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ORIGINAL STUDY

Rad-⊕-Supplemented Semimodules over Semirings

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Abstract

In this paper, Rad-⊕-supplemented semimodules are defined as generalization of ⊕-supplemented semimodules. Let R be a semiring. An R-semimodule A is called a Rad-⊕-supplemented semimodule, if each subsemimodule of A has a Rad-supplement which is a direct summand of A. Here, we investigate some properties of these semimodules and generalize some results on Rad-⊕-supplemented modules to semimodules. We prove that any finite direct sum of Rad-⊕-supplemented semimodules is Rad-⊕-supplemented. Also, we prove that if A is a subtractive semimodule with (D_3) then A is Rad-⊕-supplemented if and only if every direct summand to A is Rad-⊕-supplemented.

Keywords: Semiring, Supplemented semimodules, Rad-supplemented semimodules, Rad-⊕-Supplemented semimodules

1. Introduction

Firstly, let us point that, R will indicate a commutative semiring with identity besides A will indicate an unitary left R-semimodule throughout this article. A (left) R-semimodule A is a commutative additive semigroup which has a zero element 0_A, together with a mapping from $R\times A$ into A (sending (r, a) to ra) where $(r+s)a = ra + sa$, $r(a+b) = ra + rb$, $r(sa) = (rs)a$ and $0a = r0_A = 0$ for all $a, b \in A$ besides $r, s \in R$ [6]. Assume N is a subset of A, one says that N is an R-subsemimodule of A, precisely when N is itself a semimodule with respect to operations for A. Besides to these, for a subsemimodule X of A besides for a direct summand X of A, the notations $X \le A$ besides $X \le_{\oplus} A$ will be used respectively. $L \le A$ is said to be essential in A, indicated by $L \le_e A$, if $L \cap N \ne 0$ for all non-zero subsemimodule $N \le A$.

A subsemimodule $N \le A$ is called small in A (write $N \ll A$), if for all subsemimodule $X \le A$, with $N + X = A$ involves that $X = A$ [11]. The

radical of A, symbolized using $Rad(A)$, is the sum of all small subsemimodule of A [11]. A is named hollow, if all proper subsemimodule of A is small in A. A is named local, if it has a single maximal subsemimodul, i.e., a proper subsemimodul which contains all other subsemimoduls. A is said to be simple, if it has no nontrivial subsemimodul, besides A is said to be semisimple if it is a direct sum of its simple subsemimoduls [1,3]. The socle of A, symbolized by $Soc(A)$, is the sum of all simple subsemimoduls in A [3]. Let $L, K \le A$. K is called a supplement of L in A if it is minimal with respect to $A = L + K$. A subsemimodul K of A is a supplement (weak supplement) of L in A iff $A = L + K$ and $L \cap K \ll K$ ($L \cap K \ll A$) (see [3,15]). A is supplemented (weakly supplementd) if each subsemimodule L of A has a supplement (weak supplement) in A. Clearly, supplementd semimoduls are weakly supplementd. $L \le A$ has ample supplements in A if each subsemimodule K of A such that $A = L + K$ contains a supplement of L in A. A semimodul A is called amply supplemented if all subsemimodul of A has ample supplements in A. Hollow semimoduls are amply supplementd [9]. A semimodul A is called lifting (or D_1) if, for all $N \le A$, $A = X \bigoplus Y$ with $X \le N$ and $N \cap Y \ll A$ (see [3,9], besides [10]). $N \le A$ is a subtractive subsemimodule of A if $a, a+b \in N$ then $b \in N$ (see [2,6], and [11]). If every $N \le A$ is subtractive, at that time A is named subtractive semimodule. If C is a subtractive subsemimodule, at that time $\frac{A}{C}$ is a semimodule [6, p.165].

