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# δ-Small Intersection Graphs of Modules

Ahmed H. Alwan
*Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq*

ORIGINAL STUDY

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Ahmed H. Alwan

Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

**Abstract**

**Let $R$ be a commutative ring with unit and $U$ be a unitary left $R$-module. The $\delta$-small intersection graph of non-trivial submodules of $U$, denoted by $\Gamma_\delta(U)$, is an undirected simple graph whose vertices are the non-trivial submodules of $U$, and two vertices are adjacent if and only if their intersection is a $\delta$-small submodule of $U$. In this article, we study the interplay between the algebraic properties of $U$, and the graph properties of $\Gamma_\delta(U)$ such as connectivity, completeness and planarity. Moreover, we determine the exact values of the diameter and girth of $\Gamma_\delta(U)$, as well as give a formula to compute the clique and domination numbers of $\Gamma_\delta(U)$.**

*Keywords:* Module, δ-Small intersection graph, Connectivity, Domination, Planarity

## 1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. Bosak in 1964 [9] introduced the concept of the intersection graph of semigroups. Beck [7] introduced the concept of the zero−divisor graph of rings. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [10]. The intersection graph of ideals of submodules of modules have been investigated in [1]. Numerous other classes of graphs related with algebraic structures have been also actively examined, for instance, see [2−6].

The small intersection graph of a module [13] is another principal graph associated to a ring. The small intersection graph of submodules of a module $U$, indicated by $\Gamma(U)$ is a graph having the set of all non-trivial submodules of $U$ as its vertex set and two vertices $N$ and $L$ are adjacent if and only if $N \cap L$ is small in $U$.

Inspired by preceding studies on the intersection graph of algebraic structures, in this paper, we defined $\Gamma_\delta(U)$ the $\delta$-small intersection graph of submodules of a module.

In Section 2, we show that $\Gamma_\delta(U)$ is complete if either $U$ is a module and direct sum of two simple modules or $U$ is $\delta$-hollow module. Also, if $U$ is a $\delta$-supplemented module, then $\operatorname{diam}(\Gamma_\delta(U)) \le 2$. We proved that if $|\Gamma_\delta(U)| \ge 3$, then $\Gamma_\delta(U)$ is a star graph if and only if $\delta(U)$ is a non-zero simple $\delta$-small submodule of $U$ where every pair of non-trivial submodules of $U$ have non $\delta$-small intersection. We establish that if $|\mathbb{S}_\delta(U)| \in \{1, 2\}$ and under some condition, then $\Gamma_\delta(U)$ is a planar graph. Also, $\Gamma_\delta(U)$ is not a planar graph, whenever $|\mathbb{S}_\delta(U)| \ge 3$. In Section 3, we show that if $U = \oplus_{i=1}^{n} U_i$, with $U_i$ are distinct simple left $R$-module, then $\Gamma_\delta(U)$ is a planar graph if and only if $n \le 4$.

Throughout this paper $R$ is a commutative ring with identity besides $U$ is a unitary left $R$-module. We mean a non-trivial submodule of $U$ is a non-zero proper submodule of $U$. A submodule $N$ (we write $N \le U$) of $U$ is called small in $U$ (we write $N \ll U$), if for every submodule $L \le U$, with $N + L = U$ implies that $L = U$. A submodule $L \le U$ is said to be essential in $U$, indicated as $L \le_e U$, if $L \cap N = 0$ for every non-zero submodule $N \le U$. A module $U$ isnamed singular if $U \cong \frac{K}{L}$ for some module $K$ and an essential submodule $L \le_e K$. Following Zhou [17], a submodule $N$ of a module $U$ is called a $\delta$-small submodule (we write $N \ll_\delta U$), if, whenever $U = N + X$ with $\frac{U}{X}$ singular, we have $X = U$. It is obvious that every small submodule or projective semisimple submodule of $U$ is $\delta$-small in $U$. A nonzero $R$-module $U$ is called hollow [resp., $\delta$-hollow], if every proper submodule of $U$ is small [resp., $\delta$-small] in $U$ [14]. A non-zero module $U$ named local if it is hollow and finitely generated [16]. A submodule $P$ of a module $U$ is

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E-mail address: ahmedha_math@utq.edu.iq.

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maximal iff it is not properly contained in any other submodule of $U$. An $R$-module $U$ is said to be local if it has a unique maximal submodule. The set is of maximal submodules of $U$ is denoted by $\max(U)$. The Jacobson radical of an $R$-module $U$, indicated by $Rad(U)$, is the intersection of all maximal submodules of $U$. By $\delta(U)$ we will denote the sum of all $\delta$-small submodules of $U$ as in [17, Lemma 1.5 (1)]. Also, $\delta(R) = \delta(_R R)$. Since $Rad(U)$ is the sum of all small submodules of $U$, it follows that $Rad(U) \leq \delta(U)$ for a module $U$. A module $U$ is called $\delta$-local if $\delta(U) \ll_\delta U$ and $\delta(U)$ is maximal [14]. The module $U$ is named simple if it has no proper submodules, and $U$ is said to be semisimple if it is a direct sum of simple submodules. The socle of a module $U$, denoted by $Soc(U)$, is the sum of all simple submodules of $U$. The references for module theory are [16,17]; for graph theory is [8].

