AL-Bahir
Journal for Engineering and Pure Sciences
BJEPS

Volume 3 | Issue 1 Article 8

THE ANALYTIC SOLUTION OF NON-LINEAR BURGERS–HUXLEY EQUATIONS USING THE TANH METHOD

Kabir Oluwatobi Idowu
Department of Mathematics, Purdue University, USA.

Adedapo Chris Loyinmi
Department of Mathematics, Tai Solarin University of Education, Ijagun, Ogun State, Nigeria

ORIGINAL STUDY

The Analytic Solution of Non-linear Burgers—Huxley Equations Using the Tanh Method

Kabir O. Idowu [a,*], Adedapo C. Loyinmi [b]

[a] Department of Mathematics, Purdue University, USA
[b] Department of Mathematics, Tai Solarin University of Education, Ijagun, Ogun State, Nigeria

Abstract

The emergence of the Burgers—Huxley equation (which involves the famous Burgers equation and the Huxley equation) to predict response systems, dispersion moves, and nerve charge transmission in traffic patterns, sound, turbulent conditions theory, hydrodynamics has attracted the attention of scientists to provide reliable and efficient solutions to the problem.

The present work employed the Tanh method to solve the Burgers—Huxley nonlinear partial differential equations. In contrast to previous results with complicated and laborious solution characteristics, this method is accurate, efficient, and requires little computational work. In showing this, we solved four Burgers—Huxley case study problems using the Tanh approach and obtained the exact solution. The solutions of the four cases were presented graphically. In addition, the findings demonstrate that the Tanh method is an effective and robust approach for constructing the exact solution of nonlinear differential equations.

Keywords: Tanh methods, Burger—Huxley equation, Nonlinear partial differential equation, Analytic methods

1. Introduction

Partial differential equations (PDEs) are crucial for numerically modelling many processes in science and technology [1,2]. Understanding them is essential to gaining a thorough understanding of the actions of natural and artificial processes [3—6]. Because of its complexity [7], researchers are constantly searching for computational and analytical techniques to solve nonlinear differential equations [8,9]. Particularly, there are various approaches in the research for locating the numerical and exact solution to non-linear partial differential equations [10—17]. Numerous scholars have extensively researched nonlinear PDE, resulting in it becoming pervasive [18—21].

In 1915, Bateman proposed the Burgers' equation, which he later changed to the Burgers' equation [22,23]. The Burgers' equation is widely used in engineering and science, particularly when dealing with nonlinear equations [24]. Mathematicians and researchers are doing increasingly important and interesting things with Burgers' equation [25]. As a nonlinear partial differential equation, the Burgers—Huxley equation is essential for comprehending the connection among formation processes, flow effects, and dispersion processes [26—28]. The Burgers—Huxley equation is in the form

$$U_t = U_{xx} - \alpha U^{\delta} U_x + \beta U(1 - U^{\delta})(U^{\delta} - \gamma). \tag{1}$$

In time past, various methods have been used to solve the Burgers—Huxley equation such as the homotopy perturbation method [29], elzaki transform method [27], adomian decomposition method [30], spectral collocation method [31], G'/G-Expansion method [32], homotopy analysis method [33], etc. However, the Tanh method has not been used to solve the Burgers—Huxley equation [34,35]. The Tanh method is an effective approach for searching

Received 7 May 2023; revised 5 July 2023; accepted 6 July 2023.
Available online 7 August 2023

* Corresponding author.
E-mail address: kidowu@purdue.edu (K.O. Idowu).

https://doi.org/10.55810/2313-0083.1038

the wave propagation of one-dimensional dynamical waves and transformation equations [34,36–38].

In this view, this research work seeks to solve the Burgers–Huxley equation using the Tanh approach and confirm the accuracy and efficiency of the method. The paper is presented under the following sections: Section 2 describes the method of the solution, Section 3 presents the application of the method with four examples, Section 4 presents the graphical solution, and Section 5 presents the conclusion.

2. Method of solution

The complex wave and evolutionary models we want to study (for convenience, through space) are often expressed as

$$U_t = G(U, U_x, U_{xx}, \ldots) \text{ or } U_{tt} = G(U, U_x, U_{xx}, \ldots). \tag{2}$$

The aim is to determine a possible exact dynamical solution to (1) and provide a suitable approach to solve it [39].

Firstly, we merge the variable x and t, to arrive at a variable $\eta = k(x - Vt)$, It establishes the dynamic point of view [40]. Here k and V represent the wave number and the velocity of the travelling waves [41]. Despite that, the values of the two parameters are not known, we estimate that $k > 0$ [42]. Furthermore, the independent variable $U(x,t)$ is substituted by $U(\eta)$. Equations like (1) are then transformed into

$$-kV\frac{dU}{d\eta} = G\left(U, k\frac{dU}{d\eta}, k^2\frac{d^2U}{d\eta^2}, \ldots\right), \tag{3}$$

or

$$k^2V^2\frac{d^2U}{d\eta^2} = G\left(U, k\frac{dU}{d\eta}, k^2\frac{d^2U}{d\eta^2}, \ldots\right). \tag{4}$$

As a result, we will be dealing with Ordinary differential equations rather than Partial Difference equations [43,44]. Our primary objective is to arrive at exact solutions to such tanh-form Ordinary differential equations. If that is not feasible, estimated solutions can be obtained. As a result, we present a new independent variable $y = \tanh(\eta)$ into the Ordinary differential equations [45,46]. Hence, we can obtain the finite power series solutions in y.

$$F(y) = \sum_{n=0}^{N} a_n y^n. \tag{5}$$

Which incorporate solitary-wave and shock–wave profiles [47]. We determine the degree (N) by comparing and balancing [48]. The coefficient a_n follows from solving a nonlinear algebraic system [49,50].