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In [1], the present author introduced the concept of $\bigoplus$-supplemented semimodules. Here, we introduce Rad-$\bigoplus$-supplemented semimodules a generalization of $\bigoplus$-supplemented semimodules and investigate their properties. Section 2 is devoted to some properties of $\bigoplus$-supplemented semimodules that will be used in the sequel. In Section 3, the concept of Rad-$\bigoplus$-supplementd semimoduls is introduced. It is shown that all direct summand of subtractive Rad-$\bigoplus$-supplemented semimodule with (D_3) is a Rad-$\bigoplus$-suplemented. Also, we prove that if A is a subtractive semimodule with (D_3) at that time A is Rad-$\bigoplus$-supplementd iff every direct summand to A is Rad-$\bigoplus$-supplementd. We give an example of semimodule, which is Rad-$\bigoplus$-supplemented, but not $\bigoplus$-supplementd. In Section 4, the concept of completely Rad-$\oplus$-supplemented semimodules are introduced.

In what follows, using $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ besides $\mathbb{Z}/n\mathbb{Z}$ we symbolize, respectivly, natural numbers, non-negative integers, integers, rational numbers, the semiring of integers modulo n besides the $\mathbb{Z}$-semimodul of integers modulo n.

2. On $\bigoplus$-supplemented semimodules

In this section, $\bigoplus$-supplemented semimodules are studied. We now give the next definition.

Definition 2.1. [1] An R-semimodule A is named $\bigoplus$-supplemented if for every subsemimodule N of A there is a direct summand K of A such that $A = N + K$ and $N \cap K$ is small in K.

Remark 2.2. [1] Evidently $\bigoplus$-supplemented semimodules are supplemented. Also, Hollow (or local) semimodules and lifting semimodules are $\bigoplus$-supplemented.

Definition 2.3. [1] A semimodul A is named principally $\bigoplus$-supplementd if for each $a \in A$ there is a direct summand B of A with $A = Ra + B$ and $Ra \cap B \ll B$. A semimodule A is named a weak principaly $\bigoplus$-supplementd if for each $a \in A$ there is a direct summand B with $A = Ra + B$ and $Ra \cap B \ll A$.

Remark 2.4. Every $\bigoplus$-supplemented semimodule is principally $\bigoplus$-supplemented. Evidently, every $\bigoplus$-supplementd semimodule is supplementd, but a supplementd semimodule need not be $\bigoplus$-supplementd in general as in [8, Lem. A.4 (2)].

Example 2.5. (1) Suppose that $\mathbb{N}_0$ is the semiring of non-negative integers. As $\mathbb{N}_0$ is a local $\mathbb{N}_0$-semimodule with maximal ideal $\mathbb{N}_0 \setminus \{1\}$ [6, Example 6.60]. Then by Remark 2.2, $\mathbb{N}_0$ is $\bigoplus$-supplemented $\mathbb{N}_0$-semimodule.

(2) Assume $\mathbb{Z}_{p^n}$ as an $\mathbb{Z}$-semimodule where p is prime number besides $n \in \mathbb{N}$. Then by [1, Example 2.13], $\mathbb{Z}_{p^n}$ is $\bigoplus$-supplemented semimodule.

A commutativ semiring R is named a valuation semiring if it is a local semiring besides all finitely generate ideal is principal [5]. A semimodule A is named finitely presented if $A = \frac{F}{N}$ for certain finitely generated free semimodul F besides finitely generated subsemimodule N in F.

Factor semimodul of a $\bigoplus$-supplementd semimodul is not in general $\bigoplus$-supplementd as in [1] the next example illustration this.

Example 2.6. [1, Example 2.14] Presume R is a commutativ local semiring which is not a valuation semiring. There is a finitely presented indecomposabl semimodule $A = \frac{R^{(n)}}{K}$, which cannot be generated by fewer than n elements. So, $R^{(n)}$ is $\bigoplus$-supplementd, $n \in \mathbb{N}$, $n \geq 2$. Yet A is not $\bigoplus$-supplemented.

Theorem 2.8 deals with a special case of factor semimodules of $\bigoplus$-supplemented semimodules. First, one proves the next lemma.

Let A be a semimodul and let N be a subsemimodul of A. N is named fully invariant if $f(N) \leq N$ for every endomorphism f of$A$ ($f \in End_R(A)$).

Lemma 2.7. [1]Let A be a semimodule besides let U be a fully invariant subsemimodule of A.I$A = A_1 \bigoplus A_2$, at that time $U = U \cap A_1 \bigoplus U \cap A_2$.