For a graph $\Gamma$, $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and edges, respectively. The set of vertices adjacent to vertex $v$ of the graph $\Gamma$ is called the neighborhood of $v$ besides indicated by $N(v)$. The order of $\Gamma$ is the number of vertices of $\Gamma$ besides we indicated it by $|\Gamma|$. $\Gamma$ is finite, if $|\Gamma| < \infty$, *else*, $\Gamma$ is infinite. If $u$ and $v$ are two adjacent vertices of $\Gamma$, then we write $u - v$, i.e. $\{u,v\} \in E(\Gamma)$. The degree of a vertex $v$ in a graph $\Gamma$, indicated by $\deg(v)$, is the number of edges incident with $v$. Let $u$ and $v$ be vertices of $\Gamma$. An $u$, $v-$ path is a path (trail) with starting vertex $u$ and ending vertex $v$. For distinct vertices $u$ and $v$, $d(u,v)$ is the least length of an $u$, $v-$ path. If $\Gamma$ has no such a path, then $d(u,v) = \infty$. The diameter of $\Gamma$, indicated by diam $(\Gamma)$, is the supremum of the set $\{d(x,y)$: $u$ and $v$ are distinct vertices of $\Gamma\}$. A cycle in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. The girth of a graph $\Gamma$, indicated by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. A graph is said to be connected (or joined), if there is a path between every pair of vertices of the graph. A joined graph which does not contain a cycle is named a tree. If $\Gamma$ is a tree consisting of one vertex adjacent to all the others then $\Gamma$ is named star graph. $\Gamma$ is complete if it is connected with diam $(\Gamma) \leq 1$. A complete graph with $n$ distinct vertices, indicated by $K_n$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph $\Gamma$, symbolized by $\omega(\Gamma)$, is called the clique number of $\Gamma$.

**Lemma 1.1.** [17] Let $Z \leq U$. The next are equivalent:

(1) $Z \ll_\delta U$.
(2) If $U = W + Z$, then $U = W \oplus Y$ for a projective semisimple submodule $Y$ with $Y \leq Z$.

**Lemma 1.2.** [17, Lemma 1.3] Let $U$ be an $R$-module.

(1) For submodules $N$, $Z$, $L$ of $U$ with $Z \leq N$, we have
  i. $N \ll_\delta U$ iff $Z \ll_\delta U$ and $N/Z \ll_\delta U/Z$.
  ii. $N + L \ll_\delta U$ iff $N \ll_\delta U$ and $L \ll_\delta U$.
(2) $Z \ll_\delta U$ and $f : U \to N$ is a homomorphism, then $f(Z) \ll_\delta N$. In particular, if $Z \ll_\delta U \leq N$, then $Z \ll_\delta N$.
(3) Let $Z_1 \leq U_1 \leq U$, $Z_2 \leq U_2 \leq U$ and $U = U_1 \oplus U_2$. Then $Z_1 \oplus Z_2 \ll_\delta U_1 \oplus U_2$ iff $Z_1 \ll_\delta U_1$ and $Z_2 \ll_\delta U_2$.

**Lemma 1.3.** [17, Lemma 1.5] Let $U$ and $N$ be modules.

(1) $\delta(U) = \sum\{L \leq U | L$ is a $\delta$-small submodule of $U\}$.
(2) If $f : U \to N$ is an $R$-homomorphism, then $f(\delta(U)) \subseteq \delta(N)$. Also, $\delta\ (_R R\ )U \subseteq \delta(U)$.
(3) If $U = \oplus_{i \in I} U_i$, then $\delta(U) = \oplus_{i \in I} \delta(U_i)$.
(4) If every proper submodule of $U$ is contained in a maximal submodule of $U$, then $\delta(U)$ is the unique largest $\delta$-small submodule of $U$.

## 2. Connectedness and completeness

In this Section, we generalizing the definition of [13], we consider a graph $\Gamma_\delta(U)$ as follows:

**Definition 2.1.** Let $U$ be an $R$-module. The $\delta$-small intersection graph of $U$, symbolized by $\Gamma_\delta(U)$, is defined to be a simple graph whose vertices are in one-to-one correspondence with all non-trivial submodules of $U$ and two vertices $N$ and $L$ are adjacent, and we write $N - L$, if and only if $N \cap L \ll_\delta U$.

Remark 2.2.

(1) Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$. The nonzero proper submodules of $\mathbb{Z}_6$ are $2\mathbb{Z}_6$ and $3\mathbb{Z}_6$. Obviously, $2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = 0 \ll_\delta \mathbb{Z}_6$ and so $\Gamma_\delta(\mathbb{Z}_6)$ is $2\mathbb{Z}_6 - 3\mathbb{Z}_6$.
(2) It is clear that the graph $\Gamma(U)$ introduced in [13] is a subgraph of $\Gamma_\delta(U)$.
(3) The $\delta$-small submodules of a singular module are small submodules [17]. Clearly when $U$ is a singular module, we get that $\Gamma_\delta(U)$ is the small intersection graph $\Gamma(U)$ of $U$ introduced in [13].

A null graph is a graph whose vertices are not adjacent to each one other (i.e., edgeless graph).

**Theorem 2.3.** Let $U$ be a not simple module. Then $\Gamma_\delta(U)$ is a null graph if and only if every pair of non-trivial submodules of $U$, have non $\delta$-small intersection.

**Proof.** Assume $\Gamma_\delta(U)$ is an edgeless graph. Presume for contrary that there exist $A, B \le U$ such that $A \cap B \ll_\delta U$. At that time $A - B$, hence $\Gamma_\delta(U)$ is not null, which is a contradiction to the hypothesis "$\Gamma_\delta(U)$ is an edgeless graph". The reverse is easy.

**Example 2.4.** $\Gamma_\delta(\mathbb{Z}_4)$ and $\Gamma_\delta(\mathbb{Z})$ are edgeless graphs.

**Proposition 2.5.** Let $U$ be an $R$-module. At that point $\Gamma_\delta(U)$ is complete, if one of the following holds.

(1) If $U$ is $\delta$-hollow.
(2) If $U = U_1 \oplus U_2$ is a module, where $U_1$ and $U_2$ are simple $R$-modules.

**Proof.** (1) Let $U$ be a $\delta$-hollow module. Presume that $A_1$, $A_2$ are two different vertices of the graph $\Gamma_\delta(U)$. From this time $A_1$ and $A_2$ are two nonzero $\delta$-small submodules of $U$. As $A_1 \cap A_2 \le A_i$, for $i = 1, 2$, by Lemma 1.2, $A_1 \cap A_2 \ll_\delta U$, hence $\Gamma_\delta(U)$ is a complete graph.

