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Omar H. Taha
*General Directorate of Salah al-Din Education - Ministry of Education, Tikrit, Iraq*

Omar A. Abdullah
*Department of Mathematics - College of Computer Science and Mathematics - Tikrit University, Tikrit, Iraq.*

Ali Sh. Ajeel
*Department of Mathematics - College of Computer Science and Mathematics - Tikrit University, Tikrit, Iraq.*



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### Author Contributions

Author 1 contributed to the conceptualization and methodology; Author 2 performed the formal analysis; Author 3 was responsible for data curation and writing the original draft; all authors reviewed and edited the manuscript and approved the final version.

ORIGINAL STUDY

# Some Characterizations of Weakly Pseudo Semi 2-absorbing Submodules in Terms of Some Types of Modules

Omar H. Taha [a,*], Omar A. Abdullah [b], Ali Sh. Ajeel [b]

[a] General Directorate of Salah al-Din Education, Ministry of Education, Tikrit, Iraq
[b] Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq

**Abstract**

**The purpose of this paper is to investigate characterizations of weakly pseudo semi-2-absorbing submodules in terms of some types of modules. We provide characterizations for the class of multiplication modules with the help of some types of modules such as faithful, non-singular, Z-regular, and projective modules. Furthermore, we add some conditions to proof the residual of a weakly pseudo semi-2-absorbing submodule is a weakly pseudo semi-2-absorbing ideal.**

*JEL classification:* 16D99

*Keywords:* Multiplication modules, Weakly pseudo semi-2-absorbing

## 1. Introduction

In this paper, the ring $\mathscr{E}$ is a commutative ring with a non-zero identity, and $\mathscr{U}$ is a unitary $\mathscr{E}$-module. Over the past 13 years, the concepts of 2-absorbing submodules and weakly 2-absorbing submodules have been extensively studied by Darani and Soheilinia [1]. A submodule $\mathscr{W} \subsetneq \mathscr{U}$ of an $\mathscr{E}-$module $\mathscr{U}$ is called 2-absorbing (weakly 2-absorbing) if whenever $abu \in \mathscr{W}$ $(0 \neq abu \in \mathscr{W})$ for some $a, b \in \mathscr{E}$, $u \in \mathscr{U}$, then either $au \in \mathscr{W}$ or $bu \in \mathscr{W}$ or $ab \in [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$. Following that, Hadi and Harfash introduced the concept of semi-2-absorbing submodules as a generalization of 2-absorbing submodules [2]. A submodule $\mathscr{W} \subsetneq \mathscr{U}$ of an $\mathscr{E}-$module $\mathscr{U}$ is called semi-2-absorbing (weakly semi-2-absorbing) if whenever $a^2u \in \mathscr{W}$ $(0 \neq a^2u \in \mathscr{W})$ for some $\in \mathscr{E}$, $u \in \mathscr{U}$, then either $au \in \mathscr{W}$ or $a^2 \in [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$. Also, Abdalla and Mohammadali introduced the concept of pseudo semi-2-absorbing submodules [3]. A submodule $\mathscr{W} \subsetneq \mathscr{U}$ of an $\mathscr{E}-$module $\mathscr{U}$ is said to be a pseudo semi-2-absorbing submodule of $\mathscr{U}$, if $e^2u \in \mathscr{W}$, for $e \in \mathscr{E}$, $u \in \mathscr{U}$, implies either $ru \in \mathscr{W} + Soc(\mathscr{U})$ or $e^2 \in [\mathscr{W} + Soc(\mathscr{U}) :_{\mathscr{E}} \mathscr{U}]$. The concept of weakly pseudo semi-2-absorbing submodule is a generalization of semi-2-absorbing and pseudo semi-2-absorbing submodule introduced by Taha and Salih [4], a submodule $\mathscr{W} \subsetneq \mathscr{U}$ of an $\mathscr{E}-$module $\mathscr{U}$ is said to be a weakly pseudo semi-2-absorbing submodule of $\mathscr{U}$ (for short WPS-2AB), if $0 \neq e^2u \in \mathscr{W}$, for $e \in \mathscr{E}$, $u \in \mathscr{U}$, implies either $eu \in \mathscr{W} + Soc(\mathscr{U})$ or $e^2 \in [\mathscr{W} + Soc(\mathscr{U}) :_{\mathscr{E}} \mathscr{U}]$. An ideal $Q$ of a ring $\mathscr{E}$ is said to be a weakly pseudo semi-2-absorbing ideal of $\mathscr{E}$ if $Q$ is a weakly pseudo semi-2-absorbing submodule of the $\mathscr{E}-$module $\mathscr{E}$. An $\mathscr{E}$-module $\mathscr{U}$ is multiplication if every submodule $\mathscr{W}$ of $\mathscr{U}$ is of the from $\mathscr{W} = Q\mathscr{U}$ for some ideal $Q$ of $\mathscr{E}$. It's well known that a cyclic module is Multiplication module [5]. An $\mathscr{E}$-module $\mathscr{U}$ is called Z-regular if for any $s \in \mathscr{U}$ there exists $f \in \mathscr{U}^* = Hom_R(\mathscr{U}, \mathscr{E})$ such that $s = f(s)s$. Recall that an $\mathscr{E}$-module $\mathscr{U}$, is a projective if for every $\mathscr{E}$-epimorphism $f : M_1 \to M_2$ where $M_1$ and $M_2$ are $\mathscr{E}$-modules, and every $\mathscr{E}$-homomorphism $g : \mathscr{U} \to M_2$, there exists an $\mathscr{E}$-homomorphism $h : \mathscr{U} \to M_1$ such that $f \circ h = g$ . Recall that an

