AL-Bahir
Journal for Engineering and Pure Sciences
BJEPS

Volume 4 | Issue 1 Article 1

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Source of Funding
No external Funding

Conflict of Interest
No conflict of interest

Data Availability
Publicly available data

Author Contributions
The author solely contributed to all aspects of this work, including conceptualization, methodology, data curation, formal analysis, writing – original draft preparation, review and editing, and project administration

ORIGINAL STUDY

$\bigoplus$-Supplemented Semimodules

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Abstract

In this paper, $\bigoplus$-supplemented semimodules are defined as generalizations of $\bigoplus$-supplemented modules. Let S be a semiring. An S-semimodule A is named a $\bigoplus$-supplemented semimodule, if every subsemimodule of A has a supplement which is a direct summand of A. In this paper, we investigate some properties of $\bigoplus$-supplemented semimodules besides generalize certain results on $\bigoplus$-supplemented modules to semimodules.

Keywords: Supplemented semimodules, (Completely) $\bigoplus$-supplemented semimodules, Semiperfect semimodules

1. Introduction

Firstly, let us point that, S will indicate an associative semiring with identity besides A will indicate an unitary left S-semimodule throughout this article. A (left) S-semimodule A is a commutative additive semigroup which has a zero element 0_A, together with a mapping from $S \times A$ into A (sending (s, a) to sa) such that $(r + s)a = ra + sa$, $r(a + b) = ra + rb$, $r(sa) = (rs)a$ besides $0a = r0_A = 0$ for all $a, b \in A$ besides $r, s \in S$. Let N be a subset of A. One say that N is an S-subsemimodule of A, precisely when N is itself an S-semimodule with respect to the operations for A. A subsemimodule N of A is a direct summand of A iff there is a subsemimodule N' of A satisfying $A = N \bigoplus N'$, in particular, any element a of A can be written in a unique manner as $a + a'$, where $a \in N$ and $a' \in N'$ [7, p. 184]. Too to these, for a subsemimodule X of A besides for a direct summand X of A, the notations $X \leq A$ besides $X \leq_{\bigoplus} A$ will be used respectively. $L \leq A$ is named essential in A, indicated by $L \leq_e A$, if $L \cap N \neq 0$ for all non-zero subsemimodule $N \leq A$.

A subsemimodule $N \leq A$ is named small in A (one writes $N \ll A$), if for every subsemimodule $X \leq A$, with $N + X = A$ implies that $X = A$ [14]. The radical of A, symbolized by $Rad(A)$, is the sum of all small subsemimodules of A [14]. A is named hollow, if each proper subsemimodule of A is small in A. A is named local, if it has a single maximal subsemimodule, i.e., a proper subsemimodule which contains all other subsemimodules. A is said to be simple, if it has no nontrivial subsemimodule, besides A is said to be semisimple if it is a direct sum of its simple subsemimodules [3]. The socle of A, symbolized by $Soc(A)$, is the sum of all simple subsemimodules of A [3]. Let L, $K \leq A$. K is named a supplement of L in A if it is minimal with respect to $A = L + K$. A subsemimodule K of A is a supplement (weak supplement) of L in A iff $A = L + K$ besides $L \cap K \ll K$ ($L \cap K \ll A$) [3]. A is supplemented (weakly supplemented) if each subsemimodule L of A has a supplement (weak supplement) in A. Openly, supplemente semimodules are weakly supplemente. $L \leq A$ has ample supplements in A if each subsemimodule K of A such that $A = L + K$ contains a supplement of L in A. A semimodule A is named amply supplemented if every subsemimodule of A has ample supplements in A. Hollow semimodules are ample supplemented. A semimodule A is named lifting (or D_1) if, for all $N \leq A$, there is a decomposition $A = X \bigoplus Y$ such that $X \leq N$ and $N \cap Y$ is small in A [12]. A subsemimodule N of $\leq A$ is named a subtractive subsemimodule of A if $a, a + b \in N$ then $b \in N$ for all $a, b \in A$ ([4,7]). If every subsemimodule of A is subtractive subsemimodule, at that time A is named subtractive. If C is a subtractive subsemimodule, at that time $\frac{A}{C}$ is an R-semimodule [7, p. 165].