(2) Assume that $U = U_1 \oplus U_2$ with $U_1$ besides $U_2$ are simple $R$-modules. So, $U_1 + U_2 = U$ and $U_1 \cap U_2 = \{0\}$. Then every non-trivial submodule of $U$ is simple. Let $\mathfrak{A}$, $\mathfrak{B}$ be binary different vertices of $\Gamma_\delta(U)$. At that moment they are the non-trivial submodules of $U$ which are simple besides minimal. Furthermore, $\mathfrak{A} \cap \mathfrak{B} \le \mathfrak{A}, \mathfrak{B}$ and if $\mathfrak{A} \cap \mathfrak{B} \neq (0)$, then minimality of $\mathfrak{A}$ and $\mathfrak{B}$ implies that $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} = \mathfrak{B}$, a contradiction. Thus, $\mathfrak{A} \cap \mathfrak{B} = (0) \ll_\delta U$, henceforth $\Gamma_\delta(U)$ is complete.

By Part 1 of Proposition 2.5, we have the next corollary.

**Corollary 2.6.** Let $R$ be a ring and $U$ be a module over $R$. Then the next hold:

(1) If $V(\Gamma(U))$ is a totally ordered set, at that time a graph $\Gamma(U)$ is complete.
(2) If $U$ is a $\delta$-local module, at that point the graph $\Gamma_\delta(U)$ is complete.
(3) Every one nonzero $\delta$-small submodule of $U$ is adjacent to all other vertices of $\Gamma_\delta(U)$ besides the induced subgraphs on the sets of $\delta$-small submodules of $U$ are cliques.

**Proof.** (1) Suppose $V(\Gamma(U))$ is a totally ordered set. Then all two nontrivial submodules of $U$ are comparable. Evidently, for all $\mathscr{R} \le U$, $\mathscr{R} \ll U$, besides so $\mathscr{R} \ll_\delta U$. Hence, $U$ is a $\delta$-hollow $R$-module. So, by Proposition 2.5 (1), $\Gamma_\delta(U)$ is complete.

(2) Suppose that $U$ is a $\delta$-local $R$-module, at that time $\delta(U) \ll_\delta U$ besides $\delta(U)$ is maximal. Now, let $\mathfrak{w}$ be a nonzero submodule of $U$. To prove that $\mathfrak{w} \le \delta(U)$, by contrary way, assume $\mathfrak{w}$ is not subset of $\delta(U)$, so $\delta(U) + \mathfrak{w} = U$ since $\delta(U)$ is maximal. Hence $\mathfrak{w} = U$ since $\delta(U) \ll_\delta U$, a conflict. Thus, $\mathfrak{w} \le \delta(U)$. So, $\mathfrak{w}$ is $\delta$-small submodule of $U$. Thus, $U$ is $\delta$-hollow. So, by Proposition 2.5 (1), $\Gamma_\delta(U)$ is complete.

(3) Evident.

**Example 2.7.** For every $c \in \mathbb{Z}$ with $c \ge 2$ besides for all prime number $p$, $\mathbb{Z}_{p^c}$ is a local $\mathbb{Z}$-module, then it is hollow and so is $\delta$-hollow. Also, let $R = \mathbb{Z}$, $p$ be a prime and $U = \mathbb{Z}_{p^\infty}$, the Pr ü fer $p$-group, then every proper submodule of $R$-module $U$ is $\delta$-small in $U$. Moreover, $\delta(U) = U$. Hence for every prime number $p$, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is $\delta$-hollow. By Proposition 2.5 (1), $\Gamma_\delta(\mathbb{Z}_{p^c})$ and $\Gamma_\delta(\mathbb{Z}_{p^\infty})$ are complete graphs.

**Remark 2.8** [17]. **For a ring *R*,**

(1) $\delta(R) =$ the intersection of all maximal essential left ideals of $R$.
(2) $\delta(R) =$ the largest $\delta$-small left ideal of $R$.
(3) $\delta(R) = R$ if and only if $R$ is a semisimple ring, see [17, Corollary 1.7].

**Proposition 2.9.** Let $R$ be an integral domain with $\delta(R) \neq 0$ besides let $U$ be a finitely generated torsion-free $R$-module. Then $\Gamma_\delta(U)$ is connected and $\mathrm{diam}(\Gamma_\delta(U)) \le 2$.

**Proof.** Since $U$ is finitely generated, then $\delta(U)$ is the largest $\delta$-small submodule of $U$ according to Lemma 1.3(4). Also, the largest $\delta$-small left ideal of $R$ is $\delta(R)$ by Remark 2.8. By Lemma 1.3(2), $\delta(R)U \le \delta(U)$. Thus, $\delta(R)U \ll_\delta U$. Since $U$ is torsion-free and $\delta(R) \neq 0$ then $\delta(R)U \neq 0$. Therefore, $\delta(R)U$ is a vertex in $\Gamma_\delta(U)$. But $X \cap \delta(R)U \ll_\delta U$ for every nonzero submodule $X$ of $U$ by Lemma 1.2(1). So, there exists an edge among vertex $\delta(R)U$ besides $X$ of $\Gamma_\delta(U)$. Also, for all two vertices $X, Y$ in the graph $\Gamma_\delta(U)$, there exists a path $X - \delta(R)U - Y$ of length 2 in $\Gamma_\delta(U)$ . This completes the proof.

**Theorem 2.10.** Let a ring $R$ be a sum $R = \oplus_{i \in I} T_i$ of simple left ideals $T_i$, $i \in I$. At that point the next statements hold:

(1) $\mathrm{diam}(\Gamma_\delta(R)) = 1$,
(2) The graph $\Gamma_\delta(R)$ is a complete graph.

