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ORIGINAL STUDY

# Weakly Pseudo Primary 2-absorbing Submodules

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**Abstract**

**In this paper, we introduce the notion of a weakly pseudo primary 2-absorbing sub-module as a generalization of a 2-absorbing sub-module and a pseudo 2-absorbing sub-module. Moreover, we give many basic properties, examples, and characterizations of these notions.**

*Subj-class something like:* 13A05, 13C13, 13C60, 13C10, 13E05, 13F15

*Keywords:* Primary- 2-absorbing, Pseudo -primary- 2-absorbing, 2-absorbing

## 1. Introduction

In this paper, all rings are commutative with $0 \neq 1$, and $D$ is a unitary $R$-module. During the last 13 years, the notion of 2-absorbing sub-modules and weakly 2-absorbing sub-modules has been previously investigated by Darani and Soheilinia [1], and its generalizations to where 2-absorbing ideal was first introduced in 2007 by Badawi [2]. Let $A \subsetneq D$ be sub-module of R-module D, $A$ is called 2-absorbing (weakly 2-absorbing) if whenever $r_1 r_2 d \in A$ ($0 \neq r_1 r_2 d \in A$) for some $r_1, r_2 \in R$ , $d \in D$, then $r_1 d \in A$ or $r_2 d \in$ A or $r_1 r_2 \in [A :_R D]$. Following that, Mostafanasab, Yetkin, Tekir and Daran [3] introduced the concept primary 2-absorbing it's generalizations of primary sub-module and weakly primary sub-module, respectively, a sub-module $A \subsetneq D$ of an $R-$module $D$ is called a primary 2-absorbing, if $r_1 r_2 d \in A$, for $r_1, r_2 \in R$, $d \in D$, implies that either $r_1 d \in rad_D(A)$ or $r_2 d \in rad_D(A)$ or $r_1 r_2 \in [A + Soc(D) :_R D]$. Also, Mohammadali and Abdulla [4] introduced the concept of pseudo primary 2-absorbing sub-module, a sub-module $A \subsetneq D$ of an $R-$module $D$, is called a pseudo primary 2-absorbing sub-module $D$, if $r_1 r_2 d \in A$, for $r_1, r_2 \in R$, $d \in D$, implies that either $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D) :_R D]$. In earlier years, there has been prior research that has addressed the following themes: pseudo-2-absorbing, pseudo semi 2-absorbing [5], and weakly pseudo semi 2-absorbing [6]. This paper is based on introducing a new class of sub-modules called weakly pseudo primary 2-absorbing sub-modules. Among many other results in this paper, we show in Proposition 2 and Remark 3 the relation between the above concepts with weakly pseudo primary 2-absorbing sub-module with examples. Also, we show that the intersection of each pair of distinct weakly pseudo primary 2-absorbing sub-modules is not a weakly pseudo primary 2-absorbing sub-module in general. Also, the residual of a weakly pseudo primary 2-absorbing sub-module is a weakly pseudo primary 2-absorbing ideal. To do this, specific requirements will be conditions. In Theorem 14, we prove $A$ is a weakly pseudo primary 2-absorbing sub-module of an $R$-module $D$ if and only if $(0) \neq Q_1 Q_2 T \subseteq A$ for some ideal $Q_1, Q_2$ of $R$ and sub-module $T$ of $D$, implies that either $Q_1 T \subseteq rad_D(A) + Soc(D)$ or $Q_2 T \subseteq rad_D(A) + Soc(D)$ or $Q_1 Q_2 \subseteq [A + Soc(D) :_R D]$.

## 2. Weakly pseudo primary 2-absorbing sub-module

**Definition 1.** Let $A \subsetneq D$ be a sub-module of an $R-$module $D$, we said $A$ be a weakly pseudo primary 2-absorbing sub-module $D$, if $0 \neq r_1 r_2 d \in A$, for $r_1$,

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$r_2 \in R$, $d \in D$, implies that either $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D){:}_R D]$. An ideal $Q$ of a ring $R$ is said to be a weakly pseudo primary 2-absorbing ideal of $R$ if $Q$ is a weakly pseudo primary 2-absorbing sub-module an $R-$module $R$.

**Proposition 2.** *Let $A \subsetneq D$ be a sub-module of an R-module D, then the following holds.*

1. *If A is a 2-absorbing (weakly 2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of, generally, the converse need not hold.*
2. *If A is a pseudo-2-absorbing (weakly pseudo-2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*
3. *If A is a primary 2-absorbing (weakly primary 2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*
4. *If A is a pseudo primary 2-absorbing sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*