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* Corresponding author.
E-mail addresses: omar.h.tahamm2314@st.tu.edu.iq (O.H. Taha), omerabdulrazzaqa@tu.edu.iq (O.A. Abdullah), ali.shebl@tu.edu.iq (A.Sh. Ajeel).

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$\mathscr{E}$-module $\mathscr{U}$ it is a non-singular if $Z(\mathscr{U}) = 0$, where $Z(\mathscr{U}) = \{s \in \mathscr{U} : s\,Q = (0)\}$ for some ideal $Q$ of $\mathscr{E}$ . Recall that an $\mathscr{E}$-module $\mathscr{U}$ is faithful if $Ann_R(\mathscr{U}) = (0)$. Recall that an $\mathscr{E}$-module $\mathscr{U}$ is finitely generated if $\mathscr{U} = (x_1,x_2...,x_n) = Rx_1 + Rx_2 + ... + Rx_n$, where $x_1,x_2...,x_n \in \mathscr{U}$ [6]. In main results section of this paper, we provide characterizations for many modules, including non-singular modules, Multiplication modules, faithful finitely produced modules, projective modules, and Z-regular modules. We show, under certain conditions, the residual of weakly pseudo semi-2-absorbing submodule is a weakly pseudo semi-2-absorbing ideal, Proposition 9, 12, 15, 18. Moreover, we show that under certain conditions, if $\mathscr{W}$ is weakly pseudo semi-2-absorbing submodules $Q\mathscr{U}$ is a weakly pseudo semi-2-absorbing submodule where $Q$ is an ideal of $\mathscr{E}$ see Proposition 24, 26, 28.

## 2. Main results

**Proposition 1.** [4] *Let $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of an $\mathscr{E}$-module $\mathscr{U}$, then $\mathscr{W}$ is a WPS-2AB submodule if and only if $(0) \neq e^2T \subseteq \mathscr{W}$ for $e \in \mathscr{E}$ and $T$ is submodule of $\mathscr{U}$, implies either $eT \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $e^2 \in [\mathscr{W} + Soc(\mathscr{U}) :_{\mathscr{E}} \mathscr{U}]$.*

**Proposition 2.** [4] *Let $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of an $\mathscr{E}$-module $\mathscr{U}$ , then $\mathscr{W}$ is a WPS-2AB submodule if and only if $(0) \neq Q^2T \subseteq \mathscr{W}$ for some ideal $Q$ of $\mathscr{E}$ and submodule $T$ of $\mathscr{U}$, implies that either $QT \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $Q^2 \subseteq [\mathscr{W} + Soc(\mathscr{U}) :_{\mathscr{E}} \mathscr{U}]$.*

**Theorem 3.** *Let $\mathscr{U}$ be a Multiplication $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if whenever $(0) \neq {T_1}^2T_2 \subseteq \mathscr{W}$, for some submodules $T_1$ and $T_2$ of $\mathscr{U}$, implies that either $T_1T_2 \subseteq \mathscr{W} + Soc(\mathscr{U})$ or ${T_1}^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$.*