**Proof.** (1) Let $R = \oplus_{i \in I} T_i$, where each $T_i$ are simple left ideals, $i \in I$ . By Remark 2.8(3), we have $\delta(R) = R$. So, each $T_i$ is $\delta$-small submodule of $_R R$. Now, let $T_i$ and $T_j$ are two non-zero ideals of $R$, then $T_i \cap T_j$ is $\delta$-small in $_R R$, and thus, there exists an edge between the vertices $T_i$ and $T_j$ in $\Gamma_\delta(R)$, for all $i, j \in I$. Hence, the graph $\Gamma_\delta(R)$ is connected besides diam $(\Gamma_\delta(R)) = 1$.

(2) It follows from the proof of (1).

**Definition 2.11.** [12] Let $U$ be a module besides let $N$ and $L$ be submodules of $U$. $L$ is named a $\delta$-supplement of $N$ in $U$ if $U = N + L$ and $N \cap L \ll_\delta L$ (and so $N \cap L \ll_{\delta} U$). $N$ is named a $\delta$-supplement submodule if $N$ is a $\delta$-supplement of some submodule of $U$. $U$ is named a $\delta$-supplemented if every submodule of $U$ has a $\delta$-supplement in $U$.

**Proposition 2.12.** Let $\mathscr{h} \leq U$. Then any $\delta$-supplement of $\mathscr{h}$ in $U$ is adjacent to $\mathscr{h}$ in $\Gamma_{\delta}(U)$.

**Proof.** Let $\mathscr{h}$ be a submodule of $U$ and let $\mathscr{g}$ $\delta$-supplement of $\mathscr{h}$ in $U$. Hence $U = \mathscr{h} + \mathscr{g}$ and $\mathscr{h} \cap \mathscr{g} \ll_{\delta} \mathscr{g}$, and so $\mathscr{h} \cap \mathscr{g} \ll_{\delta} U$. Thus $\mathscr{g}$ adjacent to $\mathscr{h}$ in $\Gamma_{\delta}(U)$.

We now state-owned our next result, which gives us certain information on the structure of the $\delta$-small intersection graphs of $\delta$-supplemented modules.

**Proposition 2.13.** Let $U$ be a $\delta$-supplemented module. Then $\Gamma_{\delta}(U)$ is connected and $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 2$.

**Proof.** Let $N, L$ are submodules of $U$. Since $U$ is $\delta$-supplemented, then there exists submodule $K$ of $U$ such that $N + K = U$, $N \cap K \ll_{\delta} K$, and so $N \cap K \ll_{\delta} U$. One can consider binary likely cases for $N \cap K$.

**Case 1:** If $N \cap K = (0)$, then $N \oplus K = U$.

Now, if $L \leq N$, then $L \cap K \ll_{\delta} U$. Thus $L - K - N$ is a path of length 2 in $\Gamma_{\delta}(U)$. If $L \leq K$, then $L \cap N \ll_{\delta} U$. Thus $N$ and $L$ are adjacent vertices in the graph $\Gamma_{\delta}(U)$. Hence, $\Gamma_{\delta}(U)$ is joined besides $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 2$.

**Case 2:** If $N \cap K \neq (0)$. Since $N \cap K$ is a $\delta$-small submodule of $U$, thus $N - N \cap K - L$ is a path of length 2 in $\Gamma_{\delta}(U)$. Hence, $\Gamma_{\delta}(U)$ is joined besides $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 2$.

The next examples show there are connected graphs $\Gamma_{\delta}(U)$ with $\operatorname{diam}(\Gamma_{\delta}(U)) \geq 2$ whenever $U$ is not $\delta$-supplemented.

**Example 2.14.** (1) The $\mathbb{Z}$-module $U = \oplus_{i=1}^{\infty} U_i$ with each $U_i = \mathbb{Z}_{p^{\infty}}$ where $p$ is prime number is not $\delta$-supplemented see [12]. It is easy to see that $\Gamma_{\delta}(U)$ is connected and $\operatorname{diam}(\Gamma_{\delta}(U)) \geq 2$.

(2) The $\mathbb{Z}$-module $\mathbb{Q}$ is not $\delta$-supplemented see [12]. Now, from [12] that Let $\mathbb{Q}_1 = \{a/b \in \mathbb{Q} \mid 2 \text{ does not divide } b\}$ and $\mathbb{Q}_2 = \{a/b \in \mathbb{Q} \mid 2 \text{ divides } b\}$. Then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$. Since $\mathbb{Q}/\mathbb{Q}_1$ and $\mathbb{Q}/\mathbb{Q}_2$ are singular $\mathbb{Z}$-modules, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are not $\delta$-small submodules in $\mathbb{Q}$. Hence, any proper submodule $L$ of $\mathbb{Q}$ with $\mathbb{Q}_1 \leq L$ we have $L$ is not adjacent to $\mathbb{Q}_1$. So, $\Gamma_{\delta}(\mathbb{Q}) \geq 2$. But $\Gamma_{\delta}(\mathbb{Q})$ is connected graph.

**Lemma 2.15.** Let $U$ be a module.

(1) Let $N \leq U$ be a finitely generated submodule with $N \leq \delta(U)$. Then $N \ll_{\delta} U$.
(2) Let $N \leq U$ be a semisimple submodule with $N \leq \delta(U)$. Then $N \ll_{\delta} U$.

**Proof.** (1) Suppose that $N \leq U$ is finitely generated. Then, $N = \sum_{i=1}^{r} Rn_i$ for some $n_i \in N$, $1 \leq i \leq r$. Since $Rn_i \leq \delta(U)$, $Rn_i \ll_{\delta} U$. According to Lemma 1.2, $N \ll_{\delta} U$.

(2) By [15, Lemma 2.2].

**Proposition 2.16.** For an $R$-module $U$ with $\Gamma_{\delta}(U)$ and $\delta(U) \neq (0)$. The following conditions hold:

(1) If $N$ is a direct summand submodule of $U$ with $(0) \neq \delta(N) \ll_{\delta} U$, then $\Gamma_{\delta}(U)$ contains at least one cycle of length 3.
(2) If $T$ is a non-trivial semisimple or finitely generated submodule of $U$ contained in $\delta(U)$. At that time $d(T, \delta(U)) = 1$ and $d(T, L) = 1$ for every non-trivial submodule $L$ of $U$.