3. Existence and uniqueness of the solution

In this section, the existence and uniqueness of the Burger–Huxley solution will be discussed. To accomplish this goal, we employ the Galerkin approximation to prove the well-posedness of the equation. This is accomplished by presenting the weak formation model first. There exist

$U \in M^\infty[(0,T); M^2(\Psi)] \cap M^2[(0,T); K_0^1(\Psi)] \cap M^4[(0,T); M^4(\Psi)]$

such that $\forall u_0$,

$$\left\langle \tfrac{\partial U}{\partial t}, V\right\rangle + \left\langle \tfrac{\partial U}{\partial x}, \tfrac{\partial V}{\partial x}\right\rangle + \alpha\left\langle U\tfrac{\partial U}{\partial x}, V\right\rangle + \beta\left\langle \left(U^3 - (\alpha+1)U^2 + \alpha U\right), V\right\rangle = 0\,. \tag{6}$$

$$\left\langle U(x,0), V\right\rangle = \left\langle U_0, V\right\rangle. \tag{7}$$

We now apply the Galerkin approximation that satisfies (1), giving the orthogonal basis function $\{\tau_1, \tau_2, \tau_3, \ldots, \tau_n \subset K_0^1(\Psi) \cap K^2(\Psi)\}$ [51]. Then we get;

$$\frac{\partial U_n}{dt} - \frac{\partial^2 U_n}{dx^2} + \alpha U_n \frac{\partial U_n}{dx} + \beta P_n\left(U_n^3 - (\alpha+1)U_n^2 + \alpha U_n\right) = 0, \text{ on } \Psi \times (0,T), \tag{8}$$

where $n \in \aleph$, and $U_n = \sum_{i=1}^{n} \phi_i(t)\tau_i$

It has been demonstrated traditionally in [51–53] that the solutions to the Burgers'-Huxley Equation (1) and Equation (16) are the same. The foundation for proving the existence and uniqueness of the solution to the Burgers'-Huxley problem is provided by the preceding connections. Using the above connections, the solution of the Burgers–Huxley equations exists uniquely as shown in [51].

4. Application

4.1. Case 1

At $= 0$ $\delta = 1$, $\gamma = 1$, $\beta = 1$, then the equation becomes

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + U(1-U)(U-1). \tag{9}$$

By transforming equation (5) we have

$$-kV\frac{dU}{d\eta} = k^2\frac{d^2U(\eta)}{d\eta^2} + U(\eta)(1-U(\eta))(U(\eta)-1). \tag{10}$$

Next, we introduce $y = \tanh(\eta)$.

Also, we assume that $u(\eta) \to 0$, and $\frac{dU}{d\eta} \to \infty$ 0 as $\eta \to \infty$.

$$kV\left(1-y^2\right)\frac{dF(y)}{dy}+k^2\left(1-y^2\right)\left[\frac{d}{dy}\left[\left(1-y^2\right)\frac{dF(y)}{dy}\right]\right]+F(y)(1-F(y))(F(y)-1)=0, \tag{11}$$

From (5), $F(y) = \sum_{n=0}^{N} a_n y^n$.

Then, we substitute $F(y) = \sum_{n=0}^{N} a_n y^n$, and differentiate where necessary.

$$kV\left(1-y^2\right)\frac{d\sum_{n=0}^{N} a_n y^n}{dy}+k^2\left(1-y^2\right)\left[\frac{d}{dy}\left(1-y^2\right)\right]\frac{d\sum_{n=0}^{N} a_n y^n}{dy}+\sum_{n=0}^{N} a_n y^n\left(1-\sum_{n=0}^{N} a_n y^n\right)\left(\sum_{n=0}^{N} a_n y^n-1\right)=0, \tag{12}$$

$$kV\left(1-y^2\right)\left[\sum_{n=0}^{N}\frac{a_n y^n n}{y}\right]+k^2\left(1-y^2\right)\left[\begin{array}{l}\left[-2y\left[\sum_{n=0}^{N}\frac{a_n y^n n}{y}\right]\right]\\+\left(1-y^2\right)\left[\sum_{n=0}^{N}\left(\frac{a_n y^n n^2}{y^2}-\frac{a_n y^n n}{y^2}\right)\right]\end{array}\right]+\sum_{n=0}^{N} a_n y^n\left(1-\sum_{n=0}^{N} a_n y^n\right)\left(\sum_{n=0}^{N} a_n y^n-1\right)=0. \tag{13}$$

By expansion, we can only see that the highest power of y appears to be y^{N+2} in the second term and y^{3N} in the third.

Therefore,

$3N = N + 2$,

$2N = 2$,

$N = 1$.

Thus,

$$F(Y)=\sum_{n=0}^{1} a_n y^n = a_0 + a_1 y, \tag{14}$$

Then substituting $F(Y) = a_0 + a_1 y$ into equation (9), We have,

$$-kV\left(1-y^2\right)a_1-2k^2\left(1-y^2\right)ya_1+a_0(1-a_0-ya_1)(a_0+ya_1-1)+a_1 y(1-a_0-ya_1)(a_0+ya_1-1), \tag{15}$$

Through expansion, we have

$$-kVa_1+kVa_1y^2-2k^2a_1y+2k^2a_1y^3-a_0+2a_0^2+4a_0a_1y-a_0^3-3a_0^2a_1y+3a_0a_1^2y^2-a_1y+2y^2a_1^2-a_1^3y^3 \tag{16}$$

$y^0 : -kVa_1 - a_0 + 2a_0^2 - a_0^3 = 0,$

$y^1 : -2k^2a_1 + 4a_0a_1 - 3a_0^2a_1 - a_1 = 0,$

$y^2 : kVa_1 + 2a_1^2 + 3a_0a_1^2 = 0,$

$y^3 : 2k^2a_1 - a_1^3 = 0,$

Equality holds when each coefficient of power of y vanishes. Therefore, from y^3, we get $a_1^2 = 2k^2$.

Substituting $a_1^2 = 2k^2$ into the rest of the over-determined system of nonlinear algebraic equations and solving using Maple 13 solver, we have

$k = \frac{\sqrt{2}}{4}$, $V = \frac{\sqrt{2}}{2}$, $a_0 = \frac{1}{2}$, $a_1 = \sqrt{2k^2} = \frac{1}{2}$.