Theorem 2.8. Presume A is a nonzero semimodule besides presume U is a subtractive and fully invariant subsemimodule of A. If A is $\bigoplus$-supplemented, at that time A/U is $\bigoplus$supplemented. If, too, U is a direct summand of A, then U is $\oplus$-supplemented.

Proof. Since U is a subtractiv subsemimodul of A, so we have A/U is an R-semimodule. Assume A is $\bigoplus$-supplemented. Let $L \leq A$ and $U \leq L$. There exist $N, N' \leq A$ with $A = N \bigoplus N'$, $A = L + N$, and $L \cap N \ll N$. Using [14, *Lem.* 1.2(*d*)], *$(N+U)/U$ is a suplement of L/U in A/U. At present apply Lem.* 2.7, *$U = U \cap N \oplus U \cap N$. Therefore,*

$$(N+U) \cap (N'+U) \leq (N+U+N') \cap U + (N+U+U) \cap N'$$

Thus,

$(N+U)\cap(N'+U)\leq U+(N+U\cap N+U\cap N')\cap N'$

In that case $(N+U)\cap(N'+U)\leq U$ *besides* $((N+U)/U)\bigoplus((N'+U)/U) = A/U$. *Now* $(N+U)/U\leq_{\bigoplus}A/U$. *Thus,* A/U *is* $\bigoplus$*-supplemented.*

Now take $U\leq_{\bigoplus}A$. *Let* $\leq U$. *As* A *is* $\bigoplus$*-supplementd, there is* $K, K'\leq A$ *where* $A = K\oplus K'$, $A = V+ K$, *besides* $V\cap K\ll K$. *Henceforth* $U = V+U\cap K$. *Yet* $U = U\cap K\oplus U\cap K'$ *by Lem.* 2.7, *henceforth* $U\cap K\leq_{\bigoplus}U$. *Too,* $V\cap(U\cap K) = V\cap K\ll K$. *Just then,*$V\cap(U\cap K)\ll U\cap K$*using* [14, *Lem.* 1.1(*b*)]. *As a result*$U\cap K$ *is suplement of* V *in* U *besides*$U\cap K\leq_{\bigoplus}U$. *Hence* U *a*$\bigoplus$*-supplementd.* □

Definition 2.9. *A semimodule* A *is named distributive, if for* K, L, $N\leq A$, *we have*$N\cap(K+L) = N\cap K+N\cap L$*or* $N+(K\cap L) = (N+K)\cap(N+L)$.

Proposition 2.10. *Presume* A *is a nonzero distributive subtractive semimodul besides presume* U *is a subsemimodul of* A. *If* A *is* $\bigoplus$*-supplemented, at that time* A/U *is* $\bigoplus$*-supplementd. If, too,* U *is a direct summand of* A, *then* U *is* $\oplus$*-supplementd.*

proof. *The proof alike to that of Theorm* 2.8.

3. Rad-⊕-supplemented semimodules

In this section, the idea of Rad-$\bigoplus$-supplemented semimodules (or generalized $\bigoplus$-supplemented) is defined besides give the properties of these semimodules. In [12] Wang and Ding defined the notion of generalized supplemented modules. In [3] Khareeba and Alwan defined the notion of generalized supplement (or Rad-supplement) semimodules as follows:

Definition 3.1. Let A be an R-semimodule. A subsemimodule K of A is named Rad-supplement of N in A if $A = N+K$ and $N\cap K\leq Rad(K)$. We say that A is Rad-supplemented if every subsemimodule has a Rad-supplement in A.

Definition 3.2. A semimodule A is named Rad-$\bigoplus$-supplemented if every subsemimodule has a Rad-supplement that is a direct summand of A. i.e., for every subsemimodule $N\leq A$, $A = N+K$ and $A = K\bigoplus K'$ with $N\cap K\leq Rad(K)$ for some $K, K'\leq A$.

