Studying the continuity of a real function in Tunisian secondary schools: from an intuitive approach to a formal definition

Authors Names

Ali selmi[a]

Rahim Kouki[a,b,*]

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ABSTRACT

Continuity, as a fundamental property of real functions, occupies an important place in the teaching of Real Analysis. However, its rigorous definition remains problematic.

This study focuses on the didactic approach to continuity in Tunisian curricula, by analyzing the praxeological organizations proposed in the textbook for the 3rd high school, *Mathematics* section class. The analysis is based on the theoretical framework of the Anthropological Theory of Didactics (TAD). The results show an evolution in the order of introduction of the notions of limit and continuity, moving from limit-continuity to continuity-limit, with a formal definition of continuity. This reform seems to aim for greater mathematical rigor, while relying on graphical and kinematic intuitions. However, the study of praxeological organizations reveals difficulties in the articulation between the different registers (intuitive, graphical, and formal) around the definition of continuity. Some proposed activities aim to progressively construct this formal definition, but their implementation still seems problematic. These results raise questions about the didactic conditions that promote a satisfactory understanding of the continuity of real functions by students, particularly the balance between intuitive, graphical and formal approaches. Areas for improvement are proposed for a better articulation between the different aspects in teaching.

1. Introduction

Continuity is considered to be a simple and natural common term. It is also one of the most frequently used properties of the human mind for describing objects. It is used in various fields of thought without any particular definition or discourse, as if it were an axiom for certain mathematicians and physicists or one of the premises of human thought for philosophers. From the 16th century onwards and the appearance of the notion of the function, mathematicians have made this notion part of the edifice of the field of analysis as one of the fundamental properties of functions, which has led to its introduction into mathematics teaching since the 18th century with Lagrange in France.

In Tunisia, and in all the reforms, the continuity of a real function has occupied an important place alongside the notions of limit, derivative and integral. For students aged 17-18, it was introduced after the notion of limit, but in the latest reform (in force since 2006), the order of introduction of these two notions was reversed in the Mathematics and Experimental Science sections, with a formal definition heralding the start of a phase of mathematical rigor supported by two types of intuition: kinematic movement and graphical intuition. However, this definition seems to be a volatile object. It will not have the place imagined or desired by any of the adventurers of formalism and the rigor that goes with it.

In this article, we will approach our problem within the general framework of didactic transposition (Chevallard, Y. 1985) and its subsequent developments, in order to provide elements of a response concerning the approach to the continuity of a function and the place of its formal definition.

[a]ISEFC, Virtual University of Tunis, Tunisia, E-Mail: selmi_ali@yahoo.fr

[b]IPEIEM, University of Tunis el Manar, Tunisia, E-Mail: rahim.kouki@ipeiem.utm.tn

2. Methodology

In TAD, knowledge is present in the institution in a form of mathematical organizations (MOs) called praxeologies (Bosch & Gascon, 2004). A praxeology is composed of a quadruplet (T, τ, θ, Θ) where:

- T denotes a type of task (made up of a set of specific tasks);
- τ denotes a technique that enables task resolution,
- θ designates a technology which is a discourse that justifies the technique or enables its production.
- Θ designates a theory that supports technological discourse or enables its production.

These praxeological organizations make it possible to understand and interpret the didactic organizations proposed in the textbooks.

We will study the continuity chapter of the 3rd year "mathematics section" and "experimental science section" classes, identifying the praxeological organizations (Chevallard, 1988)[1] around the notion of continuity of a real function.

3. Work carried out

In the curricula, the content to be taught for the notion of continuity is made up of the following themes:

- Continuity at a real point
- Operations on continuous functions
- Continuity over an interval
- Image of an interval by a continuous function
- Solving equations of the form $f(x) = k$

The chapter in the textbook is divided into sections (Getting started, 9-paragraph course, MCQs-Truths-False, Mobilizing skills, Exercises and Problems, Using the computer, Maths and culture), which we will present and analyse at[2] , identifying the different praxeologies. The mathematical statements used to construct the concept of continuity of a function (definitions, theorems) are to be studied and their place and function specified.

3.1 The "Getting started" section

According to the authors of the textbook, the aim of this section is to enable pupils to consolidate their previous knowledge[3].

We analyzed the activities in terms of praxeologies. The following table shows the types of tasks, techniques and technologies involved in the three activities:

[1] School textbooks are an important stage in the process of didactic transposition (Assude & Margolinas, 2005).

[2] With the exception of the last two, this is due to the absence of any computer-based teaching of mathematics, or any real interest in the history of mathematics on the part of teachers or pupils.