*Proof.*

1. Assume that $A$ is 2-absorbing. To prove $A$ is a weakly pseudo primary 2-absorbing. Let $0 \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since $A$ is 2-absorbing, then ether $r_1 d \in A \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in A \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A \subseteq A + Soc(D)$. Now we have ether $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D){:}_R D]$. Thus $A$ is a weakly pseudo primary 2-absorbing. In regards to the contrary, the $12\mathbb{Z}$ is a proper sub-module of $\mathbb{Z}$-module $\mathbb{Z}$, $12\mathbb{Z}$ is weakly pseudo primary 2-absorbing sub-module of $\mathbb{Z}$-module $\mathbb{Z}$, since for all $0 \neq r_1 r_2 d \in 12\mathbb{Z}$ for $r_1$, $r_2$, $d \in \mathbb{Z}$, implies that either $r_1 d \in rad_{\mathbb{Z}}(12\mathbb{Z}) + Soc(\mathbb{Z}) = 6\mathbb{Z} + (0) = 6\mathbb{Z}$ or $r_2 d \in rad_{\mathbb{Z}}(12\mathbb{Z}) + Soc(\mathbb{Z}) = 6\mathbb{Z}$ or $r_1 r_2 \in [12\mathbb{Z} + Soc(\mathbb{Z}){:}_{\mathbb{Z}}\mathbb{Z}] = 12\mathbb{Z}$. For example, $0 \neq 2.2.3 \in 12\mathbb{Z}$ for $2, 3 \in \mathbb{Z}$, implies that $2.3 = 6 \in 6\mathbb{Z}$ but $2.2 = 4 \notin 12\mathbb{Z}$. But $12\mathbb{Z}$ is not 2-absorbing sub-module of $\mathbb{Z}$-module $\mathbb{Z}$ since $2.2.3 \in 12\mathbb{Z}$ for $2, 3 \in \mathbb{Z}$, but $2.3 = 6 \notin 12\mathbb{Z}$ and $2.2 = 4 \notin 12\mathbb{Z}$.
2. Assume that $A$ is pseudo-2-absorbing. To prove $A$ is weakly pseudo primary 2-absorbing. Let $0 \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since $A$ is pseudo-2-absorbing, then ether $r_1 d \in A + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in A + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A + Soc(D)$. Now we have ether $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D){:}_R D]$. Thus $A$ is a weakly pseudo primary 2-absorbing. In regards to the contrary, let $D = \mathbb{Z}$, $R = \mathbb{Z}$, and $A = 16\mathbb{Z}$. $A$ is proper sub-module of $D$, $A$ is weakly pseudo primary 2-absorbing, since for all $0 \neq r_1 r_2 d \in A$ for $r_1, r_2 \in R$, and $d \in D$ implies that either $r_1 d \in rad_D(A) + Soc(D) = 2\mathbb{Z} + (0) = 2\mathbb{Z}$ or $r_2 d \in rad_D(A) + Soc(D) = 2\mathbb{Z}$ or $r_1 r_2 \in [A + Soc(D){:}_R D] = 16\mathbb{Z}$. For example, $2.3.8 = 48 \in A$ for 2, $3 \in R$, $8 \in D$, implies that $2.8 = 16 \in 2\mathbb{Z}$ but $2.3 = 6 \notin 16\mathbb{Z}$. But $A$ is not pseudo-2-absorbing. Since $2.2.4 \in 16\mathbb{Z}$, where $2 \in R$ and $4 \in D$, but $2.4 = 8 \notin A + Soc(D) = 16\mathbb{Z} + (0) = 16\mathbb{Z}$ and $2.2 = 4 \notin [16\mathbb{Z} + Soc(D){:}_R D] = 16\mathbb{Z}$.
3. Assume that $A$ is a primary 2-absorbing sub-module of $R$-module $D$. Let $o \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since $A$ is a primary 2-absorbing, then either $r_1 d \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A \subseteq A + Soc(D)$. That is either $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D){:}_R D]$. Hence $A$ is a weakly pseudo primary 2-absorbing sub-module of $R$-module $D$. In regards to the contrary, In $\mathbb{Z}$ -module $\mathbb{Z}_{90}$, the $(\overline{30})$ is a proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}_{90}$. $(\overline{30})$ is weakly pseudo primary 2-absorbing sub-module since for all $r_1, r_2 \in \mathbb{Z}$ and $d \in \mathbb{Z}_{90}$ with $0 \neq r_1 r_2 d \in (\overline{30})$ we have either $r_1 d \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{30}) + (\overline{3}) = (\overline{3})$ or $r_2 d \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{3})$ or $r_1 r_2 \in [(\overline{30}) + Soc(\mathbb{Z}_{90}){:}_{\mathbb{Z}}\mathbb{Z}_{90}] = 3\mathbb{Z}$. Since $Soc(\mathbb{Z}_{90}) = (\overline{3})$ and $rad_D((\overline{30})) = (\overline{30})$. For example, $0 \neq 2.3.\overline{5} = \overline{30} \in (\overline{30})$, we have $3.\overline{5} = \overline{15} \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{30}) + (\overline{3}) = (\overline{3})$ and $2.3 = 6 \in 3\mathbb{Z}$.but $2.\overline{5} = \overline{10} \notin (\overline{3})$. It is clear that $(\overline{30})$ is not primary 2-absorbing sub-module since $2.3.\overline{5} = \overline{30} \in (\overline{30})$, $3.\overline{5} = \overline{15} \notin (\overline{30})$ and $2.\overline{5} = \overline{10} \notin (\overline{30})$ and $2.3 = 6 \notin 30\mathbb{Z}$.
4. Clear by Definition of weakly pseudo primary 2-absorbing sub-module.

**Remark 3.** The relation between the semi-2-absorbing (weakly semi-2-absorbing, pseudo-semi-2-absorbing and weakly pseudo-semi-2-absorbing) sub-modules with weakly pseudo primary 2-absorbing sub-module is independent. That is

1. If $A$ is a semi-2-absorbing (weakly semi-2-absorbing) sub-module of $R$-module $D$, it is not necessarily to be a weakly pseudo primary 2-absorbing sub-module of $R$-module $D$.
2. If $A$ be a weakly pseudo primary 2-absorbing sub-module of $R$-module $D$, it is not necessarily

to be semi-2-absorbing (weakly semi-2-absorbing) sub-module of $R$-module $D$.
3. If $A$ is a pseudo-semi-2-absorbing (weakly pseudo-semi-2-absorbing) sub-module of $R$-module $D$, it is not necessarily a weakly pseudo primary 2-absorbing sub-module of $R$-module $D$.
4. If $A$ is a weakly pseudo primary 2-absorbing sub-module of $R$-module $D$, it is not necessarily a pseudo-semi-2-absorbing (weakly pseudo-semi-2-absorbing) sub-module of $R$-module $D$.

The following Example 4 clear that.

**Example 4.**

1. Assume that $42\mathbb{Z}$ is proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$. $42\mathbb{Z}$ is semi-2-absorbing (weakly semi-2-absorbing)sub-module, since for all $r^2d\in 42\mathbb{Z}$, $r, d\in\mathbb{Z}$, implies that either $rd\in 42\mathbb{Z}$ or $r^2\in[42\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$. For example, $2^2.21 = 84\in 42\mathbb{Z}$, $2, 21\in\mathbb{Z}$ implies that $2.21 = 42\in 42\mathbb{Z}$ but $2^2 = 4\notin[42\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$. But $42\mathbb{Z}$ not weakly pseudo primary 2-absorbing sub-module, since $0\neq 2.3.7 = 42\in 42\mathbb{Z}$ for $2, 3, 7\in\mathbb{Z}$. But $2.7 = 14\notin rad_{\mathbb{Z}}(42\mathbb{Z}) + Soc(\mathbb{Z}) = 42\mathbb{Z} + (0) = 42\mathbb{Z}$, and $3.7 = 21\notin rad_{\mathbb{Z}}(42\mathbb{Z}) + Soc(\mathbb{Z}) = 42\mathbb{Z}$ and $2.3 = 6\notin[42\mathbb{Z} + Soc(\mathbb{Z}):_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$.
2. Assume that $16\mathbb{Z}$ is proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$. $16\mathbb{Z}$ is weakly pseudo primary 2-absorbing sub-module (by Proposition 2 [2]). But it is not a semi-2-absorbing (weakly semi-2-absorbing) sub-module, since $2^2.4 = 16\in 16\mathbb{Z}$ for $2, 4\in\mathbb{Z}$, but $2.4 = 8\notin 16\mathbb{Z}$ and $2^2 = 4\notin[16\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 16\mathbb{Z}$.
3. Similar [1] since $Soc(\mathbb{Z}) = 0$.
4. Similar [2] since $Soc(\mathbb{Z}) = 0$.

**Lemma 5.** [1] *On $m\mathbb{Z}$ is a 2-absorbing sub-module of $\mathbb{Z} -$ module $\mathbb{Z}$ if $m = 0$ or $m = p_1$ or $m = p_1p_2$ where $p_1, p_2$ are prime integers.*

**Proposition 6.** *$m\mathbb{Z}$ is a weakly pseudo primary 2-absorbing of $\mathbb{Z} -$ module $\mathbb{Z}$ if $m = p_1$, $\mathrm{m} = {p_1}^2$, $\mathrm{m} = p_1p_2$, where $p_1, p_2$ are prime integers.*

*Proof.* Since $n\mathbb{Z}$ is a 2-absorbing sub-module by Lemma 5 and since every 2-absorbing sub-module is weakly pseudo primary 2-absorbing by Proposition 2. Then $n\mathbb{Z}$ weakly pseudo primary 2-absorbing.

