(x_1) = g(x_2)$, then $\varphi(x_1) = \varphi(x_2)$ implies that $rx_1 = rx_2$ so $x_1 = x_2$.

Therefore, g is a T-monomorphism and we conclude that Ω is compressible module. Conversely, using Proposition 4.1.

Proposition 4.4: Every monoform T-module is an Endo-R.B. T-module.

Proof: Since Ω is monoform module, it is uniform prime module and every non-zero submodule A of Ω is dense. Let $\varphi \in End(\Omega)$ and define $\varphi : \Omega \to \Omega$ as follow $\varphi(x) = rx \neq 0$, $0 \neq x \in \Omega$, $0 \neq r \in T$. Then $rx \in A$ since Ω is uniform module. Thus we have that $ann_T(A) \subseteq ann_T(\varphi(x))$. Now, let $t \in ann_T(\varphi(x))$ implies that $t(rx) = 0$ and $tr(x) = 0$, but $r \notin ann_T(x)$ since A is dense submodule and $ann_T(\Omega)$ is prime ideal of T so $t \in ann_T(x) = ann_T(\Omega) \subseteq ann_T(A)$. Therefore, $ann_T(\varphi(x)) = ann_T(A)$ and Ω is an Endo-R.B module.

Proposition 4.5: Let Ω be a torsion-free multiplication T-module where (T is an integral domain). Then Ω is an Endo-R.B module if and only if Ω is monoform module.

Proof: Suppose that Ω is an Endo-R.B module, then by Corollary 2.2 Ω is scalar module and using Proposition 2.3 we have that every $0 \neq \varphi \in End(\Omega)$ is a monomorphism. Let $\varphi : \Omega \to \Omega$ be an T-homomorphism and consider the inclusion map $i : N \to \Omega$. Then the composition $\varphi \circ i : N \to \Omega$ is monomorphism and this implies that Ω is monoform module.

Conversely, assume that Ω is monoform module, then by Proposition 4.4 Ω is an Endo-R.B.

The Proposition 4.5 was proved with multiplication torsion-free module condition in order to obtain monform module. However, we need just torsion-free module as condition to prove that Ω is a monoform module in term of the density of its submodules.

Proposition 4.6: Let Ω be a torsion-free T-module. Then Ω is an Endo-R.B module if and only if Ω is monoform.

Proof: Let A be any non-zero submodule of Ω. Since Ω is a torsion-free module, then for every $x \in \Omega$ there exists $0 \neq r \in T$ such that $rx = 0$, we have $x = 0$. Ω is an Endo-R.B module, then there exists $\varphi \in End(\Omega)$ and define $\varphi(y) = ry$, $y \in M$. Thus, $ry \in A$. Now let $0 \neq x \in \Omega$ so it remains to show that $rx \neq 0$. Suppose that $rx = 0$ and this implies that $x = 0$ since Ω is torsion-free module which is a contradiction. Thus, $rx \neq 0$. Since A is an arbitrary non-zero submodule, we conclude that Ω is monoform module.

Proposition 4.7: Let Ω be a cyclic quasi-Dedekind T-module, then Ω is an Endo-R.B. T-module.

Proof: Let $\varphi \in End(\Omega)$ and define $\varphi : \Omega \to \Omega$ as $\varphi(x) = rx$, $x \in \Omega$. Let A be a proper submodule of Ω, then A is a cyclic submodule such that $A = Tx$ for some $x \in \Omega$ since Ω is cyclic. Therefore, $\varphi(x) \in A$ implies $ann_T(A) \subseteq ann_T(\varphi(x))$. Now, let $t \in ann_T(\varphi(x))$ so $t\varphi(x) = 0$, then $\varphi(tx) = 0$, but Ω is quasi-Dedekind module. Thus $tx = 0$ impies that $t \in ann_T(x) = ann_T(\Omega) = ann_T(A)$.

Proposition 4.8: Let Ω be an Endo-R.B quasi-Dedekind module, then Ω is monoform module.

Proof: It is enough to that every non-zero submodule of Ω is dense. Let N be any non-zero submodule of Ω and define $\varphi : \Omega \to \Omega$ as $\varphi(y) = ry$, $\forall y \in \Omega$. Let $0 \neq x \in \Omega$, then since Ω is an Endo-R.B module we have $ry \in N$. Now, it remains to prove that $rx \neq 0$. Suppose $rx = 0$ so $\varphi(x) = 0$ implies that $x \in \ker \varphi$, but Ω is quasi-Dedekind module means that φ is monomorphism. Thus $x = 0$ which is a contradiction. Therefore, $rx \neq 0$ and N is dense submodule and hence Ω is monoform.

Proposition 4.9: Every Endo-R.B module is retractable module.