Lifting semimodules are $\bigoplus$-supplemented. Obviously, $\bigoplus$-supplemented are supplementd and Rad-$\bigoplus$-supplemented. In addition, finitely generated Rad-$\bigoplus$-supplementd semimodules are $\bigoplus$-supplemented, similar to [13, 19.3], but it is not generally true that each Rad-$\bigoplus$-supplementd semimodule is $\bigoplus$-supplementd. Whereas supplemented besides Rad-$\bigoplus$-supplemented semimodules are Rad-supplemented.

To show a finite direct sum for Rad $\bigoplus$-supplementd semimodules is Rad-$\oplus$-supplementd, we use the next usual lemm. (in [13, 41.2]).

Lemma 3.3. *Presume* N *besides* K *is subsemimodules in* A *where* $N+K$ *has Rad-supplement* X *in* A *besides* $N\cap(K+X)$ *has Rad-suplement* Y *in* N. *At that time* $X+Y$ *isRad-supplement to* K *in* A.

Proof. *Presume* X *is a Rad-suplement to* $N+K$ *in* A. *Now* $A = (N+K)+X$ *besides* $(N+K)\cap X\leq Rad(X)$. *As* $N\cap(K+X)$ *has a Rad-supplement* Y *in* N, *one has* $N = N\cap(K+X)+Y$ *besides* $(K+X)\cap Y\leq Rad(Y)$. *Now*

$$\begin{aligned} A = N+K+X &= [N\cap(K+X)+Y]+K+X \\ &= K+(X+Y) \end{aligned}$$

as well as

$$\begin{aligned} K\cap(X+Y) &\leq X\cap(K+Y)+Y\cap(K+X) \\ &\leq X\cap(K+N)+Y\cap(K+X) \\ &\leq Rad(X)+Rad(Y) \\ &\leq Rad(X+Y) \end{aligned}$$

As a result X + Y *is a* Rad-*suplement to* K *in*A.

Theorem 3.4. For any semiring R, any finite direct sum of Rad-$\bigoplus$-supplemented R-semimodules is Rad-$\bigoplus$-supplemented.

Proof. *Assume* n *is any positiv integer besides* A_i $(1\leq i\leq n)$ *be anyy finit collection off Rad-*$\bigoplus$*-supplementd* R*-semimodules. Presume*$A = A_1\bigoplus A_2\bigoplus\cdots\bigoplus A_n$.
Assume that $n = 2$, *that is,* $A = A_1\bigoplus A_2$. *Presume* $K\leq A$. *Now* $A = A_1+A_2+K$ *besides* A_1+A_2+K *has a Rad-suplement* 0 *in* A. *As* A_1 *is Rad-*$\bigoplus$*-supplementd,* $A_1\cap(A_2+K)$ *has a Rad-supplement* X *in* A_1 *with* $X\leq_{\bigoplus}A_1$. *Using Lem.* 3.3, X *is a Rad-suplement of* A_2+K *in* A. *As* A_2 *is Rad-*$\bigoplus$*-supplementd,* $A_2\cap(K+X)$ *has a Rad-suplement* Y *in* A_2 *with* $Y\leq_{\bigoplus}A_2$. *Once more applying* Lemma 3.3, *one has*$X+Y$ *is a Rad-suplement of* K *in* A. *As* $X\leq_{\bigoplus}A_1$ *besides* $Y\leq_{\bigoplus}A_2$, *in that case* $X\bigoplus Y\leq_{\bigoplus}A$. *The* proof *is ended by induction on* n. □

We prove tthe next theorem, that is a adapted form of Theorem 2.16 *in* [1]. *We need next lemm.*

Lemma 3.5. *Suppose A is a semimodule and $N \leq A$. If F is a Rad-supplement to N in A, then $\frac{F+L}{L}$ is a Rad-supplement to $\frac{N}{L}$ in $\frac{A}{L}$ for all subtractive subsemimodule L of N.*

Proof. *Via the hypothesis, $A = N + F$ besides $F\cap N \leq Rad(F)$. Hence $\frac{A}{L} = \frac{N}{L} + \frac{F+L}{L}$ for all $L \leq N$. Consider the natural epimo. $\varphi : N \to \frac{N}{L}$. Now via [13, p. 191], $\varphi(Rad(F)) \leq Rad\left(\frac{F+L}{L}\right)$. As $F\cap N \leq Rad(F)$ it follows thatt $\frac{N}{L}\bigcap\frac{F+L}{L} = \frac{L+(N\bigcap F)}{L} = \varphi(N\bigcap F)\subseteq\varphi(Rad(F)) \leq Rad\left(\frac{F+L}{L}\right)$. As a result, $\frac{F+L}{L}$ is Rad-supplement off $\frac{N}{L}$ in $\frac{A}{L}$.* □