***Proof.*** Assume that $(0) \neq {T_1}^2T_2 \subseteq \mathscr{W}$, for some submodules $T_1$ and $T_2$ *of* $\mathscr{U}$. Since $\mathscr{U}$ is a Multiplication, We have $T_1 = D_1D$ and $T_2 = D_2D$ for some ideals $D_1$, $D_2$ in $\mathscr{E}$, it follows that $(0) \neq (D_1D)^2D_2D = D_1^2D_2D \subseteq \mathscr{W}$. *Since* $\mathscr{W}$ is a WPS-2AB submodule proposition 1 implies that $D_1(D_2D) \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $D_1^2 \subseteq [\mathscr{W} + Soc(\mathscr{U}) :_{\mathscr{E}} \mathscr{U}]$, that is either $T_1T_2 \subseteq \mathscr{W} + Soc(\mathscr{U})$ or ${T_1}^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$.

Conversely, $(0) \neq Q^2L \subseteq \mathscr{W}$, for some submodule $L$ of $\mathscr{U}$ and $Q$ an ideal of $\mathscr{E}$. Since $\mathscr{U}$ is a Multiplication, then $L = D_1D$ for some ideal $D_1$ in $\mathscr{E}$ , Hence $(0) \neq Q^2D_1D \subseteq \mathscr{W}$. Take $T = Q\mathscr{U}$, that is $T^2L \subseteq \mathscr{W}$ and it follows by hypothesis either $TL \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $T^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$, that is either $QL \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $Q^2\mathscr{U} \in \mathscr{W} + Soc(\mathscr{U})$ and Proposition 2, implies that $\mathscr{W}$ is a WPS-2AB submodule.

A straightforward consequence of Theorem 3.

**Corollary 4.** *Let $\mathscr{U}$ be a Multiplication $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if whenever $(0) \neq T^2u \subseteq \mathscr{W}$, for some submodule $T$ of $\mathscr{U}$ and $u \in \mathscr{U}$, implies that either $Tu \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $T^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$.*

**Corollary 5.** *Let $\mathscr{U}$ be a cyclic $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if whenever $(0) \neq {T_1}^2T_2 \subseteq \mathscr{W}$, for some submodules $T_1$ and $T_2$ of $\mathscr{U}$, implies that either $T_1T_2 \subseteq \mathscr{W} + Soc(\mathscr{U})$ or ${T_1}^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$.*

**Corollary 6.** Let $\mathscr{U}$ *be a cyclic* $\mathscr{E}$-module and *$\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if whenever $(0) \neq T^2u \subseteq \mathscr{W}$, for some submodules $T$ of $\mathscr{U}$ and $u \in \mathscr{U}$, implies that either $Tu \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $T^2 \subseteq \mathscr{W} + Soc(\mathscr{U})$.*

**Remark 7.** [4] If $\mathscr{W}$ is a WPS-2AB submodule of an $\mathscr{E}$-module $\mathscr{U}$. In general, $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is not a WPS-2AB ideal of $\mathscr{E}$.

**Lemma 8.** [6] *Let $\mathscr{U}$ be Z-regular of an $\mathscr{E}$-module. Then $Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$.*

Under certain conditions, the residual of $WPS - 2AB$ submodules are a $WPS - 2AB$ ideal, as shown in the following Propositions.

**Proposition 9.** *Let $\mathscr{U}$ be a Z-regular Multiplication $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is a WPS-2AB* ideal of $\mathscr{E}$.

***Proof.*** Assume that $(0) \neq e^2s \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ for some $e \in \mathscr{E}$, and $s \in \mathscr{U}$. It follows that $(0) \neq e^2(s\mathscr{U}) \subseteq \mathscr{W}$. Since $\mathscr{W}$ is WPS-2AB, Proposition 1, implies that either $e(s\mathscr{U}) \subseteq \mathscr{W} + Soc(\mathscr{U})$ *or* $e^2\mathscr{U} \subseteq \mathscr{W} + Soc(\mathscr{U})$. As $\mathscr{U}$ is a Multiplication, we have $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U}$. Furthermore, since $\mathscr{U}$ is a Z-regular, Lemma 8, implies that either $e(s\mathscr{U}) \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$ *or* $e^2\mathscr{U} \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$. This mean that either $rd \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E})$ or $e^2 \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E}) = [[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E}) :_{\mathscr{E}} \mathscr{E}]$. Then either $rd \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E})$ or $e^2 \subseteq [[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E}) :_{\mathscr{E}} \mathscr{E}]$. Hence, $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.