Proof: Let N be a non-zero submodule of an Endo-R.B Ω module, then there exists $0 \neq \varphi \in End(\Omega)$ and T-homomorphism $\varphi : \Omega \to \Omega$ define as follow $\varphi(x) = rx$, $0 \neq x \in \Omega$ such that $\varphi(x) \in N$ *and* $ann_T(\varphi(x)) = ann_T(N)$. Suppose that $Hom(\Omega, N) = 0$ and consider the inclusion map $i : N \to \Omega$. Then $\varphi = i \circ f$ where $f : \Omega \to N$, $f = 0$. Therefore, $\varphi(x) = (i \circ f)(x) = i(f(x)) = 0$ implies that $ann_T(\varphi(x)) \neq ann_T(N)$ which is a contradiction. Thus $Hom(\Omega, N) \neq 0$ for every non-zero submodule of Ω. Hence Ω is a retractable T-module.

As a result, we can say that if Ω is an Endo-R.B module then $H(\Omega, A) \neq 0$ for every non-zero submodule A of Ω.

The converse of Proposition 4.9 is not true in general, as we will show that in the next example.

Example 4.10: *Consider $\Omega = Z_4$ as Z-module and let $N = (\bar{2})$. Let $\varphi \in End(\Omega)$ and define $\varphi : Z_4 \to Z_4$ as*

$\varphi(\bar{a}) = 0,\ \forall\ \bar{a} \in \Omega$. Then we can see that Ω is a retractable module since $\varphi(\bar{a}) \in N,\ \forall\ \bar{a} \in \Omega$ implies that $im\varphi \subseteq N$. On the other hand, we see that $ann_Z(N) = ann_Z(\bar{2}) = 2Z$ is not equal to $ann_Z(\varphi(\bar{3})) = ann_Z(\bar{0}) = Z$. Therefore, Ω is not an Endo-R.B module.

The next proposition is consider to be an equivalent to Proposition 4.3 using retractable module in the proof.

Proposition 4.11: Let Ω be multiplication torsion-free T-module where (T is an integral domain). Then the following statements are equivalent:

1) Ω is an Endo-R.B module
2) Ω is compressible module.

Proof: (1) $\Rightarrow$ (2)

Suppose that Ω is an Endo-R.B module, then by Corollary 2.2 Ω is scalar module and every $\varphi \in End(\Omega)$ is monomorphism. Now, by the assumption, Ω is retractable module, then for every non-zero submodule N of Ω there exists a non-zero homomorphism $h : \Omega \to N$ and let $i : N \to \Omega$ be a inclusion map. Thus the composition $i \circ h = \varphi$ is a monomorphism and clearly h is a monomorphism. Therefore, Ω is compressible module.

(2) $\Rightarrow$ (1) By Proposition 4.1 □

Proposition 4.12: Let Ω be a retractable quasi-Dedekind T-module, then Ω is an Endo-R.B module.

Proof: Since Ω is a quasi-Dedekind module, then every T-homomorphism $\varphi \in End(\Omega)$ is monomorphism. Thus by Proposition 2.9 Ω is compressible module and by Proposition 4.1 Ω is an Endo-R.B module.

Proposition 4.13: Let Ω be a retractable self-generator T-module, then Ω is an Endo-R.B module.

Proof: Since Ω is self-generator, then $A = \sum_\psi \psi(\Omega),\ \ \psi \in \text{Hom}(\Omega, \text{A})$ so let $\psi : \Omega \to A$ be a non-zero homomorphism. Consider the inclusion homomorphism $i : A \to \Omega$, then $\varphi = i \circ \psi : \Omega \to \Omega$ is an endomorphism of Ω. Now, $\varphi(x) = (i \circ \psi)(x) = i(\psi(x)) = \psi(x) \in A,\ \ \forall\ \ x \in A$ and this implies that $ann_T(A) \subseteq ann_T(\varphi(x))$. Let $t \in ann_T(\varphi(x))$, then $t \in ann_T(\psi(x)),\ \ \forall\ \ \psi\ \ \in Hom(\Omega, A)$. Thus, $t \in ann_T(\sum_\psi \psi(\Omega)) = ann_T(A)$.
[10].

Proposition 4.14: Let Ω be an Endo-R.B torsion-free T-module, then Ω is critically compressible T-module.

Proof: By Proposition 4.6 we get that Ω is monoform T-module and hence every non-zero partial homomorphism is a monomorphism. Thus, applying Proposition 2.10 we conclude that Ω is critically compressible T-module.

Corollary 4.15: *Let Ω be an Endo-R.B module, then the following statements are equivalent:*

1) *Ω is critically compressible module.*
2) *Every non-zero partial endomorphism of Ω is monomorphism.*

Proof: Since every Endo-R.B module is retractable, then by Proposition 2.10 we obtain the result.