Theorem 3.6. Let A be a subtractive Rad-$\bigoplus$-supplemented R-semimodule besides let U be a fully invariant subsemimodule of A. At that time
(1) $\frac{A}{U}$ *is Rad-$\bigoplus$-supplemented.*
(2) *If U is a direct summand to* A ($U\leq_{\bigoplus}A$)*, then U is Rad-$\bigoplus$-supplementd.*

Proof. (1) *As A is a subtractive R-semimodule, we get $\frac{A}{U}$ is an R-semimodule* [6, p. 165]. *Let $\frac{L}{U} \leq \frac{A}{U}$. As A is Rad-$\bigoplus$-supplemented, there exist $N, N' \leq A$ wherever $A = L + N$, $L\bigcap N \leq Rad(N)$ besides $A = N\bigoplus N'$. Via Lem.* 3.5, *$\frac{N+U}{U}$ is Rad-supplementt of $\frac{L}{U}$ in $\frac{A}{U}$. As $f(U) \leq U$ to all $f\in End_R(A)$, it follows as of Lem.* 2.7, *$U = (U\bigcap N)\bigoplus(U\bigcap N')$. Henceforth $(N+U)\bigcap(N'+U) \leq U$ besides as a result $\frac{N+U}{U}\bigcap\frac{N'+U}{U} = 0$, i.e. $\frac{N+U}{U}\leq_{\bigoplus}\frac{A}{U}$. Hence $\frac{A}{U}$ is Rad-$\bigoplus$-supplementd.*

(2) *Assume $U\leq_{\bigoplus}A$ and $X \leq U$. As A is Rad-$\bigoplus$-supplementd, there exist $Y, Y' \leq A$ with $A = X + Y$, $X\bigcap Y \leq Rad(Y)$ and $A = Y\bigoplus Y'$. Henceforth $U = X + (U\bigcap Y)$. Yet again applying Lem.* 2.7, *one has $U = (U\bigcap Y)\bigoplus(U\bigcap Y')$. At this time one shows $X\bigcap(U\bigcap Y) = X\bigcap Y \leq Rad(U\bigcap Y)$. Presume $x\in X\bigcap Y$. At that time $x\in Rad(Y)$ and so $Rx\ll Y$. As $U\leq_{\bigoplus}A$, using* [13, 19.3], *$Rx\ll U$. Again using* [13, 19.3], *$Rx\ll U\bigcap Y$ because $U\bigcap Y$ is direct summand of U. As a result $x\in Rad(U\bigcap Y)$. Hence, U is Rad-$\bigoplus$-supplementd.* □

Corollary 3.7. *Presume A is a nonzero Rad-$\bigoplus$-supplementd semimodule. If $Rad(A)\leq_{\bigoplus}A$, then $Rad(A)$ is Rad-$\bigoplus$-supplemented.*

Assume R is a semiring and A be an R-semimodule. In [1] *the next condition:* (D_3) *If A_1 besides A_2 are direct summands of A with $A = A_1 + A_2$, at that time $A_1\cap A_1$ is also a direct summand of A.*

Proposition 3.8. *Presume A is a subtractive Rad-$\bigoplus$-supplementd semimodule with* (D_3). *At that time each direct summand to A is Rad-$\bigoplus$-supplemented.*

Proof. *Assume $N\leq_{\bigoplus}A$ besides $U \leq N$. Now there is a $V\leq_{\bigoplus}A$ with $A = U + V$ besides $U\cap V \leq Rad(V)$. In that case $N = U + (N\cap V)$. As A has* (D_3) *$N\cap V$ is a direct sumand of A. As a result it is as well a direct sumand of N. Note $U\cap(N\cap V) = U\cap V \leq Rad(V)$. As $N\cap V\leq_{\bigoplus}A$, it tracks $U\cap V \leq Rad(N\cap V)$. As a result N is Rad-$\bigoplus$-supplemented.* □

Similar to [7, *Propo.* 2.10], *one has the next propo.*

Proposition 3.9. *Let A be a $\bigoplus$-supplemented semimodule. Then $A = A_1\bigoplus A_2$, where A_1 is a semimodule with $Rad(A_1)$ small in A_1 and A_2 is a semimodule with $Rad(A_2) = A_2$.*

We give an alike description of this detail for Rad-$\bigoplus$-supplementd semimodules.