[3] 3rd grade textbook (Mathematics section)

Activity	Types of Tasks	Techniques	Technology/Theory
Activity 1: Solving inequalities	T1: Solve inequalities of the form: $\lvert x-\mathrm{a}\rvert \prec \mathrm{k}$ et $\lvert x-\mathrm{a}\rvert \succ \mathrm{k}$ where k is a positive real number.	τ1: Comparison rules in IR. τ2: Graphical representation to visualize the solution.	-Definition/properties of absolute values and inequalities in IR -Graphical representation of an interval on a graduated line
Activity 2: Graphical representation in an orthonormal frame of reference	T2: Graph sets of points defined by inequalities T3: Determine graphically the set of points M(x, y) in the plane satisfying conditions on x and y.	τ1: Draw lines parallel to the axes to represent sets of points τ2: Identify the intersection of the sets to determine the final set	- Definition and properties of an orthonormal reference frame - Study systems of inequalities
Activity 3: Drawing a parabola and representing points on the parabola that satisfy a property	- Draw a parabola with equation $y = x^2$ - Graph a set of points on the parabola.	τ1: Use the properties of the parabola with equation $y = x^2$ τ2: Identifying the abscissas of points from their ordinates	Definition and properties of second-degree polynomial functions Notion of graphical representation of functions

Table 1: Praxeology of prerequisites.

Comments

Activity 1 is organized around the type of task "solving inequations", the aim of which is to consolidate students' knowledge of[4] notions of distance on the real line, intervals of IR centered at a point[5] , unlimited intervals (to the left or right) and algebraic symbolism, which brings into play some semantic and syntactic aspects necessary for further learning. The algebraic solution techniques could be justified from three theoretical points of view: mathematical logic, the topology of the real line and real analysis, which provide conceptual frameworks for understanding and solving inequalities containing an absolute value in terms of relations of order, distances and intervals on the line of real numbers. In these two theoretical frameworks, solving these inequalities requires the following knowledge:

- The order of real numbers to compare algebraic expressions and establish the appropriate order relationships between these expressions.
- The notion of distance between two real numbers: the absolute value represents the distance between two points on the real line. This notion of distance is used to define the absolute value and to understand its behavior when applied to algebraic expressions.

[4] The notions of the real line, intervals, inequalities in IR, and inequations in IR were first tackled in 9e basic year (last year of secondary school, 14-15 year old students), then the work of deepening these notions continues during the first year of secondary school and the second year (especially for the two Computer Science and Technology streams).

[5]Pupils' difficulties in manipulating scripts of the form: $-\alpha < x - a < \alpha$; $a - \alpha < x < a + \alpha$; $-\beta < f(x) - f(a) < \beta$ and $f(a) - \beta < f(x) < f(a) + \beta$, are not as obvious, which is why they are the subject of study in our current thesis.

- Intervals: the solution to the inequation is expressed in the form of an interval, which represents a set of values on the real line. Intervals can be open (exclusive) or closed (inclusive) and it is the neighborhoods (open or closed in topology) that allow the solutions to be represented graphically on the real line.

In Activity 2, the same techniques are used to play with both the *y*-axis and the x-axis. The representation task can be subdivided into two sub-tasks: solving then representing. Question 3 involves finding the set of points that satisfy the two conditions on the *x*-axis and *y*-axis, which is a slightly more complex task in which the designers look for a correspondence between an interval centered at 2 and an interval centered at (-1).

In activity 3, we see an attempt by the authors to evolve the tasks of the first two activities towards a task evoking the functional relationship between two variables, one on the x-axis and the other on the y-axis.

Furthermore, the choice of processing the set of tasks is made in the graphical register. This register seems to be more relevant[6] for determining (question 2.b.) the set of real numbers x corresponding to the real numbers y in the chosen interval. Thus, by solving this task, we are witnessing the authors' topological conception of the notion of continuity of a function, or rather the beginning of the genesis of a topological definition of the notion of continuity of a function: the reciprocal image of an open neighborhood containing the $f(x)$ is an open neighborhood containing the x.

3.2 Continuity in a real: Definition

3.2.1 Analysis of the activity introducing the continuity of a function in a real number

Activity	Types of Tasks	Techniques	Technology/Theory
Objective: Study of the continuity of the function f at $x = 1$ Support: Expression of the function defined by pieces	T1: Graph the function f defined by pieces T2: Graph the set of real numbers y such that $\|y - f(1)\| < \beta$. T3: Determine graphically a sufficient condition on x so that $\|f(x) - f(1)\| < \beta$.	τ1: Plot each of the restrictions of the function f on the same reference frame τ2: Draw a horizontal band of ordinates in $]f(1) - \beta,\ f(1) + \beta[$ on the graph. τ3: Graphically identify the set of abscissas x such that the point $(x, f(x))$ is in the strip plotted.	-Implicit use of the definition of a continuous function in topological terms: f is continuous at a if, for any open interval J containing $f(a)$, there is an open interval K containing a, such that for any $x \in K,\ f(x) \in J$. -Intuitively, we can make the link between the graph of the function and the condition of its continuity: f is continuous at a if, for any $\beta > 0$, there exists $\alpha > 0$ such that : $\|x - a\| < \alpha \Rightarrow \|f(x) - f(a)\| < \beta$.