**Remark 7.** Let $A_1$ and $A_2$ are two distinct weakly pseudo primary 2-absorbing sub-modules of an $R$-module $D$. Then $A_1\cap A_2$ is not necessarily weakly pseudo primary 2-absorbing. the following Example 8 clear that.

**Example 8.** Let $D = \mathbb{Z}$, $R = \mathbb{Z}$, $A_1 = 7\mathbb{Z}$, $A_2 = 6\mathbb{Z}$ are two weakly pseudo primary 2-absorbing sub-modules of $\mathbb{Z}$-module $\mathbb{Z}$ by Proposition 6 But $A_1\cap A_2 = 42\mathbb{Z}$ is not weakly pseudo primary 2-absorbing, by *Example 4* [1].

**Remark 9.** The residual of a weakly pseudo primary 2-absorbing sub-module of an $R$-module $D$ is not necessarily to be a weakly pseudo primary 2-absorbing ideal of $R$. The following Example 10 clear that.

**Example 10.** $(\bar{0})$ is a proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}_{30}$. $(\bar{0})$ is a weakly pseudo primary 2-absorbing, by definition. But $[(\bar{0}):_{\mathbb{Z}}\mathbb{Z}_{30}] = 30\mathbb{Z}$ is not weakly pseudo primary 2-absorbing ideal of $\mathbb{Z}$. Since $2.3.5 = 30\in 30\mathbb{Z}$ for $2, 3, 5\in\mathbb{Z}$. But $2.5 = 10\notin rad_D(30\mathbb{Z}) + Soc(\mathbb{Z}) = 30\mathbb{Z} + (0) = 30\mathbb{Z}$ and $3.5 = 15\notin rad_D(30\mathbb{Z}) + Soc(\mathbb{Z}) = 30\mathbb{Z}$ and $2.3 = 6\notin[30\mathbb{Z} + Soc(\mathbb{Z}):_{\mathbb{Z}}\mathbb{Z}] = 30\mathbb{Z}$.

**Proposition 11.** *Let $A\subsetneq D$ be a sub-module of $R$-module $D$. $A$ is weakly pseudo primary 2-absorbing if and only if for any $r_1, r_2\in R$ such that $r_1r_2\notin[A + Soc(D):_R D]$ we have $[A:_D r_1r_2]\subseteq[0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$.*

*Proof.* Assume that $A$ is weakly pseudo primary 2-absorbing sub-module and let $x\in[A:_D r_1r_2]$ to prove $x\in[0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$. Then $r_1r_2x\in A$. If $0\neq r_1r_2x\in A$ and $r_1r_2\notin[A + Soc(D):_R D]$, it follows that $r_1x\in rad_D(A) + Soc(D)$ or $r_2x\in rad_D(A) + Soc(D)$ (since $A$ is weakly pseudo primary 2-absorbing). Thus either $x\in[rad_D(A) + Soc(D):_D r_1]$ or $x\in[rad_D(A) + Soc(D):_D r_2]$. If $0 = r_1r_2x\in A$ then $x\in[0:_D r_1r_2]$. Hence $x\in[0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$ That is $[A:_D r_1r_2]\subseteq[0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$.

Conversely, Assume that $0\neq r_1r_2d\in A$, for $r\in R$ , $d\in D$ and let $[A:_D r_1r_2]\subseteq[0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$ and $r_1r_2\notin[A + Soc(D):_R D]$. Then by our hypothesis $d\notin[0:_D r_1r_2]$ but $d\in[A:_D r_1r_2][0:_D r_1r_2]\cup[rad_D(A) + Soc(D):_D r_1]\cup[rad_D(A) + Soc(D):_D r_2]$. Then either $d\in[rad_D(A) + Soc(D):_D r_1]$ or $d\in[rad_D(A) + Soc(D):_D r_2]$, that is either $r_1d\in rad_D(A) + Soc(D)$ or $r_2d\in rad_D(A) + Soc(D)$. Therefore, $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$.

**Lemma 12.** *Let $A\subsetneq D$ be a sub-module of $R$-module $D$. $A$ is a weakly pseudo primary 2-absorbing sub-module of an $R$-module $D$ if and only if $(0)\neq r_1r_2T\subseteq A$ for $r_1r_2\in R$ and $T$ is sub-module of $D$, implies either $r_1T\subseteq rad_D$*

$(A) + Soc(D)$ *or* $r_2T \subseteq rad_D(A) + Soc(D)$ *or* $r_1r_2 \in [A + Soc(D):_R D]$.

*Proof.* Assume that $A$ is weakly pseudo primary 2-absorbing sub-module of an $R$-module $D$ and $(0) \neq r_1r_2T \subseteq A$ for $r_1r_2 \in R$ , $T$ is sub-module of $D$. Assume that $r_1T \nsubseteq rad_D(A) + Soc(D)$ and $r_2T \nsubseteq rad_D(A) + Soc(D)$ and $r_1r_2 \notin [A + Soc(D):_R D]$ . Then there exist $t_1$, $t_2 \in T$ such that either $r_1t_1 \notin rad_D(A) + Soc(D)$ or $r_2t_2 \notin rad_D(A) + Soc(D)$. Now since $0 \neq r_1r_2t_1 \in A$, $r_1r_2 \notin [A + Soc(D):_R D]$, and $A$ is weakly pseudo primary 2-absorbing sub-module. Then by Proposition 11, $t_1 \in [A:_D r_1r_2] \subseteq [0:_D r_1r_2] \cup [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$ implies that $t \in [0:_D r_1r_2] \cup [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$. But $r_1r_2t \neq 0$ and $r_1t_1 \notin rad_D(A) + Soc(D)$ , that is $t_1 \notin [0:_D r^2]$ and $t_1 \notin [rad_D(A) + Soc(D):_D r_1]$. Thus $t_1 \in [rad_D(A) + Soc(D):_D r_2]$, that is $r_2t_1 \in rad_D(A) + Soc(D)$. Also since $0 \neq r_1r_2t_2 \in A$ and $r_2t_2 \notin rad_D(A) + Soc(D)$ and $r_1r_2 \notin [A + Soc(D):_R D]$, it follows that $r_1t_2 \subseteq rad_D(A) + Soc(D)$. Now, $0 \neq r_1r_2(t_1 + t_2) \in A$ and $r_1r_2 \notin [A + Soc(D):_R D]$, implies that $t_1 + t_2 \in [A:_D r_1r_2]$ and $t_1t_2 \notin [0:_D r_1r_2]$. It follows by Proposition 11. we have $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$. That is either $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1]$ or $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_2]$. If $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1]$, that is $r_1(t_1 + t_2) = r_1t_1 + r_1t_2 \in rad_D(A) + Soc(D)$. and we have $r_1t_2 \subseteq rad_D(A) + Soc(D)$. That is $r_1t_1 \in rad_D(A) + Soc(D)$. which is a contradiction. If $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_2]$, that is $r_2(t_1 + t_2) = r_2t_1 + r_2t_2 \in rad_D(A) + Soc(D)$. and we have $r_2t_1 \subseteq rad_D(A) + Soc(D)$. That is $r_2t_2 \in rad_D(A) + Soc(D)$. which is a contradiction. Hence either $r_1T \subseteq rad_D(A) + Soc(D)$ or $r_2T \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \in [A + Soc(D):_R D]$.
Conversely, Assume that $0 \neq r_1r_2d \in A$, for $r_1, r_2 \in R$ , $d \in D$ that is $(0) \neq r_1r_2(d) \subseteq A$, hence by our hypothesis $r_1(d) \subseteq rad_D(A) + Soc(D)$ or $r_2(d) \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \in [A + Soc(D):_R D]$. Hence, $A$ weakly pseudo primary 2-absorbing sub-module.