Proposition 4.16: Let Ω be an Endo-R.B duo module, then Ω is fully retractable module.

Proof: Since Ω is an Endo-R.B module there exists a non-zero endomorphism φ of Ω such that $\varphi(x) \in N$ and $ann_T(N) = ann_T(\varphi(x))$ where N is a non-zero submodule of Ω. Since Ω is duo, then for every $\varphi \in End(\Omega)$, we have $\varphi(N) \subseteq N$ which implies that the partial endomorphism of Ω is not zero and this means that $0 \neq \varphi : N \to \Omega$. By Proposition 4.9 Ω is a retractable T-module so there exists a non-zero homomorphism $h : \Omega \to N$. Therefore, $h \circ \varphi \neq 0$ and hence we conclude that Ω is a fully retractable module.

Recall a T-module Ω is called polyform if every essential submodule A of Ω is dense [13]. Here, we can apply all previous propositions that are related to a monoform module since every monoform module is a polyform.

Proposition 4.17: Let Ω be an Endo-R.B duo module and End(Ω) is a domain. Then Ω is a polyform Module.

Proof: Suppose that Ω is an Endo-R.B module, then by Proposition 4.16, Ω is fully retractable module. Now, using Proposition 2.12, we obtain the result.

Proposition 4.18: Let Ω be an Endo-R.B uniform T-module and End(Ω) is a domain. Then Ω is polyform if and only if Ω is critically compressible.

Proof: Suppose that Ω is polyform and since Ω is uniform, then by [13], Ω is monoform module. Since Ω is an Endo-R.B, then Ω is retractable. Therefore, by Proposition 2.13, we have that Ω is critically compressible.

Conversely, assume that Ω is critically compressible, then by Corollary 4.15 Ω is monoform module and hence it is polyform.

Recall a T-module Ω is called fully polyform if every P-essential submodule of Ω is dense [13] and a submodule A is said to be P-essential wherever every pure submodule B with $A \cap B = (0)$ implies that B = (0) [13]. Also, if Ω is a uniform T-module, then polyform and fully polyform are identical [13].

Corollary 4.18: *If Ω is an Endo-R.B uniform T-module and End(Ω) is a domain. Then Ω is fully polyform if and only if Ω is critically compressible.*

Proof: Since Ω is uniform, then by [13], polyform and fully polyform are equivalent.

5. Further results and discussion

In this section, we present some properties and give some characterization of Endo-R.B module. Also, we point out some conditions in order to obtain Endo-R.B module.

Proposition 5.1: Let Ω be a quasi-Dedekind T-module and $A \leq \Omega$. If $\varphi(\Omega) \subsetneq \cap_{\psi} \ker \psi$, $\varphi \in Hom(\Omega, T)$, $\psi \in Hom(T, A)$. Then Ω is an Endo-R.B. T-module.

Proof: Suppose that $\varphi(\Omega) \subsetneq \cap_{\psi} \ker \psi$, then there exists $\psi_{\circ} : T \to A$ such that $0 \neq \psi_{\circ} \circ \varphi \in Hom(\Omega, A)$. Therefore, Ω is a retractable T-module and since Ω is quasi-Dedekind module, then Ω is an Endo-R.B module by Proposition 2.16.

Proposition 5.2: Let N be a torsionless submodule of T-module Ω such that $T_r(\Omega) \neq 0$, then Ω is retractable module.

Proof: Suppose that Ω is not retractable, then there exists a zero homomorphism $f : \Omega \to N$ where N is a non-zero submodule of Ω. Thus $f(m) = 0$, $\forall m \in \Omega$. Define $\psi : N \to T$ as follow $\psi(y) = r$, $\forall\ r \in T$ and clearly it is well-defined and homomorphism. Therefore, $f(\Omega) \subseteq \cap_{\psi} \ker \psi$ and N is torsionless. Hence, the composition 0= $\psi \circ f = \varphi : \Omega \to T$ implies that $T_r(\Omega) = 0$ and this is a contradiction. Thus Ω is retractable module.

Proposition 5.3: Let Ω be an Endo-R.B. T-module and $N \leq \Omega$, then $0 \neq T_r(\Omega) \subsetneq ann_T(A)$.

Proof: Suppose that $T_r(\Omega) \subseteq ann_T(A)$. Since $T_r(\Omega) \neq 0$, then there exists $\varphi \neq 0$, $\varphi \in H(\Omega, \mathrm{T})$. Define $\psi : R \to A$ *as* $\psi(r) = rx$, $\forall x \in A$, so ψ is well-defined and T-homomorphism. By our assumption, we have the following

$$\varphi(\Omega) A = 0$$

$$\varphi(\Omega) x = 0, \qquad \forall x \in A$$

$$(\psi \circ \varphi)\Omega = 0$$

$$(\psi \circ \varphi) = 0$$

$0 = (\psi \circ \varphi) : \Omega \to A$ which is a contradiction since Ω is an Endo-R.B and so it is a retractable module.