**Proof.** (1) Since $N$ is a direct summand of $U$, there is $Z \leq U$ such that $N \oplus Z = U$. Then $\delta(N) \oplus \delta(Z) = \delta(U)$, according to Lemma 1.3. Since $\delta(N) \leq N$ and $N \cap \delta(Z) \leq N \cap Z = (0)$, by the modular law, $\delta(U) \cap N = [\delta(Z) + \delta(N)] \cap N = [\delta(Z) \cap N] + \delta(N) = \delta(N)$. Thus, $\delta(U) \cap N = \delta(N)$. Then $\delta(U) \cap N \ll_{\delta} U$. Also, $\delta(N) = N \cap \delta(N) \ll_{\delta} U$ and $\delta(N) = \delta(N) \cap \delta(U) \ll_{\delta} U$ and we have, $d(N, \delta(U)) = 1$, $d(N, \delta(N)) = 1$ and $d(\delta(N), \delta(U)) = 1$. Hence, $(N, \delta(N), \delta(U))$ is a cycle. Thus, $\Gamma_{\delta}(U)$ contains at least one cycle of distance 3.

(2) Let $T \leq U$ be a non-trivial semisimple or finitely generated submodule. At that moment by Lemma 2.15, $T \ll_{\delta} U$. Since $T \leq \delta(U)$, $T = T \cap \delta(U) \ll_{\delta} U$ and since $T \cap L \leq T$, $T \cap L \ll_{\delta} U$ for every other non-trivial submodule $L$ of $U$ via Lemma 1.2. Hence $d(\delta(U), T) = 1$ and $d(L, T) = 1$.

**Proposition 2.17.** Let $U$ be a $R$-module. If $U$ has at least one non-zero $\delta$-small submodule, at that point $\Gamma_{\delta}(U)$ is a connected graph besides $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 2$.

**Proof.** Let $F \in \Gamma_{\delta}(U)$ be a non-zero $\delta$-small submodule of $U$. Let $A$ and $B$ be two non-adjacent vertices of $\Gamma_{\delta}(U)$. It is clear that $A \cap F \leq F \ll_{\delta} U$, and $F \cap B \leq F \ll_{\delta} U$. Thus $A \cap F \ll_{\delta} U$, and $F \cap B \ll_{\delta} U$ by Lemma 1.2. So, $A - F - B$ is a trail of length 2. So, $\Gamma_{\delta}(U)$ is a joined graph besides $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 2$.

**Corollary 2.18.** Let $\delta(U) \neq (0)$, if one of the next holds. Then $\Gamma_{\delta}(U)$ is a joined graph,

(1) There exists a non-trivial submodule of $U$ which is semisimple or finitely generated contained in $\delta(U)$.
(2) $U$ is a finitely generated module.

**Proof.** (1) It follows from Proposition 2.17 and Lemma 2.15. (2) Clear.

**Proposition 2.19.** If $\Gamma_{\delta}(U)$ has no isolated vertex, then $\Gamma_{\delta}(U)$ is connected and $\operatorname{diam}(\Gamma_{\delta}(U)) \leq 3$.

**Proof.** Let $A$ and $B$ be two non-adjacent vertices of $\Gamma_{\delta}(U)$. Since $\Gamma_{\delta}(U)$ has no isolated vertex, there exist submodules $A_1$ and $B_1$ such that $A \cap A_1 \ll_{\delta} U$ and

$B \cap B_1 \ll_\delta U$. Now, if $A_1 \cap B_1 \ll_\delta U$, then $A - A_1 - B_1 - B$ is a path of length 3. Otherwise $A - A_1 \cap B_1 - B$ is a trail of size 2. Showed that $\operatorname{diam}(\Gamma_\delta(U)) \leq 3$ besides $\Gamma_\delta(U)$ is a joined graph.

**Proposition 2.20.** Let $U$ be a not simple $R$-module which is semisimple $R$-module. At that point the next declarations hold:

(i) $\Gamma_\delta(U)$ has no isolated vertex.
(ii) $\Gamma_\delta(U)$ is joined besides $\operatorname{diam}(\Gamma_\delta(U)) \leq 3$.

**Proof.** (i) Let $Z$ be a vertex of the graph $\Gamma_\delta(U)$. Since $U$ is a semisimple module, then every submodule of $U$ is a direct summand of $U$ by [16, 20.2, p. 166]. Thus there exists a submodule $Y$ of $U$ such that $U = Z \oplus Y$. Hence $Z \cap Y = (0) \ll_\delta U$ besides as a result, there exists an edge among vertex $Z$ of $\Gamma_\delta(U)$ besides another vertex of $\Gamma_\delta(U)$. At that time $Z$ is non-isolated vertex. So, $\Gamma_\delta(U)$ has no isolated vertex.

(ii) By Proposition 2.19 besides Part (i).

Now we use $\mathbb{S}_\delta(U)$ which symbolizes the set of all non-zero $\delta$-small submodules of $U$.

**Proposition 2.21.** Let $n$ be a positive integer. In $R$-module $U$ with $|\mathbb{S}_\delta(U)| = n$ and $|\Gamma_\delta(U)| \geq 2$.

(i) If $N \in \mathbb{S}_\delta(U)$, then deg $(N) \neq 0$.
(ii) $\omega(\Gamma_\delta(U)) \geq n$.
(iii) If $\omega(\Gamma_\delta(U)) < \infty$, then the number of $\delta$-small submodules of $U$ is finite.

**Proof.** (i) Let $N \in \mathbb{S}_\delta(U)$. Suppose that the order of $\Gamma_\delta(U)$ is $|\Gamma_\delta(U)| = n \geq 2$ where $n$ is integer number. Let $K$ be any non-zero submodule of $U$. Then $K \cap N \leq N \ll_\delta U$. By [17, Lemma 1.3(1)], $K \cap N \ll_\delta U$ and thus an edge exists among vertex $N$ of $\Gamma_\delta(U)$ and another vertex of $\Gamma_\delta(U)$. At that point $N$ is cannot an isolated vertex. Thus, deg $(N) \neq 0$.