Proposition 3.10. *Presume A is a Rad-$\bigoplus$-supplemented semimodule. At that time $A = A_1\bigoplus A_2$, where A_1 is a semimodule with $Rad(A_1) = A_1\cap Rad(A)$ besides A_2 is a semimodul with $Rad(A_2) = A_2$.*

Proof. *As A is Rad-$\bigoplus$-supplemented, there exist subsemimodules A_1 and A_2 of A with $A = Rad(A) + A_1$, $Rad(A)\cap A_1 \leq Rad(A_1)$ and $A = A_1\bigoplus A_2$. Then $Rad(A_1) = A_1\cap Rad(A)$ and $A = A_1\bigoplus Rad(A_2)$. In that case $Rad(A_2) = A_2$.* □

We now give an example of semimodule, which is Rad-$\bigoplus$-supplementd, but not $\bigoplus$-supplementd.

Example 3.11. *Consider $A = \mathbb{Q}\oplus\frac{\mathbb{Z}}{p\mathbb{Z}}$ as a semimodule over a semiring $\mathbb{N}_0$, for any prime p. Note A has single maximal subsemimodule, i.e. $Rad(A)\neq A$. Using Theorem 3.4, A is Rad-$\bigoplus$-supplementd. If A is $\bigoplus$-supplementd, then $\mathbb{Q}$ is suplemented which is a conflict.*

Similar to [4, *Theorm* 3.12] *we give the nextt theorem in semimodule theory.*

Theorem 3.12. Presume A is a subtractive semimodule with (D_3). At that time the next statements are equivalent.
(1) A *is Rad-$\bigoplus$-supplemented.*
(2) *Each direct summand to* A *is Rad-$\bigoplus$-supplementd.*
(3) $A = A_1\bigoplus A_2$ *were* A_1 *is semisimple besides* A_2 *is a Rad-$\bigoplus$-supplemented semimodule with* $Rad(A_2)$ *essential in* A_2.
(4) $A = A_1\bigoplus A_2$ *where* A_1 *is a Rad-$\bigoplus$-supplemented semimodule besides* A_2 *is a semimodule with* $Rad(A_2) = A_2$.

Proof. (1) ⇒ (2) *It followss from Prop.* 3.8.

(2) ⇒ (3) *Using* [12, *Propo.* 2.3], *$A = A_1\bigoplus A_2$, wherever A_1 is semisimple besides A_2 is a semimodule*

with $Rad(A_2)$ *essential in* A_2. *Using* (2), A_2 *is a Rad-*$\bigoplus$*-supplementd.*

(3) ⇒ (1) *Using Theorm* 3.4, A *is Rad-*$\bigoplus$*-supplementd.*

(1) ⇒ (4) *Using Propo.* 3.10, *exist subsemimodules* A_1 *besides* A_2 *of* A *with* $A = A_1 \bigoplus A_2$ *besides* $Rad(A_2) = A_2$. *As* A *has* (D_3), *by Prop.* 3.8, A_1 *is Rad-*$\bigoplus$*-supplementd.*

(4) ⇒ (1) *As* $Rad(A_2) = A_2$, A_2 *is Rad-*$\bigoplus$*-supplemented. Using* (4) *besides Thm* 3.4, A *is Rad-*$\bigoplus$*-supplemented.* □

4. Completely Rad-$\bigoplus$-Supplemented Semimodules

In this section, the idea of completely Rad-$\bigoplus$-supplementd semimoduls is studied.