Since $F(Y) = a_0 + a_1 y$,

Then,

$$F(Y)=\frac{1}{2}+\frac{1}{2}y=\frac{1}{2}(1+y). \tag{17}$$

Recall that $y = \tanh\eta$ and $\eta = k(x - Vt)$,

Then,

$$u(x,t)=\frac{1}{2}(1+\tanh\eta)=\frac{1}{2}(1+\tanh k(x-Vt)). \tag{18}$$

Substituting the values of k and V

$$u(x,t)=\frac{1}{2}\left(1+\tanh\frac{\sqrt{2}}{4}\left(x-\frac{\sqrt{2}}{2}t\right)\right), \tag{19}$$

4.2. Case 2

At $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$, then the equation becomes;

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}+u(1-u)(u-1). \tag{20}$$

By transforming equation (5), we have

$$-kV\frac{dU}{d\eta}=k^2\frac{d^2U(\eta)}{d\eta^2}+kU(\eta)\left(\frac{dU(\eta)}{d\eta}+U(\eta)(1-U(\eta))\right.$$
$$\left.\times(U(\eta)-1).\right. \tag{21}$$

Next, we introduce $y = \tanh(\eta)$.

Also, we assume that $u(\eta)\to 0$, and $\frac{dU}{d\eta}\to\infty\ 0$ as $\eta\to\infty$.

$$kV(1-y^2)\frac{dF(y)}{dy}+k^2(1-y^2)\left[\frac{d}{dy}\left[(1-y^2)\frac{dF(y)}{dy}\right]\right]+kF(y)(1-y^2)\frac{dF(y)}{dy}+\ F(y)(1-F(y))(F(y)-1)=0, \tag{22}$$

From (5), $F(y) = \sum\limits_{n=0}^{N} a_n y^n$.

Then we substitute $F(y)=\sum_{n=0}^{N}a_n y^n$ and differentiate where necessary, we have

$$kV(1-y^2)\frac{d\sum\limits_{n=0}^{N}a_ny^n}{dy}+k^2(1-y^2)\frac{d}{dy}\left[(1-y^2)\frac{d\sum\limits_{n=0}^{N}a_ny^n}{dy}\right]+k\sum_{n=0}^{N}a_ny^n(1-y^2)\frac{d\sum\limits_{n=0}^{N}a_ny^n}{dy}+$$
$$\sum_{n=0}^{N}a_ny^n\left(1-\sum_{n=0}^{N}a_ny^n\right)\left(\sum_{n=0}^{N}a_ny^n-1\right)=0\,. \tag{23}$$

Then,

$$kV(1-y^2)\left[\sum_{n=0}^{N}\frac{a_ny^nn}{y}\right]+k^2(1-y^2)\left[\begin{array}{l}-2y\left[\sum\limits_{n=0}^{N}\frac{a_ny^nn}{y}\right]\\ +(1-y^2)\left[\sum\limits_{n=0}^{N}\left(\frac{a_ny^nn^2}{y^2}-\frac{a_ny^nn}{y^2}\right)\right]\end{array}\right]$$
$$+k\left(\sum_{n=0}^{N}a_ny^n\right)(1-y^2)\left[\sum_{n=0}^{N}\frac{a_ny^nn}{y}\right]+\sum_{n=0}^{N}a_ny^n\left(1-\sum_{n=0}^{N}a_ny^n\right)\left(\sum_{n=0}^{N}a_ny^n-1\right)=0. \tag{24}$$

By expansion, we can see that the highest power of y appears to be y^{N+2} in the second term, y^{2N+1} in the third term, and y^{3N} in the fourth.

Therefore,

$3N = N+2$, $3N = 2N+1$,
$2N = 2$, $3N-2N = 1$,
$N = 1$. $N = 1$.

Thus,

$$F(Y)=\sum_{n=0}^{1}a_ny^n = a_0 + a_1 y \tag{25}$$

Then substituting $F(Y) = a_0 + a_1 y$ into equation (17), we have

$$kV(1-y^2)a_1-2k^2(1-y^2)ya_1+k(a_0+ya_1)(1-y^2)a_1+$$
$$(a_0+ya_1)(1-a_0-ya_1)(a_0+ya_1-1), \tag{26}$$

By expansion, we have,

$$kVa_1 - kVa_1y^2 - 2k^2a_1y + 2k^2a_1y^3 + ka_0a_1 - ka_0a_1y^2 + ka_1^2y - ka_1^2y^3 + 2a_0^2 + 4a_0a_1y - a_0 - a_0^3 - 3a_0^2a_1y - 3a_0^2{a_1}^2y^2 + 2{a_1}^2y^2 - ya_1 - a_1^3y^3, \tag{27}$$

$y^0 : kVa_1 - a_0 - a_0^3 + ka_0a_1 + 2a_0^2 = 0,$
$y^1 : 4\,a_0a_1 - 2k^2a_1 - a_1 + a_1^2k - 3a_0^2a_1 = 0,$
$y^2 : -kVa_1 - ka_0a_1 + 2a_1^2 + 3a_0a_1^2 = 0,$
$y^3 : 2k^2a_1 - a_1^2k - a_1^3 = 0\,.$

Equality holds when each coefficient of power of y vanishes. Therefore, from y^3, we have

$$2k^2 - a_1k - a_1^2 = 0. \tag{28}$$

By solving the quadratic equation,

$$k = -\frac{a_1}{2} or\ k = a_1 \tag{29}$$

Substituting $a_1 = -2k$ into the rest of the over-determined system of nonlinear algebraic equations and solving using Maple 13 solver, we have
$k = \frac{1}{4}$, $V = -\frac{3}{2}$, $a_0 = \frac{1}{2}$, $a_1 = -2k = -\frac{1}{2}$.
Since $F(Y) = a_0 + a_1y$
Then $F(Y) = \frac{1}{2} - \frac{1}{2}y = \frac{1}{2}(1 - y)$.
Recall that $y = \tanh \eta$ and $\eta = k(x - Vt)$
Then,
$u(x,t) = \frac{1}{2}(1 - \tanh \eta) = \frac{1}{2}(1 - \tanh k(x - Vt))$.
Substituting the values of k and V

$$u(x,t) = \frac{1}{2}\left(1 - \tanh\frac{1}{4}\left(x + \frac{3}{2}t\right)\right). \tag{30}$$

4.3. Case 3

At $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$, then the equation becomes

(31)
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + 2u(1-u)(u-3). \tag{31}$$

By transforming equation (5), we have

(32)
$$-kV\frac{dU}{d\eta} = k^2\frac{d^2U(\eta)}{d\eta^2} + kU(\eta)\left(\frac{dU(\eta)}{d\eta} + 2U(\eta)(1-U(\eta))(U(\eta)-3).\right. \tag{32}$$

Next, we introduce $y = \tanh(\eta)$

Also, we assume that $u(\eta)\to 0$, and $\frac{dU}{d\eta}\to\infty\ 0$ as $\eta\to\infty$.