(ii) Let $\mathbb{S}_\delta(U) = \{N | N \ll_\delta U\}$ and let $|\mathbb{S}_\delta(U)| = n$. Suppose that $Z$ and $W$ are two distinct elements of $\mathbb{S}_\delta(U)$. Then $Z$ and $W$ are non-zero $\delta$-small submodules of $U$. Thus $Z \cap W \ll_\delta U$ according to [17, Lemma 1.3(1)]. So, $Z$ and $W$ are adjacent vertices. Thus, the induced subgraph on the set $\mathbb{S}_\delta(U)$ is a complete subgraph of $\Gamma_\delta(U)$. From this time, $\omega(\Gamma_\delta(U)) \geq n$.

(iii) It is clear from (ii).

**Theorem 2.22.** Let $\delta(U)$ be a non-zero simple $\delta$-small submodule of $U$ and let $|\Gamma_\delta(U)| \geq 2$. Then $\Gamma_\delta(U)$ is a star graph whenever $\Gamma_\delta(U)$ is a tree graph.

**Proof.** Since $\delta(U) \neq 0$, then $\delta(U)$ is a vertex in $\Gamma_\delta(U)$. Now, $\delta(U)$ is simple $\delta$-small, so $\delta(U)$ a unique non-zero $\delta$-small submodule of $U$. But, $\delta(U) \cap N \ll_\delta U$ for every $\in V(\Gamma_\delta(U))$ . Thus then $\Gamma_\delta(U)$ contains a vertex $\delta(U)$ which is adjacent to each other vertex. Now, suppose that $I \neq \delta(U)$ and $J \neq \delta(U)$ are two distinct vertices of $\Gamma_\delta(U)$. Now, if $I \cap J \ll_\delta U$. Then $I - \delta(U) - J$, which is a contradiction since $\Gamma_\delta(U)$ is a tree. Thus, $I \cap J$ is not a $\delta$-small submodule of $U$. So, $I$ and $J$ are not adjacent. Thus, $\Gamma_\delta(U)$ is star with center $\delta(U)$.

Let $\Gamma$ be a graph. The chromatic number of $\Gamma$ is defined to be the smallest number of colors $\chi(\Gamma)$ needed to color the vertices of $\Gamma$ so that no two adjacent vertices share the same color. One has the next corollary by Theorem 2.22.

**Corollary 2.23.** Let $U$ be a module with $0 \neq \delta(U) \ll_\delta U$ and $|\Gamma_\delta(U)| \geq 3$. Then the next conditions are equivalent:

(1) $\Gamma_\delta(U)$ is a star graph,
(2) $\Gamma_\delta(U)$ is a tree,
(3) $\chi(\Gamma_\delta(U)) = 2$,
(4) $\delta(U)$ is a simple submodule of $U$ such that every couple of non-trivial submodules of $U$, have non $\delta$-small intersection.

**Proof.** (1) → (2) and (2) → (3) The implications are obvious.

(3) → (4) On contrary, suppose $0 \neq K \leq \delta(U)$. At that point $K \ll_\delta U$. If $L \in V(\Gamma_\delta(U))$. It is easy to see that ($N$, $\delta(U)$, $L$) is a circuit (cycle) of length 3 in $\Gamma_\delta(U)$, which contradicts $\chi(\Gamma_\delta(U)) = 2$. As a result, $\delta(U)$ is simple. Now, take up that $\mathrm{Y}, \varpi \in V(\Gamma_\delta(U))$ such that $\varpi \cap \mathrm{Y} \ll_\delta U$. ($\varpi$, $\delta(U)$, Y) is a circuit in $\Gamma_\delta(U)$, which contradicts $\chi(\Gamma_\delta(U)) = 2$.

(4) → (1) It is obvious that $\delta(U)$ is adjacent to each other vertex in $\Gamma_\delta(U)$. Now, suppose that $N \neq \delta(U)$ and $L \neq \delta(U)$ are two distinct vertices of $\Gamma_\delta(U)$, such that $N$ and $L$ are adjacent. Thus, $X \cap Y \ll_\delta U$, a contradiction. Hence, $\Gamma_\delta(U)$ is a star graph.

**Proposition 2.24.** Let $U$ be a module and $|\mathbb{S}_\delta(U)| \geq 1$. If $\Gamma_\delta(U)$ does not contain a cycle, then $\Gamma_\delta(U) = K_1$ or $\Gamma_\delta(U)$ is a star graph.

**Proof.** Supposing that the graph $\Gamma_\delta(U)$ contains no a cycle. To prove $|\mathbb{S}_\delta(U)| < 2$, by contrary way, let $Z \ll_\delta U$ besides $W \ll_\delta U$. So $\mathrm{Z} + \mathrm{W} \ll_\delta U$ by Lemma 1.2, and hence, $\mathrm{Z} - (\mathrm{Z} + \mathrm{W}) - \mathrm{W}$ is a cycle of length 3, which is a illogicality. Then $|\mathbb{S}_\delta(U)| < 2$. As $|\mathbb{S}_\delta(U)| \geq 1$, then $|\mathbb{S}_\delta(U)| = 1$. Hence, $U$ has a unique non-zero $\delta$-small submodule. Let $N \in \mathbb{S}_\delta(U)$. For every vertex $L$ of $\Gamma_\delta(U)$, if $L = N$, then $\Gamma_\delta(U) \cong K_1$ and if $L \neq N$, as $L \cap N \ll_\delta U$, we deduce $\Gamma_\delta(U) \cong K_2$. Let $\Psi = \{\mathrm{v}_i | \mathrm{v}_i \neq N, i \in I\}$. At that time every two random distinct vertices $\mathrm{v}_i$ and $\mathrm{v}_j$, $i \neq j$, are not adjacent and for $i \neq j$, $\mathrm{v}_i - N - \mathrm{v}_j$ is a path besides hence $\Gamma_\delta(U)$ is a star graph.