**Proposition 13.** *Let* $A \subsetneq D$ *be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of a cyclic R-module D if and only if for* $r_1r_2 \in R$ *with* $r_1r_2 \notin [A + Soc(D):_R D]$ *we have* $[A:_R r_1r_2d] \subseteq [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$ *for some* $d \in D$.

*Proof.* Assume that $A$ is a weakly pseudo primary 2-absorbing sub-module of a cyclic $R$-module $D$ and for any $r_1r_2 \in R$ with $r_1r_2 \notin [A + Soc(D):_R D]$, let $x \in [A:_R r_1r_2d]$ to prove $x \in [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$. Implies that $r_1r_2xd \in A$, that is $r_1r_2(xd) \subseteq A$. If $0 \neq r_1r_2(xd) \subseteq A$ since $A$ is a weakly pseudo primary 2-absorbing sub-module and $r_1r_2 \notin [A + Soc(D):_R D]$, implies that either $r_1(xd) \subseteq rad_D(A) + Soc(D)$ or $r_2(xd) \subseteq rad_D(A) + Soc(D)$ (by Lemma 12) that is either $x \in [rad_D(A) + Soc(D):_R r_1d]$ or $x \in [rad_D(A) + Soc(D):_R r_2d]$. If $0 = r_1r_2(xd) \subseteq A$, implies that $x \in [0:_R r_1r_2d]$. That is $x \in [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$ Hence $[A:_R r_1r_2d] \subseteq [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$.
Conversely, Assume that $D = (d_1)$ for some $d_1 \in D$, and $0 \neq r_1r_2d \in A$, for $r_1, r_2 \in R$ , $d \in D$ with $r_1r_2 \notin [A + Soc(D):_R D]$. Since $d \in D$ then $d = xd_1$ for some $x \in R$. That is $0 \neq r_1r_2xd_1 \in A$. It follows $x \in [A:_R r_1r_2d_1] \subseteq [0:_R r_1r_2d_1] \cup [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$. Then $x \in [0:_R r_1r_2d_1] \cup [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$ but $x \notin [0:_R r_1r_2d_1]$ (sines $0 \neq r_1r_2xd_1$). Then $x \in [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$. Then either $x \in [rad_D(A) + Soc(D):_R r_1d_1]$ or $x \in [rad_D(A) + Soc(D):_R r_2d_1]$. That is either $r_1xd_1 \in rad_D(A) + Soc(D)$ or $r_2xd_1 \in rad_D(A) + Soc(D)$ Then either $r_1d \in rad_D(A) + Soc(D)$ or $r_2d \in rad_D(A) + Soc(D)$ Therefore $A$ is a weakly pseudo primary 2-absorbing sub-module of a cyclic $R$-module $D$.

**Theorem 14.** *Let* $A \subsetneq D$ *be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of D if and only if* $(0) \neq Q_1Q_2T \subseteq A$ *for some ideal* $Q_1, Q_2$ *of R and sub-module T of D, implies that either* $Q_1T \subseteq rad_D(A) + Soc(D)$ *or* $Q_2T \subseteq rad_D(A) + Soc(D)$ *or* $Q_1Q_2 \subseteq [A + Soc(D):_R D]$.

*Proof.* Assume that $(0) \neq Q_1Q_2L \subseteq A$ for some ideal $Q_1$, $Q_2$ of $R$ and sub-module $T$ of $D$. With $Q_1Q_2 \nsubseteq [A + Soc(D):_R D]$, to prove $Q_1T \subseteq rad_D(A) + Soc(D)$ or $Q_2T \subseteq rad_D(A) + Soc(D)$. Suppose that $Q_1T \nsubseteq rad_D(A) + Soc(D)$ and $Q_2T \nsubseteq rad_D(A) + Soc(D)$. That is there exist $n_1 \in Q_1$ and $m_1 \in Q_2$, such that $n_1T \nsubseteq rad_D(A) + Soc(D)$ and $m_1T \nsubseteq rad_D(A) + Soc(D)$. Now, $(0) \neq n_1m_1T \subseteq A$. Since $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$. Then by Lemma 12, either $n_1T \subseteq rad_D(A) + Soc(D)$ or $m_1T \subseteq rad_D(A) + Soc(D)$ or $n_1m_1 \in [A + Soc(D):_R D]$. Since $Q_1Q_2 \nsubseteq [A + Soc(D):_R D]$. Then there exist $n_2 \in Q_1$ and $m_2 \in Q_2$, such that $n_2m_2 \notin [A + Soc(D):_R D]$. But $0 \neq n_2m_2T \in A$ and $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$ with $n_2m_2 \notin [A + Soc(D):_R D]$. Then by Lemma 12, either $n_2T \subseteq rad_D(A) + Soc(D)$ or $m_2T \subseteq rad_D(A) + Soc(D)$. Now, we have three cases.

**Case 1.** If $n_2T \subseteq rad_D(A) + Soc(D)$ and $m_2T \nsubseteq rad_D(A) + Soc(D)$. Since $(0) \neq n_1m_2T \in A$ , $m_2T \nsubseteq rad_D(A) + Soc(D)$ and $n_1T \nsubseteq rad_D(A) + Soc(D)$, so that by Lemma 12, we have $n_1m_2 \in [A + Soc(D):_R D]$. Since $n_2T \subseteq rad_D(A) + Soc(D)$ and $n_1T \nsubseteq rad_D(A) + Soc(D)$. We get $(n_1 + n_2)T \nsubseteq rad_D(A) + Soc(D)$. on the other side $(0) \neq (n_1 + n_2)m_2T \subseteq A$, and $A$ is a weakly pseudo primary 2-absorbing sub-module, $(n_1 + n_2)T \nsubseteq rad_D$

$(A) + Soc(D)$ and $m_2T \nsubseteq rad_D(A) + Soc(D)$. Then by Lemma 12, $(n_1 + n_2)m_2 = n_1m_2 + n_2m_2 \in [A + Soc(D):_R D]$. But $n_1m_2 \in [A + Soc(D):_R D]$ we have that $n_2m_2 \in [A + Soc(D):_R D]$, which is a contradiction.

**Case 2.** If $m_2T \subseteq rad_D(A) + Soc(D)$ and $n_2T \nsubseteq rad_D(A) + Soc(D)$. Similarly, to case 1, we get a contradiction.