Table 2: Praxeology of the formal definition.

Comments

The aim of this activity is to construct, intuitively and graphically, the formal definition of the continuity of a function at a real value:

Tasks T2 and T3 are supposed to allow the idea to emerge that a function f is continuous at a real a if,

For all $\beta > 0$, there exists $\alpha > 0$ such that: $|x - a| < \alpha \Rightarrow |f(x) - f(a)| < \beta$.

What's more, graphic representation plays an essential role in giving meaning to this formal definition and helping students to make it their own.

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[6] Treating this activity as an algebraic calculation presents certain difficulties

However, on the one hand, the tasks in this activity are complex because of the implicit treatment of the notion of continuity to the right and left of the function at the point of abscissa 1. On the other hand, some of the tasks require work in the graphical register alone. However, it was possible to use the algebraic register, which the authors avoided here, to build up the idea of continuity. In addition, there is a new language, "a sufficient condition", which comes under the logical register and is not dealt with in any of the previous situations.

The authors go on to interpret[7] , all these tasks performed by an idea of movement, which explains the adjective "continuous" describing the function in question by the use of the terms *"...as close as..."* and *"...sufficiently close to..."*. An important question then arises: how do we get from this intuitive "definition-description" of the function to a rigorous definition? This question is justified by the fact that the activity leads, almost by chance, to the following formal definition of the notion of continuity:

> **Definition:** Let f be a function defined on an open interval I, and let a be a real number in I. We say that f is continuous at *a if* for all number $\beta > 0$, there exists a number $\alpha > 0$ such that, if x is in I and $|x - a| < \alpha$ then $|f(x) - f(a)| < \beta$.

Figure 1: Extract from the manual-Tome 1 p.23.

This formal definition seems to be a generalization of the various tasks of finding a sufficient condition on the real x for the difference in absolute value $f(x) - f(1)$ of to be infinitesimally small, without there being any actual work on this generalization: how do we go from decimal values of beta to any value which, implicitly, is a value as small as we want[8] ? Furthermore, from a didactic point of view, there is nothing to indicate that this definition is the result of any form of institutionalization. Furthermore, this definition seems to be the formalization of the intuitive interpretation proposed following the introductory activity: " $f(x)$ can be made as close to $f(1)$ as soon as x is sufficiently close to 1", which can also be generalized, in the same way, to the following *kinematic* definition:

> **Definition:** Let f be a function defined on an open interval I, and let a be a real number in I. We say that f is continuous at a if $f(x)$ can be made arbitrarily as close to $f(a)$ when x is close enough to a.

Figure 2: Personal interpretation of the commentary on activity 1 p.23

These two definitions are not in competition with each other, as they will both be abandoned as the course progresses, in favor of other technologies better suited to the needs of the students.

3.2.2 Discontinuity

This is the purpose of Activity 2, which is analyzed in the table below:

[7] Could this interpretation be the result of heuristic reasoning on the part of the student? And could the student, and under what conditions, convert this heuristic reasoning with rigorous arguments constituting the formal definition set out below?

[8] According to the definition, there is nothing to stop you taking $\beta = 100$ or $\beta = 1000$ for example.

Activity	Types of Tasks	Techniques	Technology/Theory
Activity 2: -Define the discontinuity of a function *f*. -Support: graph of the sign function on IR.	T 1: Plot the graphical representation of a function *f* defined piecewise T2: Calculate the absolute value of the difference between $f(x)$ and $f(0)$. T3: Determine whether the function *f* is continuous at 0 using the formal definition of continuity	τ1: Draw the various half-lines representing the restrictions of the function *f* τ2: To calculate $\lvert f(x) - f(0) \rvert$: use the definition of absolute value and the given values of f τ3: To state continuity at 0: apply the formal definition of the continuity of a function at a point.	- Properties of piecewise-defined functions. -Definition of absolute value. -Formal definition of the continuity of a function at a point and its negation. -Theory of real functions.

Table 3: Practice around the definition of discontinuity.