Conversely, Assume that $(0) \neq e^2T \subseteq \mathscr{W}$ for some submodule $T$ of $\mathscr{U}$ and $e \in \mathscr{E}$. Since $\mathscr{U}$ is a Multiplication Then $T = D_1D$, for some ideal $D_1$ in $\mathscr{E}$. That is $(0) \neq e^2D_1D \subseteq \mathscr{W}$, then $(0) \neq e^2D_1 \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$. Since $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$, then by proposition 1 either $rI \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E})$ or $e^2 \subseteq [[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E}) :_{\mathscr{E}} \mathscr{E}] = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + Soc(\mathscr{E})$. It follows that either $rD_1D \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$ or $e^2\mathscr{U} \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is a $Z$-regular, then by Lemma 8 and since $\mathscr{U}$ is a Multiplication, then $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U}$. Then either $rD_1D \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $e^2\mathscr{U} \subseteq \mathscr{W} + Soc(\mathscr{U})$. It follows that either $rT \subseteq \mathscr{W} + Soc(\mathscr{U})$ or $e^2 \subseteq [\mathscr{W} +$

$Soc(\mathscr{U}):_{\mathscr{E}}\mathscr{U}]$. Then, by Proposition 1, $\mathscr{W}$ is a WPS-2AB submodule.

A straightforward consequence of Proposition 9.

**Corollary 10.** *Let $\mathscr{U}$ be a cyclic Z-regular $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

**Lemma 11.** [6] *Let $\mathscr{U}$ be a projective $\mathscr{E}$-module, then $Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$.*

**Proposition 12.** Let *$\mathscr{U}$ be a Multiplication projective $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$.* Then *$\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

***Proof.*** Assume that $(0)\neq e^2 s \subseteq [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ for some $e\in\mathscr{E}$, and $s\in\mathscr{U}$. It follows that $(0)\neq e^2(s\mathscr{U})\subseteq\mathscr{W}$. Since $\mathscr{W}$ is WPS-2AB, Proposition 1, implies that either $e(s\mathscr{U})\subseteq\mathscr{W}+Soc(\mathscr{U})$ or $e^2\mathscr{U}\subseteq\mathscr{W}+Soc(\mathscr{U})$. As $\mathscr{U}$ is a Multiplication, we have $\mathscr{W} = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}$. Furthermore, since $\mathscr{U}$ is a projective , Lemma 11, implies that either $e(s\mathscr{U})\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$ or $e^2\mathscr{U}\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$. This mean that either $rd\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}) = [[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}]$. Then either $rd\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}]$. Hence, $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.

Conversely, Assume that $(0)\neq e^2T\subseteq\mathscr{W}$ for some submodule $T$ of $\mathscr{U}$ and $e\in\mathscr{E}$. Since $\mathscr{U}$ is a Multiplication Then $T = D_1D$, for some ideal $D_1$ in $\mathscr{E}$. That is $(0)\neq e^2D_1D\subseteq\mathscr{W}$, then $(0)\neq e^2D_1\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$. Since $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$, then by proposition 1 either $rI\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}] = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$. It follows that either $rD_1D\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$ or $e^2\mathscr{U}\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is a projective , then by Lemma 11 and since $\mathscr{U}$ is a Multiplication, then $\mathscr{W} = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}$. Then either $rD_1D\subseteq\mathscr{W}+Soc(\mathscr{U})$ or $e^2\mathscr{U}\subseteq\mathscr{W}+Soc(\mathscr{U})$. It follows that either $rT\subseteq\mathscr{W}+Soc(\mathscr{U})$ or $e^2\subseteq[\mathscr{W}+Soc(\mathscr{U}):_{\mathscr{E}}\mathscr{U}]$. Then, by proposition 1, $\mathscr{W}$ is a WPS-2AB submodule.

A straightforward consequence of Proposition 12.