**Theorem 2.25.** Let $\Gamma_\delta(U)$ be a graph of a module $U$. If $|\mathbb{S}_\delta(U)| \geq 2$, then $\Gamma_\delta(U)$ contains at least one cycle besides $\operatorname{gr}(\Gamma_\delta(U)) = 3$.

**Proof.** Presume that $|\mathbb{S}_\delta(U)| \geq 2$. At that time $U$ has at least two nonzero $\delta$-small submodules, at a guess $T_1$ and $T_2$. Since $T_1 \cap T_2 \leq T_i$, for $i = 1, 2$, by Lemma 1.2, $T_1 \cap T_2 \ll_\delta U$. Also, $T_1 \cap (T_1 \cap T_2) \ll_\delta U$ and $T_2 \cap (T_1 \cap T_2) \ll_\delta U$. We consider two probable cases for $T_1 \cap T_2$.

**Case 1:** If $T_1 \cap T_2 \neq (0)$, then $d(T_1, T_2) = 1$, $d(T_1, T_1 \cap T_2) = 1$ and $d(T_2, T_1 \cap T_2) = 1$. Thus $(T_1, T_1 \cap T_2, T_2)$ is a cycle of size 3. Also by Lemma 1.2, $T_1 + T_2 \ll_\delta U$ and since $T_1 \cap (T_1 + T_2) \ll_\delta U$ and $T_2 \cap (T_1 + T_2) \ll_\delta U$, $(T_1, T_1 + T_2, T_2)$ is a cycle of length 3. Similarly, $(T_1 \cap T_2, T_1, T_1 + T_2)$ and $(T_1 \cap T_2, T_2, T_1 + T_2)$ are cycles of length 3 and $(T_1, T_1 + T_2, T_2, T_1 \cap T_2, T_1)$ is a cycle of length 4.

**Case 2:** If $T_1 \cap T_2 = (0)$, then $(T_1, T_1 + T_2, T_2)$ is a cycle of size 3 in the graph $\Gamma_\delta(U)$. As a result, $\Gamma_\delta(U)$ contains at least one cycle and so $\mathrm{gr}(\Gamma_\delta(U)) = 3$.

**Example 2.26.** Let $U = Z \oplus F \oplus K$ be a semisimple module. Then, the subgraph $Z - F - K - Z$ is a clique. Also, $\mathrm{gr}(\Gamma_\delta(U)) = 3$.

Let $\Gamma$ is a joined graph and let $X$ is a vertex of $\Gamma$, $X$ is named a cut vertex of $\Gamma$ if there are vertices $Z$ besides $W$ of $\Gamma$ such that $X$ is in every one $Z$, $W-$path. Equally, $X$ is a cut vertex of $\Gamma$ if $\Gamma - \{X\}$ is not joined for a joined graph $\Gamma$.

**Proposition 2.27.** $\Gamma_\delta(U)$ has no cut vertex whenever $|\mathbb{S}_\delta(U)| \geq 2$.

**Proof.** Take up $T$ a cut vertex of $\Gamma_\delta(U)$, as a result $\Gamma_\delta(U) \setminus \{T\}$ is not joined. As a result there exist vertices $F$, $K$ with $T$ lies on every single trail from $F$ to $K$. Since $|\mathbb{S}_\delta(U)| \geq 2$, then $U$ has at least two nonzero $\delta$-small submodules, assume $(0) \neq N_1 \ll_\delta U$, $(0) \neq N_2 \ll_\delta U$. Thus $F \cap N_1 \ll_\delta U$, $N_1 \cap N_2 \ll_\delta U$ and $N_2 \cap K \ll_\delta U$. $F - N_1 - N_2 - K$ is a trail in $\Gamma_\delta(U) \setminus \{T\}$, a illogicality. As a result $\Gamma_\delta(U)$ has no cut vertex.

## 3. Domination and planarity of $\Gamma_\delta(U)$

In this Section, we study domination number and the planarity of $\Gamma_\delta(U)$. We recall that for a graph $\Gamma$, a subset $D$ of the vertex-set of $\Gamma$ is called a dominating set (or DS) if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number, $\gamma(\Gamma)$, of $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$, [11]. Here, a subset $D$ of the vertex set $V(\Gamma_\delta(U))$ is a DS iff for any nontrivial submodule $N$ of $U$ there is a $L$ in $D$ such that $N \cap L \ll_\delta U$.

**Lemma 3.1.** The next hold for an $R$-module $U$ with $|\Gamma_\delta(U)| \geq 2$:

(1) If $D \subseteq V(\Gamma_\delta(U))$ with either there exists a vertex $X \in D$ which $X \cap Y = (0)$, for every one vertex $Y \in V(\Gamma_\delta(U)) \setminus D$ or $D$ contains at least one $\delta$-small submodule of $U$. Then $D$ is a DS in $\Gamma_\delta(U)$.

(2) If $|\mathbb{S}_\delta(U)| \geq 1$, then for each $Z \neq 0$ with $Z \ll_\delta U$, $\{Z\}$ is a DS besides $\gamma(\Gamma_\delta(U)) = 1$.

**Proposition 3.2.** Let $U = N \oplus L$ be an $R$-module, where $N$ and $L$ are simple $R$-modules. Then $\gamma(\Gamma_\delta(U)) = 1$.

**Proof.** Assume $U = N \oplus L$, with $N$ and $L$ are simple $R$-modules. By Proposition 2.3 (1), is a complete graph $\Gamma_\delta(U)$. Let $\alpha$ be a random vertex of $\Gamma_\delta(U)$. At that time for every different vertex $Y$ of $\Gamma_\delta(U)$, $\alpha \cap Y \ll_\delta U$, so $\{\alpha\}$ is a DS besides $\gamma(\Gamma_\delta(U)) = 1$.