Proposition 5.4: Let Ω be an Endo-R.B quasi-Dedekind T-module and $A \leq \Omega$, then $ann_T(\frac{\Omega}{A}) \neq ann_T(\Omega)$.

Proof: Let $A \leq \Omega$ and suppose that $ann_T(\frac{\Omega}{A}) = ann_T(\Omega)$. Thus $[A{:}_T\Omega] = ann_T(\Omega) = ann_T(x)$, $\forall x \in \Omega$ since Ω is prime. Therefore, for every r that satisfy $r\Omega \subseteq A$ $(ry \in A)$ we have $rx = 0$ with $x \neq 0$ where $x, y \in \Omega$. This is a contradiction since Ω by Proposition 4.8 is a monoform T-module. Therefore, $ann_T(\frac{\Omega}{A}) \neq ann_T(\Omega)$.

Recall a T-module Ω is called coprime if $ann_T(\frac{\Omega}{A}) = ann_T(\Omega)$ for every proper submodule A of Ω [14].

Remark 5.5: Let Ω be an Endo-R.B quasi-Dedekind T-module, then Ω is not coprime T-module.

Proof: The proof is a direct result from Proposition 5.4.

Proposition 5.6: Let Ω /A be a quasi-Dedekind T-module where A is a proper submodule of Ω. Then Ω is an Endo-R.B module.

Proof: Let $\varphi \in End(\Omega)$ and define $\varphi : \Omega \to \Omega$ as $\varphi(m) = rm$, $m \in \Omega$. Since Ω /A is quasi-Dedekind module, then we can define $\psi : \Omega/A \to \Omega/A$ as follow $\psi(x + A) = rx + A$, $\forall x \in \Omega$. Then either $\psi = 0$ or ψ is a monomorphism. Suppose that $\psi \neq 0$ and let $m + A \in \ker \psi$, then $\psi(m + A) = A$ implies $rm + A = A$ and hence $m + A = A$, then $m \in A$ so that $rm \in A$, $\forall m \in A$ Therefore, $ann_T(A) \subseteq ann_T(\varphi(m))$. If $\psi = 0$ we get that $rm + A = A$ implies that $rm \in A$ and once again we obtain $ann_T(A) \subseteq ann_T(\varphi(m))$. Now, let $t \in ann_T(\varphi(m))$, *then* $t(rm) = 0$ implies that $t \in ann_T(rm)$, $\forall\ rm \in A$. Therefore, $t \in ann_T(A)$ and hence we conclude that $ann_T(\varphi(m)) = ann_T(A)$. The proof now is complete.

Proposition 5.7: Let Ω be an Endo-R.B module, then $r\Omega \neq \Omega$.

Proof: Suppose that $r\Omega = \Omega$. Since Ω is an Endo-R.B module, then there exists $\varphi \in End(\Omega)$ and defined as follow $\varphi(x) = rx, \ \forall x \in \Omega$, By Proposition 2.13 Ω is retractable module so that $im\varphi \subseteq A$ for every non-zero submodule A of Ω. Therefore, $\varphi(\Omega) \subseteq A$ implies that $r\Omega \subseteq A$, but $r\Omega = \Omega$, $\Omega \subseteq A$ and this is a contradiction. Hence, $r\Omega \neq \Omega$.

Proposition 5.8: Let Ω be an Endo-R.B. T-module and $0 \neq \varphi \in End(\Omega)$, then φ is not epimorphism,

Proof: Let $0 \neq \varphi \in End(\Omega)$. Since Ω is an Endo-R.B module, Ω is retractable T-module and so for every non-zero submodule A of Ω there exists a non-zero homomorphism $g : \Omega \to A$. Consider the inclusion map $i : A \to \Omega$. Thus, $\varphi = i \circ g : \Omega \to \Omega$. Therefore, it is clear that φ is not an epimorphism.

6. Conclusion

In this research article, we have introduced a new class of module called *Endo-Restricted Bounded* and explain how this type of module is stronger than bounded module provided with some examples. In addition, we present several nicely properties that join an Endo-R-B module with other important modules such as compressible modules, monoform modules, quasi-Dedekind modules, and retractable modules. Also, using an Endo-R.B. T-module as assumption lead us to come up with some statements that will be very useful for other researchers who want to search in this topic.

Authors' declaration

- Conflicts of Interest: None.
- I/We hereby confirm that all the Figures and Tables in the manuscript are mine/ours.

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