Activity 2 follows on from the introduction of the formal definition of continuity in Activity 1. Its aim is to use this definition to determine whether or not a function defined in pieces is continuous at a real point. The praxeological analysis shows that the pupils have to use the formal definition of continuity, in addition to the techniques of calculation and graphical representation. This activity represents an example where the definition of continuity doesn't work because the conditions of the definition aren't satisfied by the proposed function, which, implicitly, is a counter-example for which syntactic treatment is absent or rather could never hold because of the absence of learning about negation[9] (here the negation of the implication contained in the definition of continuity). Activity 2 p. 23 is an initial application of the definition by presenting a function which does not satisfy, at a real a, the conditions of the definition, hence the notion of discontinuity of this function (the function is not continuous at the real considered). In addition, a new definition of continuity appears as a consequence of this content on continuity and discontinuity and can be stated as follows:

> **Definition:** Let *f be* a function defined on an open interval I and a is a real number in I.
> We say that *f* is continuous on I when its representative curve C in a plane reference frame can be drawn with a continuous line, i.e. " *without lifting the pencil*".

Figure 3: Personal interpretation of the graphic consequence of the manual p.24

And it is this definition that justifies the continuity of the usual functions and the theorem established in the following paragraph.

3.3 Continuity of certain standard functions

Two activities are proposed in this part: the 1st to introduce an accepted theorem and the 2nd for apply it.

[9] This negation is as follows: $\exists \beta > 0; \forall \alpha > 0, \exists x \in I; \left(|x - 0| < \alpha \text{ et } |f(x) - f(0)| > \beta \right)$ and this technique is not available to students.

Activity	Types of Tasks	Techniques	Technology/Theory
Activity1	T1: Represent the graphs of some common functions	τ1: Plot the graphs of some common functions	-Admitted theorem on the continuity of certain usual functions
	T2: check that there are no jumps		-Graphical property of a "*curve without jumps*" function.
Activity 2	T3: Justify the continuity of certain functions	τ2: Mobilization of the theorem on the continuity of certain usual functions	-Theory of real functions, properties of usual functions

Table 4: Praxeology around the continuity of usual functions.

At this stage of the course, the aim is to extend the study of continuity to a large set of usual functions, using a graphical definition of continuity to arrive at an accepted theorem which will make it possible to declare the continuity of these functions.

3.4 Continuity of the function $|f|$

Activity	Types of Tasks	Techniques	Technology/Theory
Activity 1: Introduce the continuity theorem for absolute value functions	T1: Show that for any real numbers c and d, we have $\lvert\lvert c\rvert-\lvert d\rvert\rvert \le \lvert c-d\rvert$ T2: Show that: if f is continuous at a, then $\lvert f\rvert$ is continuous at a	τ1: Algebraic reasoning on absolute values to establish the inequality $\lvert\lvert c\rvert-\lvert d\rvert\rvert \le \lvert c-d\rvert$	-Properties of absolute values for comparison $\lvert\lvert c\rvert-\lvert d\rvert\rvert$ and $\lvert c-d\rvert$ - Theorem stating that: if f is continuous at a, then $\lvert f\rvert$ is continuous at a. - Theory of real functions[10] including properties of operations on functions (addition, product, quotient, etc.) -Formal theory of the continuity of a function at a point (part of the major fields of analysis and real topology).
Activity 2 Apply the theorem introduced	T3: Justify the continuity of functions of the form $\lvert f\rvert$ at a given point a	τ2: Use of the theorem: if f is continuous then $\lvert f\rvert$ is continuous	

Table 5: Practical aspects of the continuity of the absolute value function.

In this analysis, the aim of this organization seems to be to extend the study of continuity to functions of the form $|f(x)|$, based on a theorem establishing that the continuity of f entails the continuity of $|f|$. In addition, the emphasis is placed both on the graphical verification of continuity, and on the theoretical justification of this continuity using the theorem. This makes it possible to combine the intuitive and graphical approach of theme 1 with a more formal and theoretical dimension.

3.5 Algebraic operations on continuous functions

[10] Jules Tannery 1904, Introduction to the theory of functions of one variable

Activity	**Types of Tasks**	**Techniques**	**Technology/Theory**
Activity 1: Establish the theorem of operations on continuous functions	T1: Justify the continuity of functions obtained by algebraic operations	τ1: Check that the conditions of the theorem are fulfilled τ2: Apply the accepted continuity theorem	- Admitted theorem on the operations of continuous functions - Theory of real functions, properties of algebraic operations on functions

Table 6: Praxeology of operations on continuous functions.

This table shows that the approach adopted in the textbook aims to extend the set of continuous functions by introducing results on the preservation of continuity by the usual algebraic operations. This makes it possible to establish an important theorem which will later be used as a technique for studying certain functions in the program.

Emphasis is then placed on the application of this theorem in the justification of the continuity of more complex functions, constructed from usual functions, and this seems to be part of a more structured and deductive approach, where established theoretical results are used to extend the scope of the notion of continuity.

The graphic and intuitive dimension is still present, but it is more closely linked to an algebraic dimension using mathematical properties on the operations of functions.