(33)
$$kV(1-y^2)\frac{dF(y)}{dy} + k^2(1-y^2)\left[\frac{d}{dy}\left[(1-y^2)\frac{dF(y)}{dy}\right]\right] + kF(y)(1-y^2)\frac{dF(y)}{dy} + 2F(y)(1-F(y))(F(y)-3) = 0 \tag{33}$$

Since $F(y) = \sum_{n=0}^{N} a_ny^n$,

Then we substitute $F(y) = \sum_{n=0}^{N} a_ny^n$ and differentiate where necessary,

$$kV(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy} + k^2(1-y^2)\frac{d}{dy}\left[(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy}\right] + k\sum_{n=0}^{N} a_ny^n(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy} + 2\sum_{n=0}^{N} a_ny^n\left(1 - \sum_{n=0}^{N} a_ny^n\right)\left(\sum_{n=0}^{N} a_ny^n - 3\right) = 0 \tag{34}$$

Then

$$kV(1-y^2)\left[\sum_{n=0}^{N}\frac{a_ny^nn}{y}\right] + k^2(1-y^2)\left[-2y\left[\sum_{n=0}^{N}\frac{a_ny^nn}{y}\right] + (1-y^2)\left[\sum_{n=0}^{N}\left(\frac{a_ny^nn^2}{y^2} - \frac{a_ny^nn}{y^2}\right)\right]\right] + k\left(\sum_{n=0}^{N} a_ny^n\right)(1-y^2)\left[\sum_{n=0}^{N}\frac{a_ny^nn}{y}\right] + 2\sum_{n=0}^{N} a_ny^n\left(1 - \sum_{n=0}^{N} a_ny^n\right)\left(\sum_{n=0}^{N} a_ny^n - 3\right) = 0 \tag{35}$$

By expansion, we can see that the highest power of y appears to be y^{N+2} in the second term, y^{2N+1} in the third term and y^{3N} in the fourth.

Therefore,

$3N = N+2, \quad 3N = 2N+1,$
$2N = 2, \quad 3N-2N = 1,$
$N = 1. \quad N = 1.$

Thus,

$F(Y) = \sum_{n=0}^{1} a_n y^n = a_0 + a_1 y$

Then substituting $F(Y) = a_0 + a_1 y$ into equation (4), we have

$$kV(1-y^2)a_1 - 2k^2(1-y^2)ya_1 + k(a_0+ya_1)(1-y^2)a_1 + 2(a_0+ya_1)(1-a_0-ya_1)(a_0+ya_1-3) \tag{36}$$

Through expansion, we have

$$kVa_1 - kVa_1y^2 - 2k^2a_1y + 2k^2a_1y^3 + ka_0a_1 - ka_0a_1y^2 + ka_1^2y - ka_1^2y^3 + 8a_0^2 + 16a_0a_1y - 6a_0 - 2a_0^3 - 6a_0^2a_1y - 6a_0^2a_1{}^2y^2 + 8a_1{}^2y^2 - 6ya_1 - 2a_1^3y^3$$
$$y^0 : kVa_1 - 6a_0 - 2a_0^3 + ka_0a_1 + 8a_0^2 = 0,$$
$$y^1 : 16\,a_0a_1 - 2k^2a_1 - 6a_1 + a_1^2k - 6a_0^2a_1 = 0,$$
$$y^2 : -kVa_1 - ka_0a_1 + 8a_1^2 + 6a_0a_1^2 = 0,$$
$$y^3 : 2k^2a_1 - a_1^2k - 2a_1^3 = 0. \tag{37}$$

Equality holds when each coefficient of power of y vanishes. Therefore, from y^3, we have $2k^2 - a_1k - 2a_1^2 = 0$.

By solving the quadratic equation.

$k = \left(\frac{1}{4} + \frac{\sqrt{17}}{4}\right)a_1$ or $k = \left(\frac{1}{4} - \frac{\sqrt{17}}{4}\right)a_1$.

Substituting $k = \left(\frac{1}{4}+\frac{\sqrt{17}}{4}\right)a_1$ into the rest of the over-determined system of nonlinear algebraic equations and solving using Maple 13 solver, we have

$V = \frac{5\sqrt{17}+7}{4}$, $a_0 = \frac{1}{2}$, $a_1 = -2k = -\frac{1}{2}$, $k = \frac{1+\sqrt{17}}{8}$.

Since $F(Y) = a_0 + a_1 y$,

Then, $F(Y) = \frac{1}{2} - \frac{1}{2}y = \frac{1}{2}(1-y)$.

Recall that $y = \tanh\eta$ and $\eta = k(x-Vt)$,

Then, in its original variable.

$u(x,t) = \frac{1}{2}(1-\tanh\eta) = \frac{1}{2}(1-\tanh k(x-Vt))$.

Substituting the values of k and V

$$u(x,t) = \frac{1}{2}\left(1 - \tanh\frac{1+\sqrt{17}}{8}\left(x - \frac{5\sqrt{17}+7}{4}t\right)\right) \tag{38}$$

4.4. Case 4

At $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$, then the equation becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} + u(1-u)(u-3) \tag{39}$$

By transforming equation (5) we have

$$-kV\frac{dU}{d\eta} = k^2\frac{d^2U(\eta)}{d\eta^2} + 2kU(\eta)\left(\frac{dU(\eta)}{d\eta} + U(\eta)(1-U(\eta))\right) \times (U(\eta)-3) \tag{40}$$

Next, we introduce $y = \tanh(\eta)$

Also, we assume that $u(\eta)\to 0$, and $\frac{dU}{d\eta}\to\infty$ 0 as $\eta\to\infty$.

$$kV(1-y^2)\frac{dF(y)}{dy} + k^2(1-y^2)\left[\frac{d}{dy}\left[(1-y^2)\frac{dF(y)}{dy}\right]\right] + 2kF(y)(1-y^2)\frac{dF(y)}{dy} + F(y)(1-F(y))(F(y)-3) = 0 \tag{41}$$

Since $F(y) = \sum_{n=0}^{N} a_n y^n$.