In this paper, we introduce $\bigoplus$-supplemented semimodules and investigate their possessions. New characterizations of semiperfect semimodules

Received 4 June 2023; accepted 7 August 2023.
Available online 6 November 2023

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https://doi.org/10.55810/2313-0083.1044

are obtained using $\bigoplus$-supplemente semimodules. In Section 2, we define $\bigoplus$-supplemented semimodules. Furthermore, for any semiring S, we show that any finite direct sum of $\bigoplus$-supplemented S-semimodules is $\bigoplus$-supplemented. In Section 3, we define completely $\bigoplus$-supplemented semimodules. We also show that any $\bigoplus$-supplemented semimodule has D_3 property is completely $\bigoplus$-supplemented.

In what follows, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we indicate, respectively, natural numbers, non-negative integers, integers, rational numbers, the semiring of integers modulo n besides the $\mathbb{Z}$-semimodule of integers modulo n.

2. $\bigoplus$-Supplemented Semimodules

In this part, we introduce $\bigoplus$-supplemente semimodules. Mohamed and Müller [10] call a module A $\bigoplus$-supplementd if each submodule N of A has a supplement that is a direct summand of A. Openly, each $\bigoplus$-supplementd module is supplementd, nonetheless a supplementd modul need not be $\bigoplus$-supplemente in general (see [10, Lem. A.4 (2)]). Alike to [10] we have the next definition of $\bigoplus$-supplemented semimodules.

Definition 2.1. An S-semimodule A is named $\bigoplus$-supplemented if for every subsemimodule N of A there is a direct summand K of A such that $A = N + K$ and $N \cap K$ is small in K.

Remark 2.2. Obviously $\bigoplus$-supplemented semimodules are supplemented. In addition, Hollow semimodules and lifting semimodules are $\bigoplus$-supplemented.

Definition 2.3. [2] A semimodule A is named principally $\bigoplus$-supplemented if for each $a \in A$ there exists a direct summand B of A such that $A = Sa + B$ and $Sa \cap B$ is small in B.

Definition 2.4. [2] A semimodule A is named a weak principally $\bigoplus$-supplemented if for each $a \in A$ there exists a direct summand B such that $A = Sa + B$ and $Sa \cap B \ll A$.
Each $\bigoplus$-supplemented semimodule is supplemented. All $\bigoplus$-supplemented semimodules are principally $\bigoplus$-supplemented.

Definition 2.5. [14] A homomorphism $f : A \to B$ of left S-semimodules is named k-quasiregular if whenever $K \le A$, $a \in A \setminus K$, $a' \in K$, and $f(a) = f(a')$ there exists $s \in \text{Ker}(f)$ such that $a = a' + s$.

Definition 2.6. [14] Let A be a left S-semimodule. A left S-semimodule P together with an S-homomorphism $f : P \to A$ is named a projective cover of A if:

(1) P is projective,
(2) f is small, epimorphism besides k-quasiregular.

By [13], a semiring is named perfect (or semiperfect) if every S-semimodule (or every simple S-semimodule) has a projective cover. Too, a semiring is named semiperfect if each finitely generated S-semimodule has a projective cover. Now alike to [13] the next definition are given.

Definition 2.7. A semimodule A is named semiperfect if each factor semimodule of A has a projective cover.
Mohamed and Müller [10, Coro. 4.43] call a projective module A is semiperfect, iff A is discrete (if A has the conditions (D_1) and (D_2)), iff every submodule of A has a supplement.
Let A be a semimodule. Similar to [10], we consider the next conditions in semimodule theory.

(D_1) For each subsemimodule N of A, A has a decomposition with $A = A_1 \bigoplus A_2$, $A_1 \le N$ and $A_2 \cap N \ll A_2$.
(D_2) If N is a subsemimodule of A is such that $\frac{A}{N}$ is isomorphic to a summand of A, then N is a summand of A.
(D_3) If A_1 besides A_2 are direct summands of A with $A = A_1 + A_2$, then $A_1 \cap A_2$ is besides a direct summand of A.