The approach that emerges here is therefore an approach to continuity based on a repertoire of so-called reference functions known to be continuous, while being able to extend this property to more complex functions (operations of continuous operations: Admitted Theorem) using algebraic reasoning.

3.6 Continuity of the square root of a continuous function

Activity	**Types of Tasks**	**Techniques**	**Technology/Theory**
Activities 1 and 2 Establish the continuity theorem for the function $\sqrt{f}$	T1: Show that: if f is continuous at a, then $\sqrt{f}$ is continuous at a	τ1: Algebraic reasoning and calculus on inequalities to establish that : $\sqrt{f}$ is continuous if f is continuous.	Theorem stating that if f is continuous and positive at a, then $\sqrt{f}$ is continuous at a -Algebraic properties and monotonicity of the square root function, enabling you to reason about continuity. -Theory of real functions, including properties of operations on functions (square root, etc.)
Activity 3 Apply the continuity theorem to the function $\sqrt{f}$	T2: Study the continuity of functions of the form: $x \mapsto \sqrt{f(x)}$ at given points.	τ2: Theorem establishing that if f is continuous and positive in a, then $\sqrt{f}$ is continuous at a	
Activities 4 and 5 Modeling geometric situations using continuous functions	T3: Show that functions linked to geometric constructions are continuous	τ3: Geometric and algebraic reasoning to show the continuity of certain functions	

Table 7: Practical aspects of the continuity of the square root of a continuous function.

This analysis shows a progression in the study of continuity by looking at the square root of a continuous function. The emphasis is on establishing a general theoretical result, which can then be used to extend the analysis of continuity to functions of the form: $x \mapsto \sqrt{x}$. This is based both on algebraic reasoning and on the properties of the square root function.

The concept conveyed is a geometric concept of the continuity of the square root function resulting from the geometric construction which highlights a line without jumps. This shows a more diversified and structured approach to continuity.

3.7 Continuity on the right. Continuity on the left

Activity	**Types of Tasks**	**Techniques**	**Technology/Theory**
Activities1-2 -Establishing definitions -Establish the link between continuity in a real situation and continuity to the left and to the right	T1: Graph functions and study their continuity at given points T2: Study the continuity to the right and left of a function	τ1: use the graph of a function to study its continuity at a given point τ2: Give a graphical condition on an algebraic inequality	Definition of left-hand and right-hand continuity of a function at a point -Theorem establishing the link between continuity at a point and continuity to the right and left
Activity 3 Applications	T3: justify the continuity of positively defined functions	Application of the definitions of left-hand and right-hand continuity to analyse the continuity of functions τ3:Use of the established theorem	-Theory of real functions, properties of continuity to the right and left at a real number
Activity 4 Establish the continuity theorem for the function $\sqrt{f}$ left and right	T4: Draw the curve of a function T5: graphically justify the continuity of the function to the right of a	τ: Graphical property of a continuous function	

Table 8: Practice around continuity on the left and right.

Comments

In this section, the emphasis is on using the definitions of left-hand and right-hand continuity, and on applying an accepted theorem which relates continuity at a point to left-hand and right-hand continuity at that point.

The activities will help you to understand these concepts in greater depth by applying them to given functions, and by using graphical and algebraic reasoning to study continuity.

The concept of continuity conveyed is a topological one, linked as it is to the position of one object in relation to another (right, left) in the study of continuity, and highlights the importance of these notions in characterizing the behavior of functions at particular points. However, it should be noted that the order in which Activity 4 is presented in this part may influence the students' answers concerning the continuity of the function. Students might say that the function is not continuous at 0 because it is continuous only to the right at 0. However, we know that the square root of a number is continuous at 0 and we only need to consider the topology induced on the set of its positive reals, which is not being taught at this stage.

It would then be a good idea to choose other examples for the activity to deal with, relating to the continuity on the left (or right) of the function $\sqrt{f}$.

3.8 Continuity on an interval

Activity	Types of Tasks	Techniques	Technology/Theory
Activities1-2 Definition: Continuity on an interval	T1: Determine the continuity of a function on open, closed or semi-open intervals T2: Graph functions and analyse their continuity over specific intervals T3: Justify the continuity of a function using the definitions and properties of continuity on an interval.	τ1: Use the definitions of continuity on an open, closed or semi-open interval to analyse the continuity of a function. τ2: Application of continuity properties to determine continuity on specific intervals τ3: Graphical representation of functions to visualize their behavior and continuity over given intervals	Definitions of continuity on different types of interval (open, closed, semi-open) Continuity properties to characterize the continuity of a function on a given interval Theory of real functions, continuity on intervals -Properties of polynomial and rational functions in relation to their continuity over intervals

Table 9: Practical aspects of continuity on an interval.