Then we substitute $F(y) = \sum_{n=0}^{N} a_n y^n$ and differentiate where necessary

$$kV(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy} + k^2(1-y^2)\frac{d}{dy}\left[(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy}\right] + 2k\sum_{n=0}^{N} a_ny^n(1-y^2)\frac{d\sum_{n=0}^{N} a_ny^n}{dy} + \sum_{n=0}^{N} a_ny^n\left(1-\sum_{n=0}^{N} a_ny^n\right)\left(\sum_{n=0}^{N} a_ny^n - 3\right) = 0 \tag{42}$$

Then

$$kV\left(1-y^2\right)\left[\sum_{n=0}^{N}\frac{a_n y^n n}{y}\right]+k^2\left(1-y^2\right)\left[-2y\left[\sum_{n=0}^{N}\frac{a_n y^n n}{y}\right]+\left(1-y^2\right)\left[\sum_{n=0}^{N}\left(\frac{a_n y^n n^2}{y^2}-\frac{a_n y^n n}{y^2}\right)\right]\right]$$
$$+2k\left(\sum_{n=0}^{N}a_n y^n\right)\left(1-y^2\right)\left[\sum_{n=0}^{N}\frac{a_n y^n n}{y}\right]+\sum_{n=0}^{N}a_n y^n\left(1-\sum_{n=0}^{N}a_n y^n\right)\left(\sum_{n=0}^{N}a_n y^n-3\right)=0 \tag{43}$$

By expansion, we can see that the highest power of y appears to be y^{N+2} in the second term, y^{2N+1} in the third term and y^{3N} in the fourth.

Therefore,

$3N = N+2,\quad 3N = 2N+1,$
$2N = 2,\qquad 3N-2N = 1,$
$N = 1.\qquad N = 1.$

Thus,

$F(Y) = \sum_{n=0}^{1} a_n y^n = a_0 + a_1 y.$

Then, substituting $F(Y) = a_0 + a_1 y$ into equation (4), we have

$$kV\left(1-y^2\right)a_1-2k^2\left(1-y^2\right)ya_1+2k(a_0+ya_1)\left(1-y^2\right)a_1+(a_0+ya_1)(1-a_0-ya_1)(a_0+ya_1-3)\,. \tag{44}$$

Through expansion, we have

$$kVa_1-kVa_1y^2-2k^2a_1y+2k^2a_1y^3+2ka_0a_1-2ka_0a_1y^2+2ka_1^2y-2ka_1^2y^3+4a_0^2+8a_0a_1y-3a_0-a_0^3-3a_0^2a_1y-3a_0^2{a_1}^2y^2+4{a_1}^2y^2-3ya_1-a_1^3y^3 \tag{45}$$

$y^0 : kVa_1-3a_0-a_0^3+2ka_0a_1+4a_0^2=0,$
$y^1 : 8a_0a_1-2k^2a_1-3a_1+2a_1^2k-3a_0^2a_1=0,$ s
$y^2 : -kVa_1-2ka_0a_1+4a_1^2+3a_0a_1^2=0,$
$y^3 : 2k^2a_1-2a_1^2k-a_1^3=0\,.$

Equality holds when each coefficient of power of y vanishes. Therefore, from y^3, we have

$$2k^2-2a_1k-a_1^2=0. \tag{46}$$

By solving the quadratic equation,
$k=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)a_1$ or $k=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)a_1$.

Substituting $k=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)a_1$ into the rest of the over-determined system of nonlinear algebraic equations and solving using Maple 13 solver, we have

$V=\frac{\sqrt{3}-5}{2}$, $a_0=\frac{3}{2}$, $a_1=\frac{3}{2}$, $k=\frac{3-3\sqrt{3}}{4}$.
Since $F(Y)=a_0+a_1y$,
Then $F(Y)=\frac{3}{2}-\frac{3}{2}y=\frac{3}{2}(1-y)$
Recall that $y=\tanh\eta$, and $\eta=k(x-Vt)$.

Then, in its original variable,

$$u(x,t)=\frac{3}{2}(1-\tanh\eta)=\frac{3}{2}(1-\tanh k(x-Vt)). \tag{47}$$

Substituting the values of k and V

$$u(x,t)=\frac{3}{2}\left(1-\tanh\frac{3-3\sqrt{3}}{4}\left(x-\frac{\sqrt{3}-5}{2}t\right)\right). \tag{48}$$

5. Numerical simulations

In this section, we present the result of the four cases graphically. The result from the Tanh method is the same as the exact solution found in the literature [27,29].

6. Discussion

In this section, using 3D plots, we show how the exact solutions and the Tanh Method results relate to one another for each of the four cases of the Burgers–Huxley equation. Figures 1–4 show the graphs of the solutions derived from the Tanh method at $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$. The result shows that the Tanh method is an appropriate, efficient, and accurate method for solving the Burger–Huxley equation. The method is also suitable for finding the exact solution directly instead of using semi-analytic methods [29,54] and numerical

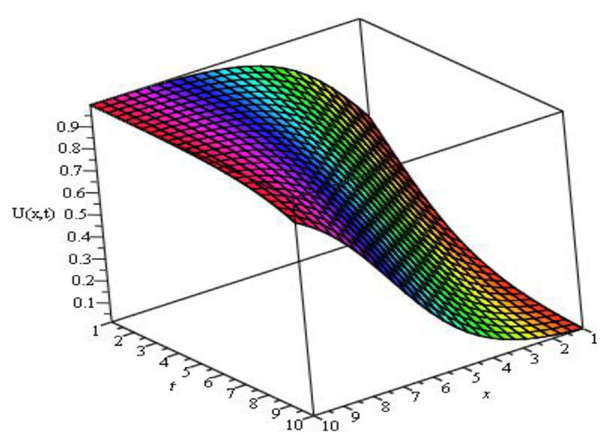

Fig. 1. Solution plot for case 1 at = 0 δ = 1, γ = 1, β = 1.