Similar to [10], we call a projective subtractive semimodule A is semiperfect, if and only if A is discrete (if A has the conditions (D_1) and (D_2)), if and only if each subsemimodule of A has a supplement. Now, alike to [8, Lemma 1.2], we give the next lemma.

Lemma 2.8. Assume A is a projective subtractive semimodule. Now the next statements are equivalent.

(1) A is semiperfect.
(2) A is supplemented.
(3) A is $\bigoplus$-supplemented.

Proof: (1) ⇔ (2) Using [10, Coro. 4.43]. (1) ⇔ (3) as in the proof of [5], Propo. 1.4]. □
Let A be a semimodule. Similar to [10, Proposition 4.8], A has (D_1) iff A is amply supplementd besides each supplement subsemimodule of A is a direct

summand. As a result, every (D_1)-semimodule is $\bigoplus$-supplemented.

Lemma 2.9. Suppose that N and L are sub-semimodules of A with $N+L$ has a supplement H in A besides $N\cap(H+L)$ has a supplement G in N. At that time $H+G$ is a supplement of L in A.

Proof: Lease H be a supplement of $N+L$ in A besides, G be a supplement of $N\cap(H+L)$ in N. Now $A=(N+L)+H$ such that $(N+L)\cap H\ll H$ and $N=[N\cap(H+L)]+G$ such that $(H+L)\cap G\ll G$. As $(H+G)\cap L\leq[(G+L)\cap H]+[(H+L)\cap G]$, $H+G$ is a supplement of L in A. □

Theorem 2.10. For any semiring S, any finite direct sum of $\bigoplus$-supplementd S-semimodules is $\bigoplus$-supplementd.

Proof: Lease m be a positive integer besides A_i be a $\bigoplus$-supplemente S-semimodule for all $1\leq i\leq m$. Let $A=A_1\bigoplus\cdots\bigoplus A_m$. To show that A is $\bigoplus$-supplemente it is sufficient by induction on m to show this is the case when $m=2$. So, take $m=2$.
Let $L\leq A$. Then $A=A_1+A_2+L$ thus that A_1+A_2+L has a supplement 0 in A. Let H be a supplement of $A_2\cap(A_1+L)$ in A_2 with H is a direct summand of A_2. Using Lem 2.9, H is a supplement of A_1+L in A. Lease K be a supplement of $A_1\cap(L+H)$ in A_1 with K is a direct summand of A_1. For a second time applying Lem 2.9, we get that $H+K$ is a suplement to L in A. Since $H\leq_{\bigoplus}A_2$ and $K\leq_{\bigoplus}A_1$ so, $H+K=H\bigoplus K\leq_{\bigoplus}A$. As a result $A=A_1\bigoplus A_2$ is $\bigoplus$-supplemented. □

Corollary 2.11. A finite direct sum of semimodules with (D_1) is $\bigoplus$-supplementd.

Corollary 2.12. Any finite direct sum of hollow (or local) semimodules is $\bigoplus$-supplementd.

Example 2.13.
(1) Consider $\mathbb{N}_0$ is the semiring of non-negative integers. As $\mathbb{N}_0$ is a local $\mathbb{N}_0$-semimodule. Now by Corollary 2.12, $\mathbb{N}_0$ is $\bigoplus$-supplemented $\mathbb{N}_0$-semimodule.
(2) Consider $\mathbb{Z}_{p^n}$ as an $\mathbb{Z}$-semimodule where p is prime number and $n\in\mathbb{N}$. Now by Corollary 2.12, $\mathbb{Z}_{p^n}$ is $\bigoplus$-supplemented.

A commutativ semiring S is named a valuation semiring if it is a local semiring besides each finitely generated ideal is principal [6]. A semimodule A is named finitely presented if $A=\frac{F}{N}$ for certain finitely generated free semimodule F besides finitely generated subsemimodule N of F.
Similar to [9, Example 2.2] we have the next example show this a factor semimodule of a $\bigoplus$-supplemented semimodule is not in general $\bigoplus$-supplemented.

Example 2.14. Assume S is a commutativ local semiring which is not a valuation semiring. As in [9, Example 2.2], there is an indecomposable finitely presented semimodule $A=\frac{S^{(n)}}{K}$, which cannot be generated by fewer than n elements. Using [9] $S^{(n)}$ is $\bigoplus$-supplemente, $n\in\mathbb{N}$. However A is not $\bigoplus$-supplemente.
Theorem 2.16 deals with a special case of factor semimodules of $\bigoplus$-supplemented semimodules. First, we show the next lemma.