**Corollary 13.** *Let $\mathscr{U}$ be a cyclic projective $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$.* Then *$\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

**Lemma 14.** [7] *Let $\mathscr{U}$ be a non-singular $\mathscr{E}$-module, then $Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$.*

**Proposition 15.** *Let $\mathscr{U}$ be a non-singular Multiplication $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

***proof.*** The proof is similar to the proof of Proposition 9 and by Lemma 14.

Assume that $(0)\neq e^2 s\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ for some $e\in\mathscr{E}$, and $s\in\mathscr{U}$. It follows that $(0)\neq e^2(s\mathscr{U})\subseteq\mathscr{W}$. Since $\mathscr{W}$ is WPS-2AB, Proposition 1, implies that either $e(s\mathscr{U})\subseteq\mathscr{W}+Soc(\mathscr{U})$ or $e^2\mathscr{U}\subseteq\mathscr{W}+Soc(\mathscr{U})$. As $\mathscr{U}$ is a Multiplication, we have $\mathscr{W} = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}$. Furthermore, since $\mathscr{U}$ is a non-singular, Lemma 14, implies that either $e(s\mathscr{U})\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$ or $e^2\mathscr{U}\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$. This mean that either $rd\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}) = [[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}]$. Then either $rd\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}]$. Hence, $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.

Conversely, Assume that $(0)\neq e^2T\subseteq\mathscr{W}$ for some submodule $T$ of $\mathscr{U}$ and $e\in\mathscr{E}$. Since $\mathscr{U}$ is a Multiplication Then $T = D_1D$, for some ideal $D_1$ in $\mathscr{E}$. That is $(0)\neq e^2D_1D\subseteq\mathscr{W}$, then $(0)\neq e^2D_1\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$. Since $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$, then by proposition 1 either $rI\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$ or $e^2\subseteq[[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}] = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]+Soc(\mathscr{E})$. It follows that either $rD_1D\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$ *or* $e^2\mathscr{U}\subseteq[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}+Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is a non-singular, then by Lemma 14 and since $\mathscr{U}$ is a Multiplication, then $\mathscr{W} = [\mathscr{W}:_{\mathscr{E}}\mathscr{U}]\mathscr{U}$. Then either $rD_1D\subseteq\mathscr{W}+Soc(\mathscr{U})$ or $e^2\mathscr{U}\subseteq\mathscr{W}+Soc(\mathscr{U})$. It follows that either $rT\subseteq\mathscr{W}+Soc(\mathscr{U})$ *or* $e^2\subseteq[\mathscr{W}+Soc(\mathscr{U}):_{\mathscr{E}}\mathscr{U}]$. Then, by proposition 1, $\mathscr{W}$ is a WPS-2AB submodule.

A straightforward consequence of Proposition 15 is the following outcome

**Corollary 16.** *Let $\mathscr{U}$ be a cyclic non-singular $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$. Then $\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

**Lemma 17.** [5] *Let $\mathscr{U}$ be a faithful Multiplication $\mathscr{E}$-module. Then $Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$.*

**Proposition 18.** *Let $\mathscr{U}$ be a faithful Multiplication $\mathscr{E}$-module and $\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$.* Then *$\mathscr{W}$ is a WPS-2AB submodule if and only if $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.*

***proof.*** The proof is similar to the proof of Proposition 9 and by Lemma 17
A straightforward consequence of Proposition 18.

**Corollary 19.** *Let $\mathscr{U}$ be a faithful cyclic $\mathscr{E}$*-module and *$\mathscr{W} \subsetneq \mathscr{U}$ be a submodule of $\mathscr{U}$.* Then *$\mathscr{W}$ is a WPS-*

*2AB submodule if and only if* $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ *is a WPS-2AB ideal of* $\mathscr{E}$.

**Lemma 20.** [5] *Let* $\mathscr{U}$ *be a finitely generated Multiplication* $\mathscr{E}$*-module, and* $Q_1, Q_2$ *be ideals of* $\mathscr{E}$, then $Q_1D\subseteq Q_2D$, *if and only if* $Q_1\subseteq Q_2 + ann_{\mathscr{E}}(\mathscr{U})$.