**Case 3.** If $n_2T \subseteq rad_D(A) + Soc(D)$ and $m_2T \subseteq rad_D(A) + Soc(D)$, since $m_2T \subseteq rad_D(A) + Soc(D)$ and $m_1T \nsubseteq rad_D(A) + Soc(D)$, we get $(m_1 + m_2)T \nsubseteq rad_D(A) + Soc(D)$. But $(0) \neq n_1(m_1 + m_2)T \subseteq A$ and $A$ is a weakly pseudo primary 2-absorbing sub-module with $n_1T \nsubseteq rad_D(A) + Soc(D)$ and $(m_1 + m_2)T \nsubseteq rad_D(A) + Soc(D)$. Then by Lemma 12, we get $n_1(m_1 + m_2) \in [A + Soc(D):_R D]$. Since $n_1m_1 \in [A + Soc(D):_R D]$ and $n_1m_1 + n_1m_2 \in [A + Soc(D):_R D]$, implies that $n_1m_2 \in [A + Soc(D):_R D]$. Since $(0) \neq (n_1 + n_2)m_1T \in A$ and $m_1T \nsubseteq rad_D(A) + Soc(D)$ and $(n_1 + n_2)T \nsubseteq rad_D(A) + Soc(D)$ and $A$ is a weakly pseudo primary 2-absorbing sub-module. Then by Lemma 12, we get $(n_1 + n_2)m_1 \in [A + Soc(D):_R D]$. But $n_1m_1 + n_1m_1 \in [A + Soc(D):_R D]$, and since $n_1m_1 \in [A + Soc(D):_R D]$, we get $n_2m_1 \in [A + Soc(D):_R D]$. Since $(0) \neq (n_1 + n_2)(m_1 + m_2)T \subseteq A$ and $(n_1 + n_2)T \nsubseteq [A + Soc(D):_R D]$ and $(m_1 + m_2)T \nsubseteq [A + Soc(D):_R D]$. Then by Lemma 12, we get $(n_1 + n_2)(m_1 + m_2) = n_1m_1 + n_1m_2 + n_2m_1 + n_2m_2 \in [A + Soc(D):_R D]$. Since $n_1m_1 \in [A + Soc(D):_R D]$, $n_2m_1 \in [A + Soc(D):_R D]$ and $n_1m_2 \in [A + Soc(D):_R D]$, we get $n_2m_2 \in [A + Soc(D):_R D]$, which is a contradiction. Consequently either $Q_1T \subseteq rad_D(A) + Soc(D)$ or $Q_2T \subseteq rad_D(A) + Soc(D)$ .

Conversely, assume that $0 \neq r_1r_2d \subseteq A$ for $r_1, r_2 \in R$ and $d \in D$, that is $0 \neq (r_1)(r_2)T \subseteq A$. Then by hypothesis either $(r_1)T \subseteq rad_D(A) + Soc(D)$ or $(r_2)T \subseteq rad_D(A) + Soc(D)$ or $(r_1)(r_2) \subseteq [A + Soc(D):_R D]$. That is either $r_1T \subseteq rad_D(A) + Soc(D)$ or $r_2T \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \subseteq [A + Soc(D):_R D]$. Then by Lemma 12, we get $A$ is a weakly pseudo primary 2-absorbing sub-module of a $R$-module $D$.

The following corollaries are direct consequences of a *Theorem 14*.

**Corollary 16.** *Let $A \subsetneq D$ be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0) \neq Q_1Q_2d \subseteq A$ for some ideals $Q_1.Q_2$ of R and $d \in D$, implies that $Q_1d \subseteq rad_D(A) + Soc(D)$ or $Q_2d \subseteq rad_D(A) + Soc(D)$ or $Q_1Q_2 \subseteq [A + Soc(D):_R D]$.*

**Corollary 17.** *Let $A \subsetneq D$ be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0) \neq rQT \subseteq A$, for some $r \in R$ and for some ideal Q of R and sub-module T of D, implies that $rT \subseteq rad_D(A) + Soc(D)$ or $QT \subseteq rad_D(A) + Soc(D)$ or $rQ \subseteq [A + Soc(D):_R D]$.*

**Corollary 18.** *Let $A \subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0) \neq rQd \subseteq A$, for some $r \in R$ and for some ideal Q of R and $d \in D$, implies that $rQ \subseteq rad_D(A) + Soc(D)$ or $Qd \subseteq rad_D(A) + Soc(D)$ or $rQ \subseteq [A + Soc(D):_R D]$.*

**Remark 19.** [11] Suppose that $R$-module $D$ is semi-simple if and only if for any sub-module $A$ of $D$, $Soc\left(\frac{D}{A}\right) = \frac{Soc(D)+A}{A}$.

**Proposition 20.** *Let $A \subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of D with D is semi-simple, and T is sub-module of D with $T \subseteq A$, then $\frac{A}{T}$ is weakly pseudo primary 2-absorbing sub-module of an R-module $\frac{D}{T}$.*

*Proof.* Assume that $A$ is a weakly pseudo primary 2-absorbing sub-module of an $R$- module $D$ and $T$ is a sub-module of $D$ and let $T \subseteq A$. let $0 \neq r_1r_2(d + T) = r_1r_2d + T \in \frac{A}{T}$ with $r_1, r_2 \in R$ and $d + T \in \frac{D}{T}$, $d \in D$, implies that $r_1r_2d \in A$ If $r_1r_2d = 0$ then $r_1r_2(d + T) = 0$ which is contradiction. Thus $0 \neq r_1r_2d \in A$ and since $A$ is a weakly pseudo primary 2-absorbing sub-module then either $r_1d \in rad_D(A) + Soc(D) = rad_D(A) + T + Soc(D)$ or $r_2d \in rad_D(A) + Soc(D) = rad_D(A) + T + Soc(D)$ or $r_1r_2D \subseteq A + Soc(D)$. Since ($T \subseteq A \subseteq rad_D(A)$ then $rad_D(A) + T = rad_D(A)$). It follows that is either $r_1(d + T) \in \frac{rad_D(A)+T+Soc(D)}{T}$ or $r_2(d + T) \in \frac{rad_D(A)+T+Soc(D)}{T}$ or $r_1r_2\frac{D}{T} \subseteq \frac{A+T+Soc(D)}{T}$. That is either $r_1(d + T) \in \frac{rad_D(A)}{T} + \frac{T+Soc(D)}{T}$ or $r_2(d + T) \in \frac{rad_D(A)}{T} + \frac{T+Soc(D)}{T}$ or $r_1r_2\frac{D}{T} \subseteq \frac{A}{T} + \frac{A+Soc(D)}{T}$. since $D$ is semi-simple then $Soc\left(\frac{D}{T}\right) = \frac{T+Soc(D)}{T}$ by Remark 19 that is either $r_1(d + T) \subseteq rad_{\frac{D}{T}}\frac{A}{T} + Soc\left(\frac{D}{T}\right)$ or $r_2(d + T) \subseteq rad_{\frac{D}{T}}\frac{A}{T} + Soc\left(\frac{D}{T}\right)$ or $r_1r_2 \in \left[\frac{A}{T} + Soc\left(\frac{D}{T}\right):_R \frac{D}{T}\right]$ Hence $\frac{A}{T}$ is weakly pseudo primary 2-absorbing sub-module of an $R$-module $\frac{D}{T}$.