Comments

This analysis shows that the aim of this part of the course is to study the continuity of functions over given intervals, using the definitions and properties associated with continuity over an interval. The activities proposed aim to reinforce students' understanding of the continuity of functions in different interval contexts, using a variety of examples to illustrate the theoretical concepts. The activities are organized in such a way as to enable students to put into practice the definitions and properties of continuity over an interval, so that they can develop their skills in graphing and justifying the continuity of functions.

3.9 Image of an interval by a continuous function

In this section, the praxeological analysis is summarized in the table below:

Activity	Types of Tasks	Techniques	Technology/Theory
Activities1-2-3	T1: Justify the continuity of a function on a given interval T2: Graphically represent the image of an interval by a continuous function T3: Graphically solve equations involving continuous functions.	τ1: Definition of the continuity of a function on an interval τ2: Graphical representation of sets of images of an interval by a continuous function τ3: Graphically solve equations related to the function and its image intervals.	-Theorem admitting that the image of an interval by a continuous function is an interval -Theory of real functions, including the properties of continuity and the image of an interval by a continuous function.

Table10: Practical aspects of the mapping of an interval by a continuous function.

Comments

The activities proposed are aimed at studying the image of an interval by a continuous function, highlighting the continuity of the function and the properties of image sets. These activities enable students to reinforce their understanding of the concepts of continuity, image of an interval and solving equations related to the function. Graphical representation is used to visualize image sets and to illustrate theoretical concepts.

The activities are designed to develop students' skills in justifying continuity, graphing and solving equations, while highlighting the theorem on the image of an interval by a continuous function.

3.10 Solving equations of the form : $f(x)=k$

Activity	Types of Tasks	Techniques	Technology/Theory
Activity 1: Solving equations of the form: $f(x)=k$	T1:Determine the solutions of equations of the form : $f(x)=k$ with given amplitude squares	τ1:Use the properties of functions to show their monotonicity and determine the image intervals τ2: Identify the solutions to the equation $f(x)=k$ as antecedents of the real k by f	-Using function graphs to find solutions to equations and IVT for the existence of solutions of equations of the form $f(x)=k$ in a given interval. - Real function theory and properties, image intervals, solutions of equations
Activities2-3-4: Application of the theorem in exercises and problems	T1: Study of functions to determine monotonicity, image intervals and solve equations. T2: Use function graphs to solve equations and determine solutions in given intervals.	τ3:Use the properties of functions to show their monotonicity and determine the image intervals (Activity 2) τ4:Solving equations graphically to find solutions in specific intervals (Activities 3 and 4) τ5:Using IVT[11] to prove that the equation $f(x)=k$ has at least one solution in a given interval (Activity 4)	-IVT and equations of the form $f(x)=k$, -Properties of functions and graphical representations.

Table11: Practice in solving equations of the form: $f(x) = k$.

Comments

This analysis shows that theme 10, which aims to solve equations of the form $f(x)=k$, represents a practical application of the properties of continuous functions, the graphical representations of these functions and the theorem giving a sufficient condition for the existence of solutions in a given interval (the intermediate value theorem).

The activities on offer aim to develop pupils' skills in solving equations, studying functions and using graphical representation and its properties to find solutions in given intervals. The use of graphs helps to visualize solutions and deepen understanding of theoretical concepts.

3.11 Analysis of MCQ and True/False Questions

According to the authors of the textbook, *the MCQ section is designed to enable students to assess their own work. The True or False section is designed to help students gradually learn logical rules.*

[11] Intermediate Value Theorem.

	Types of Tasks	Activity	Technology/Theory
MCQS	T1: Identification of the discontinuity at a point T2: Determining the continuity of a function T3: Finding solutions to equations	τ1:Use the continuity properties to identify the discontinuity of a function at a point τ2:Application of continuity definitions to determine continuity at specific points τ2:Solve equations to find solutions in given intervals	-Use graphs to visualize the curves of functions and assess continuity -Use the properties of functions to determine continuity at given points -Use equations to find solutions in given intervals Application of theorems and sufficient conditions for the existence of solutions of equations -Theory of real functions, properties -Intermediate value theorem -Theorems on the continuity of a function and the logical validation of true and false mathematical propositions
True-False	T1: Evaluating the continuity of a function over given intervals	τ:Graphical analysis to assess the continuity of a function over specific intervals	

Table 12: Practice in MCQs and True-False questions.

Comments

This analysis details the different techniques, technologies and theories involved in solving MCQ questions and True/False questions about the continuity of functions and finding solutions to equations in given intervals.

3.12 Analysis of the "Mobilizing your skills" section

According to the authors, *"this section is devoted to solving problems, most of which are integrative, in mathematical or environmental situations"*.