**Proposition 21.** *Let* $\mathscr{U}$ *be a finitely generated Z-regular Multiplication* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$, *then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

***Proof.*** Assume that $(0)\neq {T_1}^2T_2\subseteq Q\mathscr{U}$ for $T_1, T_2$ some submodules of $\mathscr{U}$. As $\mathscr{U}$ is Multiplication $\mathscr{E}$-module, we have $T_1 = D_1D, T_2 = D_2D$ for some ideals $D_1$, $D_2$ of $\mathscr{E}$. That is $(0)\neq D_1^2D_2D\subseteq Q\mathscr{U}$. Since $\mathscr{U}$ is a finitely generated Multiplication $\mathscr{E}$-module, then by Lemma 20 we have $(0)\neq D_1^2D_2\subseteq Q + ann_{\mathscr{E}}(\mathscr{U})$. Since $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$, then $Q + ann_{\mathscr{E}}(\mathscr{U}) = Q$, that is $(0)\neq D_1^2D_2\subseteq Q$. Since $Q$ is a WPS-2AB ideal of $\mathscr{E}$, then by Proposition 2 $ID_2\subseteq Q + Soc(\mathscr{E})$ or $D_1^2\subseteq[Q + Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}] = Q + Soc(\mathscr{E})$. It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is Z-regular, then by Lemma 8, ($Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$). It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. Then either that $T_1T_2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ *or* ${T_1}^2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. By Proposition 3, we have $Q\mathscr{U}$, a WPS-2AB submodule of $\mathscr{E}$.

**Lemma 22.** [7] *Every cyclic* $\mathscr{E}$*-module is finitely generated.*

A straightforward consequence of Proposition 21.

**Corollary 23.** *Let* $\mathscr{U}$ *be a Z-regular cyclic* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$, *then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

**Proposition 24.** *Let* $\mathscr{U}$ *be a finitely generated Multiplication projective* $\mathscr{E}$*-module and Q be a WPS-2AB* ideal of $\mathscr{E}$ with $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. *Then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

***Proof.*** Assume that $(0)\neq {T_1}^2T_2\subseteq Q\mathscr{U}$ for $T_1, T_2$ some submodules of $\mathscr{U}$. As $\mathscr{U}$ is Multiplication $\mathscr{E}$-module, we have $T_1 = D_1D, T_2 = D_2D$ for some ideals $D_1$, $D_2$ of $\mathscr{E}$. That is $(0)\neq D_1^2D_2D\subseteq Q\mathscr{U}$. Since $\mathscr{U}$ is a finitely generated Multiplication $\mathscr{E}$-module, then by Lemma 20 we have $(0)\neq D_1^2D_2\subseteq Q + ann_{\mathscr{E}}(\mathscr{U})$. Since $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$, then $Q + ann_{\mathscr{E}}(\mathscr{U}) = Q$, that is $(0)\neq D_1^2D_2\subseteq Q$. Since $Q$ is a WPS-2AB ideal of $\mathscr{E}$, then by Proposition 2 $ID_2\subseteq Q + Soc(\mathscr{E})$ or $D_1^2\subseteq[Q + Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}] = Q + Soc(\mathscr{E})$. It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is *a finitely generated*, then by Lemma 11, ($Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$). It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. Then either that $T_1T_2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ *or* ${T_1}^2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. By Proposition 3, we have $Q\mathscr{U}$, a WPS-2AB submodule of $\mathscr{E}$.

A straightforward consequence of Proposition 24.

**Corollary 25.** *Let* $\mathscr{U}$ *be a cyclic projective* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. *Then,* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

**Proposition 26.** *Let* $\mathscr{U}$ *be a finitely generated Multiplication non-singular* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. *Then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

***Proof.*** Assume that $(0)\neq {T_1}^2T_2\subseteq Q\mathscr{U}$ for $T_1, T_2$ some submodules of $\mathscr{U}$. As $\mathscr{U}$ is Multiplication $\mathscr{E}$-module, we have $T_1 = D_1D, T_2 = D_2D$ for some ideals $D_1$, $D_2$ of $\mathscr{E}$. That is $(0)\neq D_1^2D_2D\subseteq Q\mathscr{U}$. Since $\mathscr{U}$ is a finitely generated Multiplication $\mathscr{E}$-module, then by Lemma 20 we have $(0)\neq D_1^2D_2\subseteq Q + ann_{\mathscr{E}}(\mathscr{U})$. Since $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$, then $Q + ann_{\mathscr{E}}(\mathscr{U}) = Q$, that is $(0)\neq D_1^2D_2\subseteq Q$. Since $Q$ is a WPS-2AB ideal of $\mathscr{E}$, then by Proposition 2 $ID_2\subseteq Q + Soc(\mathscr{E})$ or $D_1^2\subseteq[Q + Soc(\mathscr{E}):_{\mathscr{E}}\mathscr{E}] = Q + Soc(\mathscr{E})$. It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{E})\mathscr{U}$. Since $\mathscr{U}$ is a non-singular, then by Lemma 14, ($Soc(\mathscr{U}) = Soc(\mathscr{E})\mathscr{U}$). It follows that either $ID_2D\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ or $D_1^2\mathscr{U}\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. Then either that $T_1T_2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$ or ${T_1}^2\subseteq Q\mathscr{U} + Soc(\mathscr{U})$. By Proposition 3, we have $Q\mathscr{U}$, a WPS-2AB submodule of $\mathscr{E}$.