Lemma 2.15. Assume A is a semimodule besides let $U\leq A$ such that $f(U)\leq U$ for every $f\in End_S(A)$. If $A=A_1\bigoplus A_2$, then $U=U\cap A_1\bigoplus U\cap A_2$.

Proof: Assume $\pi_i:A\to A_i$ $(i=1,2)$ indicate the canonical projections. Take $x\in U$. Now $x=\pi_1(x)+\pi_2(x)$. Using supposition, $\pi_i(U)\leq U$ for $i=1, 2$. Hence $\pi_i(x)\in U\cap A_i$ for $i=1, 2$. Thus $U\leq U\cap A_1\bigoplus U\cap A_2$. Hence $U=U\cap A_1\bigoplus U\cap A_2$. □

Theorem 2.16. Let A be a subtractive semimodule besides let $U\leq A$ with $f(U)\leq U$ for all $f\in End_S(A)$. If A is $\bigoplus$-supplemented, at that time A/U is $\bigoplus$-supplemented. If, also, U is a direct summand of A, at that time U is also $\bigoplus$-supplemented.

Proof: As A is a subtractive S-semimodule, we get A/U is an S-semimodule [7, p. 165]. Assume A is a $\bigoplus$-supplemented semimodule. Let L be a sub-semimodule of A which contains U. There is $N, N'\leq A$ with $A=N\bigoplus N'$, $A=L+N$, and $L\cap N\ll N$. By [16], Lem. 1.2(d)], $(N+U)/U$ is a supplement of L/U in A/U. Currently apply Lem. 2.15 to have this $U=U\cap N\bigoplus U\cap N'$. As a result,

$$(N+U)\cap(N'+U)\leq(N+U+N')\cap U+(N+U+U)\cap N'$$

So,

$$(N+U)\cap(N'+U)\leq U+(N+U\cap N+U\cap N')\cap N'$$

From now $(N+U)\cap(N'+U)\leq U$ and $((N+U)/U)\bigoplus((N'+U)/U)=A/U$. Now $(N+U)/U$ is a direct summand of A/U. Therefore, A/U is $\bigoplus$-supplemented.
At the present asume U is a direct summand to A. Let V be a subsemimodule in U. As A is $\bigoplus$-supplemented, there exist $K, K'\leq A$ with $A=K\bigoplus K'$,

$A = V + K$, and $V \cap K \ll K$. Hence $U = V + U \cap K$. However $U = U \cap K \bigoplus U \cap K'$ by Lem. 2.15, hereafter $U \cap K$ is a direct summand of U. As well, $V \cap (U \cap K) = V \cap K \ll K$. Now, $V \cap (U \cap K) \ll U \cap K$ by [16, Lem. 1.1(b)]. So $U \cap K$ is a supplement of V in U besides it is a direct summand of U. Henceforth U is $\bigoplus$-supplementd. □

Corollary 2.17. Assume A is a subtractive S-semimodule besides $P(A)$ the sum of all its radical subsemimodules. If A is $\bigoplus$-supplemente, at that time $A/P(A)$ is $\bigoplus$-supplemente. If, furthermore, $P(A)$ be a direct sumand to A, at that time $P(A)$ is also $\bigoplus$-supplemented.

3. Completely $\bigoplus$-supplemented semimodules

Even though the properties lifting (or D_1), amply supplementd besides supplementd are inherited by summands, it is unknown (and improbable) that the same is correct for the property $\bigoplus$-supplemented since it is not true in modules as in [8].

Similar to [8] we give the next definition of completely $\bigoplus$-supplemente semimodules.

Definition 3.1. A semimodule A is named completely $\bigoplus$-supplemented if every direct summand of A is $\bigoplus$-supplemented.
Remarked that an S-semimodule A is supplemented if and only if A is $\bigoplus$-supplemented whenever S is Dedekind semidomain. Thus an S-semimodule A is $\bigoplus$-supplementd if and only if A is completely $\bigoplus$-supplementd. For more information about semidomains, see [4,6].
Clearly, every lifting (or D_1) semimodule is completely $\bigoplus$-supplemented.