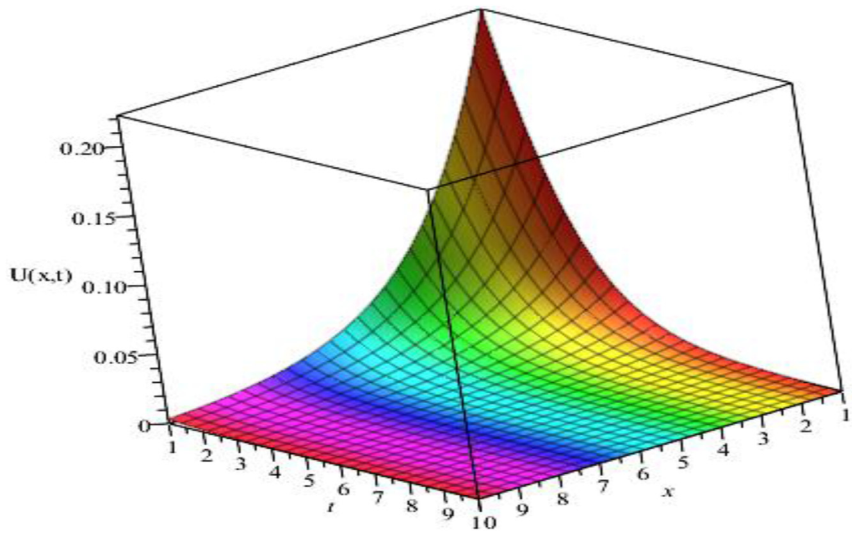

Fig. 2. Exact Solution plot for case 2 at $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$.

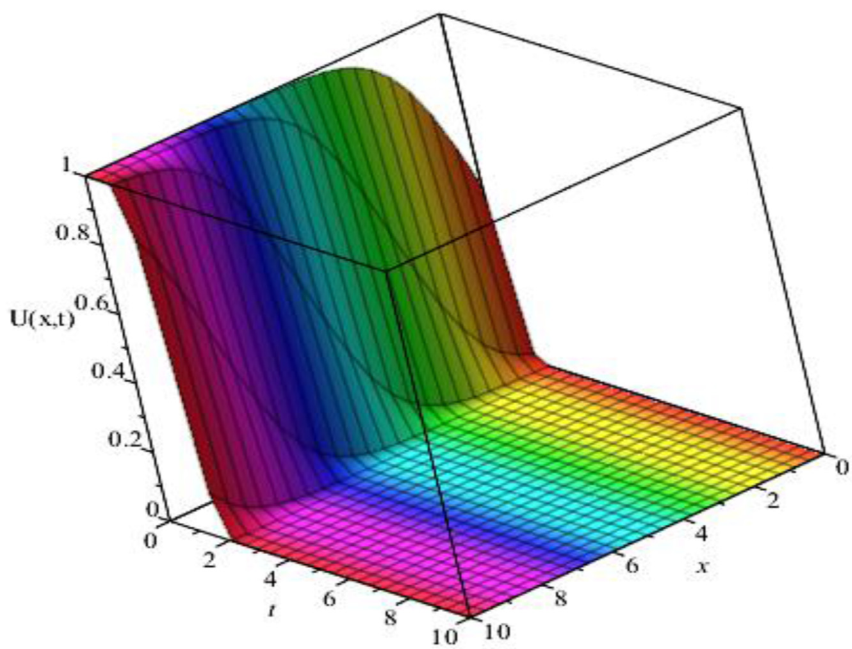

Fig. 3. Solution plot for case 3 at $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$.

methods [32,55], which will only result in an series and approximate solution of the Burger–Huxley equation. The method will be advantageous to engineers and researchers because it provides easier and more accurate solutions in less time compared with other methods.

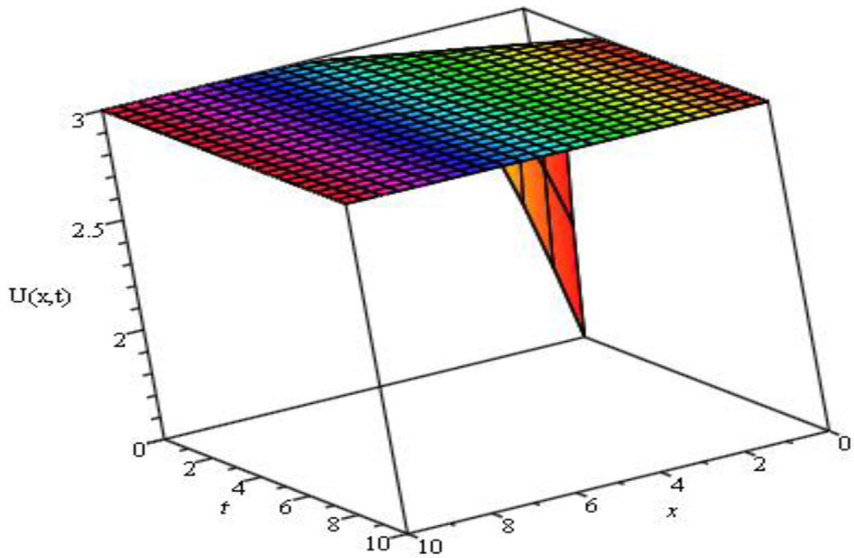

Fig. 4. Solution plot for case 4 at $\alpha = -1$, $\delta = 1$, $\gamma = 1$, $\beta = 1$.