A straightforward consequence of proposition 26.

**Corollary 27.** *Let* $\mathscr{U}$ *be a cyclic non-singular* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. *Then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

**Proposition 28.** *Let* $\mathscr{U}$ *be a finitely generated faithful Multiplication* $\mathscr{E}$*-module and Q be a WPS-2AB* ideal of $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. Then $Q\mathscr{U}$ *is a WPS-2AB. submodule of* $\mathscr{E}$.

***proof.*** The proof is similar to the proof of Proposition 21 and by Lemma 20.
A straightforward consequence of Proposition 28.

**Corollary 29.** *Let* $\mathscr{U}$ *be a faithful cyclic* $\mathscr{E}$*-module and Q be a WPS-2AB ideal of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U})\subseteq Q$. *Then* $Q\mathscr{U}$ *is a WPS-2AB submodule of* $\mathscr{E}$.

**Lemma 30.** [7] *If* $\mathscr{U}$ *is a Multiplication* $\mathscr{E}$-module, then $\mathscr{U}$ *is cancellation if and only if* $\mathscr{U}$ *is a finitely generated faithful.*

**Proposition 31.** *Let* $\mathscr{U}$ *be a faithful finitely generated Multiplication* $\mathscr{E}$*-module, and* $\mathscr{W}\subsetneq\mathscr{U}$ *be a submodule of* $\mathscr{U}$. *The statements below are considered to be equivalent*:

(*a*) $\mathscr{W}$ *is a WPS-2AB submodule* .
(*b*) $[\mathscr{W}:_{\mathscr{E}}\mathscr{U}]$ *is a WPS-2AB ideal of* $\mathscr{E}$ .

(c) $\mathscr{W} = Q\mathscr{U}$ *for some WPS-2AB ideal Q of* $\mathscr{E}$ .

*Proof* $(a)\Leftrightarrow(b)$ Follows by Proposition 18
$(b)\Rightarrow(c)$ Assume that $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$. since $\mathscr{U}$ is Multiplication, then $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U}$. Put $Q = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ then Q is a WPS-2AB ideal of $\mathscr{E}$ and $\mathscr{W} = Q\mathscr{U}$.
$(c)\Rightarrow(b)$ Assume that $\mathscr{W} = Q\mathscr{U}$ for some Q which is a WPS-2AB ideal of $\mathscr{E}$. Since $\mathscr{U}$ is Multiplication, then $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} = Q\mathscr{U}$. Since $\mathscr{U}$ is faithful finitely generated $\mathscr{E}$-module, then by Lemma 30, $\mathscr{U}$ is a cancellation. Therefore $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] = Q$ is a WPS-2AB ideal of $\mathscr{E}$.

A straightforward consequence of Proposition 31.

**Corollary 32.** *Let* $\mathscr{U}$ *be a faithful cyclic* $\mathscr{E}$*-module and* $\mathscr{W} \subsetneq \mathscr{U}$ *be a submodule of* $\mathscr{U}$. *The statements below are considered to be equivalent*:

(a) $\mathscr{W}$ *is a WPS-2AB submodule.*

(b) $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ *is a WPS-2AB ideal of* $\mathscr{E}$.

(c) $\mathscr{W} = Q\mathscr{U}$ *for some WPS-2AB ideal Q of* $\mathscr{E}$.