**Proposition 21.** *Let D be a semi-simple R- module, and $A, B$ are sub-modules of R- module D with $B \subseteq A$ and $rad_D(B) \subseteq rad_D(A)$. If B and $\frac{A}{B}$ are weakly pseudo primary 2-absorbing sub-modules of D, $\frac{D}{B}$ respectively, then A is weakly pseudo primary 2-absorbing sub-module.*

*Proof.* Assume that $0 \neq r_1r_2d \in A$ for $r_1, r_2 \in R$ and $d \in D$, then $0 \neq r_1r_2(d + B) \in \frac{A}{B}$ If $0 \neq r_1r_2d \in B$ and $B$ is weakly pseudo primary 2-absorbing sub-module, implies that either $r_1d \in rad_D(B) + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_2d \in rad_D(B) + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_1r_2D \subseteq B + Soc(D) \subseteq A + Soc(D)$, thus $A$ is a weakly pseudo primary 2-absorbing sub-module. Assume that $r_1r_2d \notin B$, it follows that $0 \neq r_1r_2(d+B) \in \frac{A}{B}$, since $\frac{A}{B}$ is weakly pseudo primary 2-absorbing sub-module of $\frac{D}{B}$, implies that either $r_1(d+B) \subseteq rad_{\frac{D}{B}}(\frac{A}{B}) + Soc(\frac{D}{B})$ or $r_2(d+B) \subseteq rad_{\frac{D}{B}}(\frac{A}{B}) + Soc(\frac{D}{B})$ or $r_1r_2\frac{D}{B} \subseteq \frac{A}{B} + Soc(\frac{D}{B})$, since $D$ is semi-simple then $Soc(\frac{D}{B}) = \frac{B+Soc(D)}{B}$ by Remark 19 it follows that either $r_1(d+B) \subseteq \frac{rad_D(A)}{B} + \frac{B+Soc(D)}{B}$ or $r_2(d+B) \subseteq \frac{rad_D(A)}{B} + \frac{B+Soc(D)}{B}$ or $r_1r_2\frac{D}{B} \subseteq \frac{A}{B} + \frac{B+Soc(D)}{B}$, since $B \subseteq A$ implies that $B + Soc(D) \subseteq A + Soc(D)$, hence either $r_1(d+B) \subseteq \frac{rad_D(A)}{B} + \frac{A+Soc(D)}{B} = \frac{rad_D(A)+Soc(D)}{B}$ or $r_2(d+B) \subseteq \frac{rad_D(A)}{B} + \frac{A+Soc(D)}{B} = \frac{rad_D(A)+Soc(D)}{B}$ or $r_1r_2\frac{D}{B} \subseteq \frac{A}{B} + \frac{A+Soc(D)}{B}$, implies that either $r_1(d+B) \subseteq \frac{rad_D(A)+Soc(D)}{B}$ or $r_2(d+B) \subseteq \frac{rad_D(A)+Soc(D)}{B}$ or $r_1r_2\frac{D}{B} \subseteq \frac{A+Soc(D)}{B}$, it follows either $r_1d \subseteq rad_D(A) + Soc(D)$ or $r_2d \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \in [A + Soc(D) :_R D]$ Therefore, $A$ is a weakly pseudo primary 2-absorbing sub-module.

**Lemma 22.** [7] *Suppose that $D$ be an R-module, $T$ is essential sub-module of $D$, Then $Soc(D) = Soc(T)$.*

**Proposition 23.** *Suppose that $A$ and $T$ are sub-modules of R-modules $D$, with $A$ is a proper sub-module of $T$, $rad_D(T) \subseteq rad_T(A)$, and $T$ an essential sub-module of $D$. If $T$ is a weakly pseudo primary 2-absorbing sub-module of $D$, then $A$ is a weakly pseudo primary 2-absorbing sub-module of $T$.*

*Proof.* Assume that $(0) \neq r_1r_2L \in A$ for $r_1, r_2 \in R$, $L$ is sub-module of $T$, that is $L$ is a sub-module of $D$ and $(0) \neq r_1r_2L \in T$. Since $T$ is weakly pseudo primary 2-absorbing sub-module of $D$, it follows that by *Lemma 16* either $r_1L \subseteq rad_D(T) + Soc(D)$ or $r_2L \subseteq rad_D(T) + Soc(D)$ or $r_1r_2 \in [T + Soc(D) :_R D]$. But $T$ is essential sub-module of $D$, then by Proposition *Lemma 26* $Soc(D) = Soc(T)$, and since $rad_D(T) \subseteq rad_T(A)$ Thus either $r_1L \subseteq rad_T(A) + Soc(T)$ or $r_2L \subseteq rad_T(A) + Soc(T)$ or $r_1r_2 \in [A + Soc(T) :_R D] \subseteq [A + Soc(T) :_R T]$. Hence, $A$ is a weakly pseudo primary 2-absorbing sub-module of $T$.

**Lemma 24.** [7] *[Module Law] Suppose that A, B and C are a sub-module of an R−module D, with $B \subseteq C$. Then $(A + B) \cap C = (A \cap C) + B = (A \cap C) + (B \cap C)$.*

**Lemma 25.** [7] *Suppose that $A$ is a sub-module of an R−module D, then $Soc(A) = A \cap Soc(D)$.*

**Proposition 26.** *Assume that $A$ and $B$ are sub-modules of R-module D, with B is no contained in A and $Soc(D) \subseteq B$. If A is a weakly pseudo primary 2-absorbing sub-module of D, Then $A \cap B$ is a weakly pseudo primary 2-absorbing sub-module of B.*

*Proof.* Since $B$ is no contained in $A$, then $A \cap B$ is a proper sub-module of $B$. Let $(o) \neq r_1r_2L \subseteq A \cap B$ for some $r_1, r_2 \in R$ and $L$ is a sub-module of $B$, that is $L$ is a sub-module of $D$. It follows that $o \neq r_1r_2L \subseteq A$, But $A$ is a weakly pseudo primary 2-absorbing sub-module, then by Lemma 12 either $r_1L \subseteq rad_D(A) + Soc(D)$ or $r_2L \subseteq rad_D(A) + Soc(D)$ or $r_1r_2D \subseteq A + Soc(D)$. so then either $r_1L \subseteq (rad_D(A) + Soc(D)) \cap B$ or $r_2L \subseteq (rad_D(A) + Soc(D)) \cap B$ or $r_1r_2D \subseteq (A + Soc(D)) \cap B$. since $Soc(D) \subseteq B$, then by module law (Lemma 24). either $r_1L \subseteq (rad_D(A) \cap B) + (B \cap Soc(D))$ or $r_2L \subseteq (rad_D(A) \cap B) + (B \cap Soc(D))$ or $r_1r_2D \subseteq (A \cap B) + (B \cap Soc(D))$. Thus by Lemma 25 $B \cap Soc(D) = Soc(B)$. it follows that either $r_1L \subseteq (rad_D(A) \cap B) + Soc(B)$ or $r_2L \subseteq (rad_D(A) \cap B) + Soc(B)$ or $r^2D \subseteq (A \cap B) + Soc(B)$. Since $B \subseteq rad_D(B)$ and $rad_D(A) \cap rad_D(B) = rad_D(A \cap B)$. Then either $r_1Q \subseteq rad_D(A \cap B) + Soc(B)$ or $r_2Q \subseteq rad_D(A \cap B) + Soc(B)$ or $r^2D \subseteq (A \cap B) + Soc(B)$. Hence, $A \cap B$ is a weakly pseudo primary 2-absorbing sub-module of $B$.