**Proposition 3.3.** Let $\delta(U) \neq 0$ of a finitely generated $R$-module $U$. Then $\{\delta(U)\}$ is a dominating set of $\Gamma_\delta(U)$ and so the graph $\Gamma_\delta(U)$ is joined (=connected).

**Proof.** Assume $\Re \in \Gamma_\delta(U)$. If $\Re$ is $\delta$-small then $\delta(U)$ is adjacent to $\Re$. Now, if $\Re$ is not $\delta$-small. Since $\delta(U) \neq 0$ in finitely generated module, at that point $\delta(U) \ll_\delta U$. So, $\Re \cap \delta(U) \ll_\delta U$. So, $\Re$ is adjacent to $\delta(U)$. This implies that $\{\delta(U)\}$ is a dominating set of $\Gamma_\delta(U)$, so $\Gamma_\delta(U)$ is connected as obligatory.

**Theorem 3.4.** Let $|\mathbb{S}_\delta(U)| \geq 2$ besides $|\Gamma_\delta(U)| \geq 3$ of a module $U$. We have:

(1) If $\mu$ and $\lambda$ are two $\delta$-small submodules of $U$ then there exists $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$.

(2) The graph $\Gamma_\delta(U)$ has at least one triangle.

**Proof.** It is clear.

**Proposition 3.5.** The next statements are equivalent for an $R$-module $U$:

(1) If $\{\mu, \lambda\} \in E(\Gamma_\delta(U))$, then there is no $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$.

(2) $U$ has at most one nonzero $\delta$-small submodule such that $\hbar \cap h$ is not a $\delta$-small for every couple of non-$\delta$-small nontrivial submodules $\hbar, h$ of $U$.

(3) The graph $\Gamma_\delta(U)$ has no triangle.

**Proof.** (1) $\Rightarrow$ (2) Take up that for all two adjacent vertices of $\Gamma_\delta(U)$, there is no $\psi \in V(\Gamma_\delta(U))$ with $\psi \in N(\mu) \cap N(\lambda)$. Assume there exist nonzero submodules $N_1 \ll_\delta U$ and $N_2 \ll_\delta U$. Since $N_1 \cap N_2 \ll_\delta U$, they are adjacent vertices of the graph $\Gamma_\delta(U)$ besides too, there is no $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$, which is a illogicality by Theorem 3.4(1).

(2) $\Rightarrow$ (3) Presume there is no nonzero $\delta$-small submodules in $U$. As $\hbar \cap h$ is not $\delta$-small for every couple of non-$\delta$-small nontrivial submodules $\hbar, h$ of $U$, $\Gamma_\delta(U)$ has no triangle. Besides, Let $S$ be the unique nonzero $\delta$-small submodule of $U$. At that point for every three random vertices $N_1, N_2$, and $N_3$ of the graph $\Gamma_\delta(U)$, at least two of them are not $\delta$-small. Let $S = N_1$. As $N_2 \cap N_3$ is not a $\delta$-small

submodule of $U$, then $N_2 - S - N_3$ is a path. Also if $S \neq N_i$, for $i = 1, 2, 3$. Since $N_i \cap N_j$ is not a $\delta$-small submodule of $U$, for $i, j = 1, 2, 3$ and $i \neq j$, then $N_1, N_2$, and $N_3$ are not adjacent vertices in the graph $\Gamma_\delta(U)$. Hence, the graph $\Gamma_\delta(U)$ has no any triangle.

(3) ⇒ (1) It is clear.

**Proposition 3.6.** Let $\delta(U) \neq 0$ of a finitely generated $R$-module $U$, then the graph $\Gamma_\delta(U)$ has a triangle.

**Proof.** Since $U$ is finitely generated, from this time $(0) \neq \delta(U) \ll_\delta U$ according to Lemma 1.3(4). Now consider two possible cases for $\delta(U)$.

**Case I:** If $\delta(U)$ is a simple submodule of $U$, because $\delta(U) = \sum_{i \in \Lambda} U_i$, where $U_i \ll_\delta U$, $\forall i \in \Lambda$, we choose $\Gamma = \sum_{i \in \Lambda - \{1\}} U_i$. Then $\{U_1, \delta(U), \Gamma\}$ is a triangle in $\Gamma_\delta(U)$.

**Case II:** If $\delta(U)$ is a non-simple submodule of $U$, at that point there exists a non-trivial submodule $Z \leq U$ which $Z \subset \delta(U)$. Since $\delta(U) \ll_\delta U$, then, $Z \ll_\delta U$. Thus for each vertex H of $\Gamma_\delta(U)$, $\{Z, \delta(U), H\}$ is a triangle in $\Gamma_\delta(U)$.

**Definition 3.7.** [8] If a graph $\Gamma$ has a drawing in a plane without crossings, then $\Gamma$ is said to be planar.

**Theorem 3.8.** [8, Th. 10.30] A graph is planar if it contains no subdivision of either $K_5$ or $K_{3,3}$.

**Proposition 3.9.** If $|\mathbb{S}_\delta(U)| = 1$ or $|\mathbb{S}_\delta(U)| = 2$, and the intersection of every pair of non-small submodules of $U$ is a non-small submodule, then $\Gamma_\delta(U)$ is a planar graph.

**Proof.** Similar to that in [13, Theorem 2.15].

**Proposition 3.10.** For any module $U$, if $|\mathbb{S}_\delta(U)| \geq 3$, then $\Gamma_\delta(U)$ is not a planar graph.

**Proof.** Suppose $|\mathbb{S}_\delta(U)| \geq 3$. Then $U$ has at least three nonzero $\delta$-small submodules, at a guess $M$, $N$ and $P$. Any one of the vertices $M + N$, $N + P$ and $M + P$ are non-zero submodules and adjacent to all of submodules $M$, $N$ and $P$ in $\Gamma_\delta(U)$. $\Gamma_\delta(U)$ contains a complete graph $K_5$ for example the subgraph induced on the set $\{M, N, P, M + N, N + P\}$. By Th. 3.8, $\Gamma_\delta(U)$ is not planar.

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