Recall $\mathscr{U}$ be a weak cancellation of $\mathscr{E}$ -module if whenever $Q_1\mathscr{U} = Q_2\mathscr{U}$, for $Q_1, Q_2$ are ideal of $\mathscr{E}$, implies that $Q_1 + ann_{\mathscr{E}}(\mathscr{U}) = Q_2 + ann_{\mathscr{E}}(\mathscr{U})$ [6].

**Lemma 33.** [7] *If* $\mathscr{U}$ *be a Multiplication* $\mathscr{E}$*-module, then* $\mathscr{U}$ *is finitely generated if and only if* $\mathscr{U}$ *is a weak cancellation.*

**Proposition 34.** *Let* $\mathscr{U}$ *be a finitely generated Multiplication* (cyclic) *Z-regular* $\mathscr{E}$*-module and* $\mathscr{W} \subsetneq \mathscr{U}$ *is a submodule of* $\mathscr{U}$ *with* $ann_{\mathscr{E}}(\mathscr{U}) \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$. *The statements below are considered to be equivalent*:

(a) $\mathscr{W}$ *is a WPS-2AB submodule.*

(b) $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ *is a WPS-2AB ideal of* $\mathscr{E}$.

(c) $\mathscr{W} = Q\mathscr{U}$ *for some WPS-2AB ideal Q of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U}) \subseteq Q$.

***Proof.*** $(a)\Leftrightarrow(b)$ Follows from Proposition 9
$(b)\Rightarrow(c)$ Let $\mathscr{W}$ be submodule of an $\mathscr{E}$-module $\mathscr{U}$, then $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U}$ . But $Q = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U}$ implies that Q is a WPS-2AB ideal of $\mathscr{E}$ with $ann_{\mathscr{E}}(\mathscr{U}) = [0 : \mathscr{U}] \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}] = Q$, that is $ann_{\mathscr{E}}(\mathscr{U}) \subseteq Q$.
$(c)\Rightarrow(b)$ Assume that $\mathscr{W} = Q\mathscr{U}$ for some WPS-2AB ideal Q of $\mathscr{E}$, with $ann_{\mathscr{E}}(\mathscr{U}) \subseteq Q$. Since $\mathscr{U}$ is Multiplication then $\mathscr{W} = [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]\mathscr{U} = Q\mathscr{U}$. Since $\mathscr{U}$ is faithful finitely generated $\mathscr{E}$-module, then by Lemma 33 $\mathscr{U}$ is a weak cancellation. Therefore $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] + ann_{\mathscr{E}}(\mathscr{U}) = Q + ann_{\mathscr{E}}(\mathscr{U})$. But $ann_{\mathscr{E}}(\mathscr{U}) \subseteq Q$ and $ann_{\mathscr{E}}(\mathscr{U}) \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$, that is $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}] = Q$. Since Q is a WPS-2AB ideal of $\mathscr{E}$. Then $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ is a WPS-2AB ideal of $\mathscr{E}$.

**Proposition 35.** *Let* $\mathscr{U}$ *be a finitely generated Multiplication (cyclic) projective* $\mathscr{E}$*-module and* $\mathscr{W} \subsetneq \mathscr{U}$ *be a submodule of* $\mathscr{U}$ *with* $ann_{\mathscr{E}}(\mathscr{U}) \subseteq [\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$. *The statements below are considered to be equivalent:*

(a) $\mathscr{W}$ *is a WPS-2AB submodule.*

(b) $[\mathscr{W} :_{\mathscr{E}} \mathscr{U}]$ *is a WPS-2AB ideal of* $\mathscr{E}$.

(c) $\mathscr{W} = Q\mathscr{U}$ *for some WPS-2AB ideal Q of* $\mathscr{E}$ *with* $ann_{\mathscr{E}}(\mathscr{U}) \subseteq Q$.

*Proof* $(a)\Leftrightarrow(b)$ Follows from Proposition 9.
$(b)\Leftrightarrow(c)$ Follows from Proposition 34.

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## Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Data Availability

The data and fundamental definitions used in this study are available in the published research article: Taha OH, Salih MA. *Weakly Pseudo Semi-2-absorbing Submodule.* The International Journal of Mathematics and Computer Science. 2024; 19 (4):927-32.

## Author Contributions

Author 1 contributed to the conceptualization and methodology; Author 2 performed the formal analysis; Author 3 was responsible for data curation and writing the original draft; all authors reviewed and edited the manuscript and approved the final version.

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