Definition 4.1. [1] A semimodul A is called completely $\bigoplus$-supplemented if each direct summand of A is $\bigoplus$-supplemented.
Obviously, lifting (or D_1) semimodule is completely $\bigoplus$-supplemented [1].

Definition 4.2. A semimodule A is called completely Rad-$\oplus$-supplemented semimodule if each direct summand of A is Rad-$\oplus$-supplementd semimodule.

Definition 4.3. [1] Given a positive integer m, the semimoduls A_i $(1 \le i \le m)$ are named relatively projective if A_i is A_j-projective for all $1 \le i \ne j \le m$.

Lemma 4.4. [3, *Lemma* 1] *Presume* A *is a semimodul besides* K *supplement subsemimodule of* A. *At that time* $K \cap Rad(A) = Rad(K)$.

Proposition 4.5. [6, *Proposition* 14.22] *Presume* A *is an R-semimodule and let* $N, K \le A$. *Let* L *be a subtractive subsemimodul of* A *with* $N \le L$. *At that time* $L \cap (N + K) = N + (L \cap K)$.

Theorem 4.6. *Presume* A_i $(1 \le i \le m)$ *is a finite collection of relatively projectiv subtractive semimodules. Now the semimodule* $A = A_1 \bigoplus \cdots \bigoplus A_n$ *is Rad-*$\bigoplus$*-supplementd if and only if* A_i *is Rad-*$\bigoplus$*-supplementd for each* $1 \le i \le n$.

Proof. *In Theorm* 3.4 *the sufficiency is showed. In opposition,* A_1 *to be Rad-*$\bigoplus$*-supplementd just is shown. Assume* $F \le A_1$. *Now there is* $K \le A$ *with* $A = F + K$, $K \le_{\bigoplus} A$ *besides* $F \cap K \le Rad(K)$. *As* $A = F + K = A_1 + K$, *using* [8, Lemma 4.47], *there is* $K_1 \le K$ *with* $A = A_1 \bigoplus K_1$. *Now* $K = K_1 \bigoplus (A_1 \cap K)$ *by using* Proposition 4.5, *as* $K_1 \le K$ *besides* K *is a subtractive subsemimodule of* A. *Note* $A_1 = F + (A_1 \cap K)$ *besides* $A_1 \cap K \le_{\bigoplus} A_1$. *Henceforth,* $F \cap K = F \cap (A_1 \cap K)$ *besides* $F \cap K \le Rad(A)$, $F \cap K \le A_1 \cap K$, *at that time* $F \cap K \le (A_1 \cap K) \cap Rad(A) = Rad(A_1 \cap K)$ *using* Lemma 4.4. *As a result* A_1 *is Rad-*$\bigoplus$*-supplemented semimodule.* □

Proposition 4.7. *Presume* A *is a Rad-*$\bigoplus$*-supplementd semimodul with* (D_3). *At that time* A *is completely Rad-*$\bigoplus$*-supplementd semimodule.*

Proof. *Suppose* $N \le_{\bigoplus} A$ *and* $K \le N$. *One shows* K *hass Rad-suplement in* N *that is direct summandd of* N. *As* A *is Rad-*$\bigoplus$*-supplementd semimodule, there is* $B \le_{\bigoplus} A$ *with* $A = K + B$ *and* $K \cap B \le Rad(B)$. *From here* $N = K + (N \cap B)$. *Also,* $N \cap B \le_{\bigoplus} A$ *as* A *has* (D_3). *Now* $K \cap (N \cap B) = K \cap B$ *and* $K \cap B \le Rad(A)$, $K \cap B \le N \cap B$, *then* $K \cap B \le (N \cap B) \cap Rad(A) = Rad(N \cap B)$ *by* Lemma 4.4. □

5. Conclusion

In this paper, we have defined besides studied the concept of Rad-$\bigoplus$-supplemented semimodules over semirings. We observed that if U is a fully invariant subsemimodule of a subtractive Rad-$\bigoplus$-supplemented semimodule A, at that time $\frac{A}{U}$ is Rad-$\bigoplus$-supplemented. Too, if A is a subtractive Rad-$\bigoplus$-supplementd semimodule with (D_3), at that time each direct summand to A is Rad-$\bigoplus$-supplemented.

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