Under certain conditions, the intersection of each pair of distinct weakly pseudo primary 2-absorbing sub-module is a weakly pseudo primary 2-absorbing sub-module in general.

**Proposition 27.** *Assume that A and B are weakly pseudo primary 2-absorbing sub-modules of R-module D, with B is no contained in A and either $Soc(D) \subseteq A$ or $Soc(D) \subseteq B$. Then $A \cap B$ is a weakly pseudo primary 2-absorbing sub-module of D.*

*Proof.* Since $B$ is not contained in $A$ and $B$ is a proper sub-module of $D$, it implies that $A \cap B$ is a proper sub-module of $D$. Assume that $Soc(D) \subseteq A$ But $Soc(D) \nsubseteq B$. Let $o \neq Q_1Q_2L \subseteq A \cap B$ for some $Q_1, Q_2$ be an ideal in $R$ and $L$ is a sub-module of $B$, that is, $L$ is a sub-module of $D$. It follows that $o \neq Q_1Q_2L \subseteq A$ and $o \neq Q_1Q_2L \subseteq B$, But $A$ and $B$ are a weakly pseudo primary 2-absorbing sub-module, then by Theorem 14 either $Q_1L \subseteq rad_D(A) + Soc(D)$ or $Q_2L \subseteq rad_D(A) + Soc(D)$ or $Q_1Q_2D \subseteq A + Soc(D)$ and either $Q_1L \subseteq rad_D(B) + Soc(D)$ or $Q_2L \subseteq rad_D(B) + Soc(D)$ or $Q_1Q_2\ D \subseteq B + Soc(D)$. it follows that either $Q_1L \subseteq (rad_D(A) + Soc(D)) \cap (rad_D(B) + Soc(D))$ or $Q_2L \subseteq (rad_D(A) + Soc(D)) \cap (rad_D(B) + Soc(D))$ or $Q_1Q_2D \subseteq (A + Soc(D)) \cap (B + Sco(D))$. since $Soc(D) \subseteq B$. Then, by module law (Lemma 24). either $Q_1L \subseteq (rad_D(A) \cap rad_D(B)) + Soc(D)$ or $Q_2L \subseteq (rad_D(A) \cap rad_D(B)) + Soc(D)$ or $Q_1Q_2D \subseteq (A \cap B) + Soc(D)$. That is either $Q_1L \subseteq rad_D(A \cap B) + Soc(D)$ or $Q_2L \subseteq rad_D(A \cap B) + Soc(D)$ or $Q_1Q_2D \subseteq (A \cap B) + Soc(D)$. By Theorem 14 Hence, $A \cap B$ is a weakly pseudo primary 2-absorbing sub-module of $D$. similar if $Soc(D) \subseteq B$.

**Lemma 28.** [7] *Suppose that M and N be R−module and $f : M \to N$ be an R-epimorphism, then $f(Soc(M)) \subseteq Soc(N)$.*

**Lemma 29.** [7] *Suppose that* $f: D_1 \to D_2$ *be an R-epimorphism, with* $kef(f) \subseteq A$ *then* $f(rad_{D_1}(A)) \subseteq rad_{D_2}(f(A))$.

**Proposition 30.** *Suppose that* $f: D_1 \to D_2$ *be an R-epimorphism and A is a weakly pseudo primary 2-absorbing sub-module of* $D_1$, *with* $ker(f) \subseteq A$ *then* $f(A)$ *is a weakly pseudo primary 2-absorbing sub-module of* $D_2$.

*Proof.* It's clear that $f(A)$ is a sub-module of $D_2$. Let $o \neq r_1 r_2 d_2 \in f(A)$ for some $r_1, r_2 \in R$ and $d_2 \in D_2$, and let $r_1 r_2 \notin [f(A) + Soc(D_2) :_R D_2]$ to prove either $r_1 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$ or $r_2 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. Since $f$ is onto then $o \neq r_1 r_2 f(d_1) \in f(A)$ for some $d_1 \in D_1$ that is $f(r_1 r_2 d_1) = f(q)$ for some $q \in A$. That is $r_1 r_2 d_1 - q \in ker(f) \subseteq A$. implies that $r_1 r_2 d_1 \in A$, that is $d_1 \in [A :_{D_1} r_1 r_2]$, it follows by Proposition 11 then $d_1 \in [A :_{D_1} r_1 r_2] \subseteq [0 : D_1 r_1 r_2] \cup [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_1] \cup [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_2]$. But $A$ is a weakly pseudo primary 2-absorbing sub-module of $D_1$ so $r_1 r_2 d_1 \neq 0$ that is $d_1 \notin [0 :_{D_1} r_1 r_2]$, therefore either $d_1 \in [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_1]$, or $d_1 \in [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_2]$. That is either $r_1 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_2 d_1 \in rad_{D_1}(A) + Soc(D_1)$. So either $r_1 f(d_1) \in f(rad_{D_1}(A)) + f(Soc(D_1)) \subseteq rad_{D_2}(f(A)) + Soc(D_2)$ or $r_2 f(d_1) \in f(rad_{D_1}(A)) + f(Soc(D_1)) \subseteq rad_{D_2}(f(A)) + Soc(D_2)$ (by Lemma 28 $f(Soc(D_1)) \subseteq Soc(D_2)$ and Lemma 29 $f(rad_{D_1}(A)) \subseteq rad_{D_2}(f(A))$). Thus either $r_1 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. $r_2 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. Therefore, $f(A)$ is a weakly pseudo primary 2-absorbing sub-module of $D_2$.

**Proposition 31.** *Suppose that* $f: D_1 \to D_2$ *be an R-epimorphism and A is a weakly pseudo primary 2-absorbing sub-module of* $D_2$, *then* $f^{-1}(A)$ *is a weakly pseudo primary 2-absorbing sub-module of* $D_1$.

*Proof.* Its clear that $f^{-1}(A)$ is sub-module of $D_1$. Let $o \neq r_1 r_2 d \in f^{-1}(A)$ for some $r_1, r_2 \in R$ and $d \in D_1$, then $0 \neq r_1 r_2 f(d) \in A$ since $A$ is a weakly pseudo primary 2-absorbing sub-module of $D_2$, implies that either $r_1 f(d) \in rad_{D_2}(A) + Soc(D_2)$ or $r_2 f(d) \in rad_{D_2}(A) + Soc(D_2)$ or $r_1 r_2 D_2 \subseteq A + Soc(D_2)$. Since $f$ is $R$-epimorphism then $D_2 = f(D)$. It follows that either $r_1 d \in f^{-1} rad_{D_2}(A) + f^{-1}(Soc(D_2)) \subseteq rad_{D_1}(f^{-1}(A)) + Soc(D)$ or $r_2 d \in f^{-1} rad_{D_2}(A) + f^{-1}(Soc(D_2)) \subseteq rad_{D_1}(f^{-1}(A)) + Soc(D)$ or $r_1 r_2 D \subseteq f^{-1}(A) + Soc(D)$. (by Lemma 28 and Lemma 29). Hence $f^{-1}(A)$ is a weakly pseudo primary 2-absorbing sub-module of $D_1$.