Example 3.2. Assume x is any integer besides indicate A the $\mathbb{Z}$-semimodule $(\mathbb{Z}/x^i\mathbb{Z}) \bigoplus (\mathbb{Z}/x^j\mathbb{Z})$ ($i, j \in \mathbb{N}$). At that time A is completely $\bigoplus$-supplemented (see [8, Example 2.16]).

Definition 3.3. Given a positive integer m, the semimodules A_i $(1 \le i \le m)$ are named relatively projective if A_i is A_j- projective for all $1 \le i \neq j \le m$.

Proposition 3.4. [7, Proposition 14.22] (Semimodularity Law) Let A be a semimodule over semiring S besides let N and K be subsemimodules of A. Let L be a subtractive subsemimodule of A with $N \subseteq L$. At that point $L \cap (N + K) = N + (L \cap K)$.

Theorem 3.5. Let A_i $(1 \le i \le m)$ be a finite collection of relatively projective subtractive semimodules. Now the semimodule $A = A_1 \bigoplus \cdots \bigoplus A_m$ is $\bigoplus$-supplementd iff A_i is $\bigoplus$-supplementd for each $1 \le i \le m$.

Proof: The sufficiency is showed in Thm 2.10. In opposition, we just show A_1 to be $\bigoplus$-supplemented. Let $F \le A_1$. Now there is $K \le A$ with $A = F + K$, K is a direct sumand to A besides $F \cap K \ll K$. Since $A = F + K = A_1 + K$, by [10, Lemma 4.47], there exists $K_1 \le K$ such that $A = A_1 \bigoplus K_1$. Now $K = K_1 \bigoplus (A_1 \cap K)$ by using Proposition 3.4, since $K_1 \le K$ and K is a subtractive subsemimodule of A. Note that $A_1 = F + (A_1 \cap K)$ and $A_1 \cap K$ is a direct sumand to A_1. Henceforth, $F \cap K = F \cap (A_1 \cap K) \ll A_1 \cap K$ as in modules see [10, Lemma 4.2]. Thus A_1 is $\bigoplus$-supplement. □

Theorem 3.6. Let A be a $\bigoplus$-supplemented semimodule with (D_3). At that time A is completely $\bigoplus$-supplemented.

Proof: Assume N is a direct sumand to A besides $F \le N$. We show F has a supplement in N that is direct sumand of N. As A is $\bigoplus$-supplemente, there exists a direct summand K of A with $A = F + K$ besides $F \cap K \ll K$. As a result $N = F + (N \cap K)$. Moreover, $N \cap K$ is a direct sumand of A has (D_3). Now $F \cap (N \cap K) = F \cap K \ll N \cap K$. □

Definition 3.7. [1] Let A be a semimodule. A subsemimodule N of A is closed in A if N has no proper essential extensions in A.

Definition 3.8. [1] A semimodule A is named extending semimodule if every closed subsemimodule of A is a direct summand of A. A is said to be extending (CS-semimodul) if every subsemimodul of A is essential in a direct summand of A.
In [11] P. F. Smith calls a module A is named UC-module if each submodule of A has a unique closure in A. Similar to [11], we have the next definition.

Definition 3.9. A semimodule A is named UC-semimodule if each subsemimodul of A has a unique closure in A.

Lemma 3.10. Let A be a UC extending semimodule. Then A has (D_3).

Proof: Assume A_1, A_1 are direct summands of A with $A = A_1 + A_2$. Using [15], Proposition 1.1], $A_1 \cap A_2$ is a closed subsemimodule of A. As A is extending, $A_1 \cap A_2$ is a direct summand of A. As a result A has (D_3). □

Proposition 3.11. Assume A is a UC extending semimodule. Now A is $\bigoplus$-supplemented iff A is completely $\bigoplus$-supplemented.

Proof: The sufficiency is evidence. Conversly, supposing A is $\bigoplus$-supplemente. Using Lemma 3.10, A has (D_3). Thus A is completely $\bigoplus$-supplemente from Theorem 3.6. □

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