7. Conclusion

In this paper, we have successfully and efficiently used the Tanh method to solve the Burger–Huxley equation. Four cases are presented to show the efficiency and accuracy of the method to solve Burger–Huxley. In each case, we arrived at the exact solution to the equations presented. Therefore, the result in this paper is sufficient to conclude that the Tanh method is a suitable method for solving nonlinear partial differential equations, and in particular, the Burger–Huxley equation. Hence, the Tanh approach is highly recommended for use in the solution of models involving fluid dynamics, technology, nonlinear dynamics, noise, convection, dispersion, advection-diffusion, etc. Analytical solutions to Burgers-Huxley-type equations and related nonlinear partial differential equations may also be obtained using this approach.

Acknowledgement

The authors of this study are extremely appreciative of the reviewers who remained anonymous for their contributions to the improvement of the quality of the study's scientific methodology. The authors would also like to thank Mrs. Loyinmi Adetoun and Mrs. Idowu Elizabeth Olawumi for their support throughout the preparation of this manuscript.

References

[1] Overton CE, Wilkinson RR, Loyinmi A, Miller JC, Sharkey KJ. Approximating quasi-stationary behaviour in network-based SIS dynamics. Bull Math Biol 2022;84:1–32.
[2] Idowu KO, Akinwande TG, Fayemi I, Adam UM, Loyinmi AC. Laplace homotopy perturbation method (LHPM) for solving systems of N-dimensional non-linear partial differential equation. 2023. https://doi.org/10.55810/2313-0083.1031.
[3] Samaniego E, Anitescu C, Goswami S, Nguyen-Thanh VM, Guo H, Hamdia K, et al. An energy approach to the solution of partial differential equations in computational mechanics via machine learning: concepts, implementation and applications. Comput Methods Appl Mech Eng 2020;362:112790.
[4] Loyinmi AC, Erinle-Ibrahim LM, Adeyemi AE. The new iterative method (NIM) for solving telegraphic equation. J. Niger. Assoc. Math. Phys. 2017;43:31–6.
[5] Loyinmi AC, Lawal OW, Sottin DO. Reduced differential transform method for solving partial integro-differential equation. J. Niger. Assoc. Math. Phys. 2017;43:37–42.
[6] Loyinmi AC, Lawal OW. The asymptotic solution for the steady variable-viscosity free convection flow on a porous plate. J. Niger. Assoc. Math. Phys. 2011;19:273–6.
[7] Mohanty RK, Sharma S. High-accuracy quasi-variable mesh method for the system of 1D quasi-linear parabolic partial differential equations based on off-step spline in compression approximations. Adv Differ Equ 2017;2017:1–30.
[8] Akinfe KT, Loyinmi AC. The implementation of an improved differential transform scheme on the schrodinger equation governing wave-particle duality in quantum physics and optics. 2021. Available SSRN 4098920.
[9] Akinfe TK, Loyinmi AC. An improved differential transform scheme implementation on the generalized Allen–Cahn equation governing oil pollution dynamics in oceanography. Partial Differ. Equations Appl. Math. 2022;6:100416. https://doi.org/10.1016/j.padiff.2022.100416.
[10] Kudryashov NA. Exact solutions of the equation for surface waves in a convecting fluid. Appl Math Comput 2019;344:97–106.
[11] Rezazadeh H, Tariq H, Eslami M, Mirzazadeh M, Zhou Q. New exact solutions of nonlinear conformable time-fractional Phi-4 equation. Chin J Phys 2018;56:2805–16.
[12] Tang B, Wang X, Wei L, Zhang X. Exact solutions of fractional heat-like and wave-like equations with variable coefficients. Int J Numer Methods Heat Fluid Flow 2014;24:455–67.
[13] Erinle-Ibrahim LM, Adewole AI, Loyinmi C, Sodeinde OK. AN optimization scheme using linear programming in a production line of rite foods limited ososa. FUDMA J. Sci. 2020;4:502–10.
[14] Odulaja DO, Erinle-Ibrahim LM, Loyinmi AC. Numerical computation and series solution for mathematical model of