It comprises two situations analyzed as follows:

Activity	Types of Tasks	Techniques	Technology/Theory
Situation 1	T1: Graphical representation of sets of points satisfying a given condition T2: Use the properties of functions to demonstrate the existence of intervals with given values	τ1:Use continuity properties and function values to determine the intervals where the function remains positive or negative	-Use the graph of a function to visualize the sets of points with a given condition - Theory of real functions, sign of a function.
Situation 2	T1: Solving equations and determining the number of solutions to the proposed equation T2: Numerical calculation for precise framing of solutions to a given equation T3: Determine the intervals containing the solutions to the equation with a given accuracy	τ2:Properties of functions for finding solutions to equations and determining their number. τ3:Numerical calculation (dichotomy)	-Numerical calculation to obtain precise frames for solutions to polynomial equations. - Theory of polynomial functions and equations for solving equation.

Table 13: Problem-solving practices.

Comments

In this part, the emphasis is on using the algebraic, numerical and graphical registers by combining calculation techniques and using the intermediate value theorem to solve equations and give approximate values of solutions.

3.13 Analysis of exercices and problems

This section includes 20 exercises designed, according to the authors, "to *enable pupils to use their skills independently*".

The results of the analysis of the various financial years are presented in the table below:

Types of Tasks	Techniques	Technology/Theory
T1: Justify the continuity of a function f at a point .	τ1: Check graphically that the curve of the function shows a continuous line (with no jumps) at the point $(a, f(a))$ τ2: A more rigorous technique is to calculate the values of the real $\lvert f(x)-f(a)\rvert$ and then check that $\lvert f(x)-f(a)\rvert$ approaches 0 when x approaches a.	- Definition of the continuity of a function at a real point. -Propriétés des fonctions continues (théorème des valeurs intermédiaires, etc.)
T2: Draw the representative curve of a function f	τ3:Graphical representation of real functions with real variables τ4:Use the characteristics of the function (monotonicity, extremums, etc.)	-Properties of reference functions (polynomials, rationals, etc.) -Methods for the graphical representation of functions
T3: Justify the continuity of a function f on an interval	τ5:Visually identify any gaps in the graph -Check continuity on the definition sub-intervals	-Definition of continuity on an interval -Properties of continuous functions (theorem on the continuity of reference functions, theorem of operations on continuous functions, restrictions, etc.).
T4: Determine graphically the solutions to the equation $f(x)=0$	τ6:Study the sign of on an interval to identify changes in sign of $f(x)$ τ7:Locate graphically the abscissas of the points where the graph of f intersects the x-axis.	-Properties of continuous functions (existence of solutions to equations: intermediate value theorem) -Graphical methods for solving equations
T5: Give an approximate value for the solutions to the equation $f(x)=0$	τ8:Graphical reading of the approximate abscissa values of the points of intersection of the curve of the function with the abscissa axis. τ9:Approximation of roots by calculation	-Properties of continuous functions (existence of solutions) -Methods of numerical approximation of solutions

Table 14: Practice in solving exercises and problems.

Comments

This table shows that the exercises cover a wide range of task types, involving graphical, analytical and numerical aspects of the continuity of functions. In the 20 exercises studied, five main types of task are identified:

- T1: Justify the continuity of a function f at a point
- T2: Draw the representative curve of a function f

- T3: Justify the continuity of a function f on an interval
- T4: Determine graphically the solutions to the equation $f(x)=0$
- T5: Give an approximate value for the solutions to the equation $f(x)=0$

These 5 types of task cover all the activities proposed in the 20 exercises on continuity of functions. Each type of task calls on specific techniques, based on technologies and theories linked to the properties of continuous functions and their graphical and analytical representations. For example, justifying continuity at particular points and on intervals and then using this property of continuity to solve equations makes it possible to mobilise different techniques relating to continuity (theorems established, graphical representation).

The recurring techniques used are :

- Check graphically that the function passes through the point (a, f(a)) without *jumping*.
- Visually identify any points of discontinuity on the graph
- Use a theorem from the course to justify continuity at a point
- Use a theorem from the course to justify continuity on an interval
- Analyse the sign of the function on an interval to deduce the existence of solutions to the equation $f(x)=0$.
- Approximate graphical reading of solutions

4. Discussion and results

The table below summarizes the different activities and exercises (including multiple-choice and true/false questions, as well as skill-testing situations).