**Lemma 32.** [8] *Let* $V = \bigoplus_{i \in \wedge} V_i$ *be an R-module and* $V_i$ *is an R-module for each* $i \in \wedge$, *then* $Soc(V) = \bigoplus_{i \in \wedge} Sco(V_i)$.

**Proposition 33.** *Suppose that* $D = D_1 \oplus D_2$ *be an R-module with* $D_1, D_2$ *be R-module and* $A = A_1 \oplus A_2$ *be sub-module of D, where* $A_1, A_2$ *sub-modules of* $D_1, D_2$ *respectively and* $rad_D(A) \subseteq Soc(D)$. *If A is a weakly pseudo primary 2-absorbing sub-module of D, then* $A_1$ *and* $A_2$ *are weakly pseudo primary 2-absorbing sub-modules of* $D_1$ *and* $D_2$ *respectively.*

*Proof.* Assume that $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$ and let $o \neq r_1 r_2 d_1 \in A_1$ for some $r_1 r_2 \in R$ and $d_1 \in D_1$, it follows that $(0,0) \neq r_1 r_2 (d_1, 0) \in A_1 \oplus A_2$ , but $A = A_1 \oplus A_2$ is a weakly pseudo primary 2-absorbing sub-module of $D$, then either $r_1(d_1, 0) \in rad_D(A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$ or $r_2(d_1, 0) \in rad_D(A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$ or $r_1 r_2 D \subseteq (A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$. But $rad_D(A) \subseteq Soc(D)$, then $rad_D(A) + Soc(D) = Soc(D) = Soc(D_1 \oplus D_2) = Soc(D_1) \oplus Soc(D_2)$. (by Lemma 32). Thus either $r_1(d_1, 0) \in Soc(D_1) \oplus Soc(D_2)$ or $r_2(d_1, 0) \in Soc(D_1) \oplus Soc(D_2)$ or $r_1 r_2 (D_1 \oplus D_2) \subseteq Soc(D_1) \oplus Soc(D_2)$, it follows that either $rd_1 \in Soc(D_1) \subseteq rad_{D_1}(A_1) + Soc(D_1)$ or $rd_2 \in Soc(D_1) \subseteq rad_{D_1}(A_1) + Soc(D_1)$ or $r_1 r_2 D \subseteq Soc(D_1) \subseteq A_1 + Soc(D_1)$. Hence, $A_1$ is a weakly pseudo primary 2-absorbing sub-module of $D_1$. In the same way $A_2$ is a weakly pseudo primary 2-absorbing sub-module of $D_2$.

**Proposition 34.** *Suppose that* $D = D_1 \oplus D_2$ *be an R-module with* $D_1, D_2$ *be R-module and A is a proper sub-module of* $D_1$, *with* $rad_{D_1}(A) \subseteq Soc(D_1)$, *and* $Soc(D_2) = D_2$. *if A is a weakly pseudo primary 2-absorbing sub-module of* $D_1$ *then* $A \oplus D_2$ *is a weakly pseudo primary 2-absorbing sub-module of D.*

*Proof.* Assume that $(0,0) \neq r_1 r_2 (d_1, d_2) \in A \oplus D_2$ for $r_1, r_2 \in R$ $(d_1, d_2) \in D_1 \oplus D_2$ where $0 \neq d_1 \in D_1$, $0 \neq d_2 \in D_2$, implies that $0 \neq r_1 r_2 d_1 \in A$ but $A$ is a weakly pseudo primary 2-absorbing sub-module of $D_1$, implies that either $r_1 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_2 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_1 r_2 D_1 \subseteq A + Soc(D_1)$. Since $rad_{D_1}(A) \subseteq Soc(D_1)$, then $rad_{D_1}(A) + Soc(D_1) = Soc(D_1)$, it follows that either $r_1 d_1 \in Soc(D_1)$ or $r_2 d_1 \in Soc(D_1)$ or $r_1 r_2 D_1 \subseteq Soc(D_1)$. But $Soc(D_2) = D_2$, so we have either $r_1(d_1, d_2) \in Soc(D_1) \oplus Soc(D_2) = Soc(D_1 \oplus D_2) \subseteq rad_D(A \oplus D_2) + Soc(D)$ or $r_2(d_1, d_2) \in Soc(D_1) \oplus Soc(D_2) = Soc(D_1 \oplus D_2) \subseteq rad_D(A \oplus D_2) + Soc(D)$ or $r_1 r_2 (D_1 \oplus D_2) \subseteq Soc(D_1) \oplus Soc(D_2) = Soc(D) \subseteq A \oplus D_2 + Soc(D)$. (by *Lemma 34*) Hence, $A \oplus D_2$ is a weakly pseudo primary 2-absorbing sub-module of $D$.

**Proposition 35.** *Let* $A \subsetneq D$ *is a sub-module of R- module D, with* $rad_D(A)$ *is 2-absorbing sub-module of D. Then A is a weakly pseudo primary 2-absorbing sub-module of D.*

*Proof.* Assume that $0 \neq rQd \in A$ for $r \in R$, $Q$ be an ideal of $R$ and $d \in D$ with $rQD \nsubseteq A + Soc(D)$. Then $rQd \in rad_D(A)$, since $rad_D(A)$ is 2-absorbing sub-module of $D$, by Corollary 18 that is either $rd \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $Qd \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $rQD \subseteq A \subseteq A + Soc(D)$ then either $rd \in$

$rad_D(A) + Soc(D)$ or $Qd \subseteq rad_D(A) + Soc(D)$ or $rQD \subseteq A + Soc(D)$ Hence, by Corollary 18, $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$.

**Proposition 36.** *Let D be an R- module, with $Soc(A)$ is a 2-absorbing sub-module of D. If $A \subsetneq D$ is a sub-module of D such that $A \subseteq Sco(D)$, Then A is a weakly pseudo primary 2-absorbing sub-module of D.*

*Proof.* Assume that $0 \neq rQT \subseteq A$ for $r \in R$, $Q$ be an ideal of $R$, and T is sub-module of $D$, since $A \subseteq Sco(D)$ Then $rQT \subseteq Soc(D)$, since $Soc(D)$ is 2-absorbing sub-module of $D$, and by Corollary 17 that is either $rT \in Soc(D) \subseteq rad_D(A) + Soc(D)$ or $QT \in Soc(D) \subseteq rad_D(A) +\ Soc(D)$ or $rQD \subseteq Soc(D) = A + Soc(D)$. then either $rT \subseteq rad_D(A) + Soc(D)$ or $QT \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A + Soc(D)$. Hence, by Corollary 17, $A$ is a weakly pseudo primary 2-absorbing sub-module of $D$.

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