HIV/AIDS Computation and series solution for mathematical model of HIV/AIDS, online). Scienpress Ltd; 2013.
[15] Morenikeji E-IL, Babajide AO, Oluwatobi IK. Application of homotopy perturbation method to the mathematical modelling of temperature rise during microwave hyperthermia. FUDMA J. Sci. 2021;5:273–82.
[16] Babajide AO, Oluwatobi IK. On the elzaki substitution and homotopy pertubation methods for solving partial differential equation involving mixed partial derivatives. FUDMA J. Sci. 2021;5:159–68.
[17] Erinle-Ibrahim Latifat M, Oluwatobi IK, Sulola Abigail I. Mathematical modelling of the transmission dynamics of malaria infection with optimal control. Kathmandu Univ J Sci Eng Technol 2021;15.
[18] Akinfe KT, Loyinmi AC. Stability analysis and semi-analytic solution to a SEIR-SEI Malaria transmission model using He's variational iteration method. 2020.
[19] Lawal OW, Loyimi AC. Application of new iterative method for solving linear and nonlinear initial boundary value problems with non local conditions. Sci World J 2019;14:100–4.
[20] Lawal OW, Loyimi AC. Erinle-ibrahim, algorithm for solving a generalized hirota-satsuma coupled KDV equation using homotopy perturbation transform method. Sci World J 2018; 13. www.scienceworldjournal.org.
[21] Lawal OO, Loyinmi AC, Sowunmi OS. Homotopy perturbation algorithm using laplace transform for linear and nonlinear ordinary delay differential equation. J. Niger. Assoc. Math. Phys. 2017;41:27–34.
[22] Alesemi M, Iqbal N, Hamoud AA. The analysis of fractional-order proportional delay physical models via a novel transform. Complexity 2022;2022. https://doi.org/10.1155/2022/2431533.
[23] Suleman M, Elzaki T, Wu Q, Anjum N, Rahman JU. New application of Elzaki projected differential transform method. J Comput Theor Nanosci 2017;14:631–9.
[24] Mohanty RK, Sharma S. A new high-accuracy method based on off-step cubic polynomial approximations for the solution of coupled Burgers' equations and Burgers–Huxley equation. Eng Comput 2021;37:3049–66.
[25] Yasmin H, Iqbal N. A comparative study of the fractional-order nonlinear system of physical models via analytical methods. Math Probl Eng 2022;2022. https://doi.org/10.1155/2022/7488996.
[26] Akinfe TK, Loyinmi AC. A solitary wave solution to the generalized Burgers-Fisher's equation using an improved differential transform method: a hybrid scheme approach. Heliyon 2021;7:e07001.
[27] Loyinmi AC, Akinfe TK. An algorithm for solving the Burgers–Huxley equation using the Elzaki transform. SN Appl Sci 2020;2:1–17.
[28] Loyinmi AC, Akinfe TK. Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method. Eng. Reports. 2020;2: e12084.
[29] Nourazar SS, Soori M, Nazari-Golshan A, Nourazar SS, Soori M, Nazari-Golshan A. On the exact solution of burgers-huxley equation using the homotopy perturbation method. J Appl Math Phys 2015;3:285–94. https://doi.org/10.4236/JAMP.2015.33042.
[30] Alomari AK, Noorani MSM, Nazar R. Solutions of heat-like and wave-like equations with variable coefficients by means of the homotopy analysis method. Chin Phys Lett 2008;25: 589–92. https://doi.org/10.1088/0256-307x/25/2/064.
[31] Darvishi MT, Kheybari S, Khani F. Spectral collocation method and Darvishi's preconditionings to solve the generalized Burgers-Huxley equation. Commun Nonlinear Sci Numer Simul 2008;13:2091–103. https://doi.org/10.1016/J.CNSNS.2007.05.023.
[32] Zhu M, Zhu M. Solving the burgers-huxley equation by G'/G expansion method. J Appl Math Phys 2016;4:1371–7. https://doi.org/10.4236/JAMP.2016.47146.
[33] Molabahrami A, Khani F. The homotopy analysis method to solve the Burgers–Huxley equation. Nonlinear Anal R World Appl 2009;10:589–600. https://doi.org/10.1016/J.NONRWA.2007.10.014.
[34] Lawal OW, Loyinmi AC, Ayeni BO. Laplace homotopy perturbation method for solving coupled system of linear and nonlinear partial differential equationvol. 46; 2019. p. 83–91.
[35] Lawal OW, Loyinmi AC, Hassan AR. Finite difference solution for magnetohydrodynamics thin film flow of a third-grade fluid down inclined plane with ohmic heating. Abacus 2019;46:92–7.
[36] Lawal OW, Loyinmi AC, Aruba DA. Approximate solutions of higher dimensional linear and nonlinear initial boundary valued problems using new iterative method, J Niger. Assoc Math Phys 2017;41:35–40.
[37] Agbomola JO, Loyinmi AC. Modelling the impact of some control strategies on the transmission dynamics of Ebola virus in human-bat population: an optimal control analysis. Heliyon 2022;8:e12121.
[38] Loyinmi AC, Idowu KO. Semi-analytic approach to solving rosenau-hyman and korteweg-de vries equations using integral transform. Tanzan J Sci 2023;49:26–40.
[39] Erinle-Ibrahim LM, Idowu KO. Mathematical modelling of pneumonia dynamics of children under the age of five. Abacus (Mathematics Sci. Ser. 2021;48.
[40] Idowu KO, Loyinmi AC. Impact of contaminated surfaces on the transmission dynamics of corona virus disease (Covid-19). Biomed. J. Sci. Tech. Res. 2023;51:42280–94. https://doi.org/10.26717/BJSTR.2023.51.008046.
[41] Oluwatobi IK, Chris LA. Qualitative analysis of the transmission dynamics and optimal control of covid-19 (preprint). 2023.
[42] Malfliet W, Hereman W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. Phys Scripta 1996;54:563.
[43] Loyinmi AC, Oredein AI. The unsteady variable viscosity free convection flow on a porous plate. J. Niger. Assoc. Math. Phys. 2011;19:229=232.
[44] Lawal OW, Loyinmi AC. Magnetic and porosity effect on MHD flow of a dusty visco-elastic fluid through horizontal plates with heat transfer. J Nig Assoc Math Phys 2012;21:95–104.
[45] Agbomola J, Loyinmi A. A mathematical model for the dynamical behavior of ebola virus transmission in human-bat population: implication of immediate discharge of recovered individuals. 2022.
[46] Loyinmi AC, Oredein AI, Prince SU. Homotopy Adoman decomposition method for solving linear and nonlinear partial differential equations. Tasued J. Pure Appl. Sci. 2018;1:254–60.
[47] Lawal OW, Loyinmi AC. The effect of Magnetic field on MID Viscoelastic flow and heat transfer over a stretching sheet. Pioneer J. Adv. Appl. Math. 2011;3:83–90.
[48] Lawal OW, Loyinmi AC. Oscillating flow on a visco-elastic fluid under exponential pressure gradient with heat transfer. Pioneer J. Adv. Appl. Math. 2011;3:33–82.
[49] Akinfe KT. A reliable analytic technique for the modified prototypical Kelvin–Voigt viscoelastic fluid model by means of the hyperbolic tangent function. Partial Differ. Equations Appl. Math. 2023;7:100523. https://doi.org/10.1016/J.PADIFF.2023.100523.
[50] Erinle-Ibrahim LM, Adebimpe O, Lawal WO, Agbomola JO. A mathematical model and sensitivity analysis of Lassa fever with relapse and reinfection rate. Tanzan J Sci 2022;48:414–26.
[51] Chin PWM. The analysis of the solution of the Burgers – Huxley equation using the Galerkin method. 2023. p. 2787–807. https://doi.org/10.1002/num.22987.
[52] Temam R, Temam R. Infinite-dimensional dynamical systems in mechanics and physics. xvi+ 500pp., 応用数理 Springer 1988;1(1991):350–1.
[53] Evans LC. Partial differential equations graduate studies in mathematicsvol. 19. (American Mathematical Society: Providence, Rhode Island); 1998.
[54] Yefimova OY, Kudryashov NA. Exact solutions of the Burgers-Huxley equation. J Appl Math Mech 2004;3:413–20.
[55] Tomasiello S. Numerical solutions of the Burgers–Huxley equation by the IDQ method. Int J Comput Math 2010;87: 129–40.