Nature of statements	Activities	Exercices	Total
Workforce	27	24	51
Number of statements using the formal definition	3	0	3
Number of statements using theorems from the course or graphical properties of curves to justify the continuity or discontinuity of a proposed function.	21	24	45

Table 15: Statistics of different activities and exercises

This table indicates a minimal presence of the formal definition in the didactic organizations related to continuity. This raises doubts about the level of rigor required in mathematical activity relating to this subject. Indeed, a study of the "Continuity" chapter shows that the formal definition of the continuity of a function at a point does not appear explicitly in the technologies used in most of the activities and exercises, and the approach adopted is based on graphical intuition and on the continuity theorem for usual functions and the theorem of operations on continuous functions.

Moreover, in solving the various exercises, the link between the graphic approach and the analytical approach is a central point, in line with the aim of establishing solid praxeologies around the notion of continuity.

The technologies identified are mainly limited to the qualitative properties of continuous functions, without involving the formal definition of continuity in. It is clear, then, that the aim of the exercises proposed is not to develop a formal understanding of continuity, but rather to grasp it intuitively and graphically. Moreover, the technologies used are:

- The theorem on the continuity of usual functions.
- The theorem of operations on continuous functions
- Graphical representation of functions
- The definition of continuity on an interval
- The existence theorem for solutions to equations (intermediate value theorem)
- Graphical and numerical methods for solving equations

The fact that the formal definition of continuity is only used in two theorem-proving activities, without being applied in the exercises, raises questions about the approach adopted in these programs:

1) What are the reasons in the textbook for the marginal use of the formal definition in α, β of continuity in a real?

2) Does this low level of use underline any perverse effects in the progressive construction of the notion of continuity through activities and exercises?

A plausible answer to these questions is that the marginal use of formal definition as a task-solving technique throughout the chapter can be interpreted for the following three reasons:

A. Focus on practical skills

In the introduction to the syllabuses, emphasis is placed on the development of certain skills (aptitudes). In line with these objectives, the didactic organization of the 'Continuity' chapter of the textbook seems to be geared towards developing certain skills such as:

- Recognize the continuity of a function at a point or over an interval from its algebraic expression or from a graph,
- Determine an exact or approximate value of a solution to an equation of the form where the function is continuous over an interval, $f(x) = k$
- Exploiting the properties of continuous functions, etc.

In this context, the formal definition could be considered too theoretical at this stage, the aim being to focus on the acquisition of certain practical skills that can be improved as the various apprenticeships relating to Real Analysis progress.

B. Gradual preparation for rigor

The limited function prescribed for formal definition could be part of a more progressive approach, aimed at preparing students for more in-depth use of formal definition at higher levels of education.

The two activities proposed (to establish the two theorems) would then introduce the students to the use of the formal definition and enable them to appreciate its role in the proof of theorems, without going too deeply into it.

C. Balance between intuition and formalism

The program designers are probably seeking to strike a balance between the development of geometric intuition and the gradual introduction of mathematical formalism. An intuitive definition (either kinematic or graphical) of continuity can be developed, while an overly limited (or even useless) use of the formal definition can be part of an intuitive approach that simplifies the students' work while preparing them for a more rigorous approach.

However, one may question the coherence of this approach. If the formal definition is not further utilized in the exercises and activities given to students, it could create an imbalance between the theoretical introduction and its practical application. Indeed, this inconsistency might lead to varied teaching practices among teachers.

5. Conclusion

In our article, we showed that the majority of the activities and all the exercises and problems proposed do not involve a formal definition of the continuity of a function at a point. In fact, an analysis of the types of task and the techniques used to solve them shows that they are mainly based on the graphical register and the general properties of continuous functions (theorems admitted or demonstrated), without explicitly using the formal definition of continuity in α, β .

This detailed praxeological analysis has enabled us to characterize the knowledge and skills targeted by the different praxeological organizations of the continuity of functions. The ultimate aim of these organizations seems to be to consolidate pupils' knowledge of the notion of continuity of functions, using both graphical representations and established theorems.

In this context, the presence of the formal definition of continuity seems to be part of a desire to establish a balance in the teaching of this notion, which was destabilized in the previous reform because of the changes made in 1998. However, this definition is too limited in scope and has only been used to prove two theorems. However, it might be thought that a better integration of this definition into the exercises could reinforce the coherence of the teaching of continuity. This was not the case either in the official textbook or in the parallel documents.

This could be interpreted by the assumption that the program designers, followed in this by the teachers, are probably aiming for a balance between the acquisition of practical skills and the gradual preparation for a more rigorous approach: enabling all students to develop a geometric and pragmatic conception of continuity, before confronting those who will go on to university studies where the formalism of mathematics is fundamental with the rigorous definition.

We believe that a move towards a more rigorous approach, involving formal definition, would be necessary to establish a sound mathematical understanding of the continuity of functions.

From this work we wish to highlight the importance of formal definition in an optimal organization that will support the development of a suitable form of rigor at the level of secondary school students, preparing them for more rigorous teaching at university.

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