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On Li-yorke Measurable Sensitivity of Topological Dynamics

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Abstract

This paper provides an overview of the relevant definitions and introduces the notion of Li-Yorke measurable sensitivity. It investigates the sensitivity of Li-Yorke measures under group actions, leading to a result that is consistent with the conservative ergodic case. Additionally, the paper employs the Lipschitz metric to establish several results.

Introduction:

Research three period[1] implies chaos in 1975 by Li-Yorke caused widespread interest in the dynamical system[1].In 2003, researchers Ethan -Akin and Sergil Kolyada presented a "Li-Yorke Sensitive" paper] 2]. In 2004, researcher S.F. Kolyada presented a paper entitled Li-Yorke Sensitive and other concepts of chaos [6]. In 2012, researchers Jared, Lucas Manuelli, and Cesar E. Saliva published a paper tidied Li-Yorke Hallett measurable Sensitive [4].In this paper, we review some preliminary definitions and introduce the concept of Li-Yorke measurable sensitive, examine Li-Yorke's measurable sensitivity to group action, resulting in a concordance that implies in the conservative ergodic case, and also use the Lipshatiz metric to prove some results.

2. Preliminaries

Definition (2. 1) [4]:

A nonsingular dynamical system (X, S, μ, T) where:

1- (X, S) is standard Borel space
2- μ is σ-finte nonatomic measure on X.
3- $T: X \to X$ is a nonsingular endomorphism, which means that for all

$A \in S, T^{-1}(A) \in S$ and $\mu A = 0$ if and only if $\mu\ (T^{-1}(A))$ =0.

Definition (2. 2) [4]:

Let $T: X \to X$ be endomorphism. Then the set A is invariant if $T^{-1}A = A$.

Definition(2.3)[4]:

T be conservative if $\forall A$ of positive measure $\exists n > 0$ such that, $\mu\ (T^{-n}\ (A) \cap A) > 0$.

Definition (2. 4) [4]:A transformation T is ergodic if whenever A is invariant set, then $\mu(A) = 0$ or $\mu(X - A) = 0$.

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Definition (2. 5) [3]:

We say a nonsingular dynamical system (X, μ, T) is measurable sensitive if every

Isomorphic $mod\ 0$ dynamical system (X_1, μ_1, T_1) and μ_1- compatible metric d on X_1, there exists $\delta > 0$ such that $\forall x \in X_1$ and $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x): d(T_1^n(x), T_1^n(y) > \delta\} > 0.$$

Definition (2.6) [3]:

For a pseudo-metric d define

a) a function $D^d: X \to \mathbb{R}$ by $D^d(x) = max\{\epsilon \geq 0: \mu(B^d(x, \epsilon)) = 0\}$
b) a sub set $Dis(d)$ of X by $Dis(d) = \{x \in X : D^d(x) > 0\}$.

Remark (2.7) [4]:

If T is conservative and A positively invariant, then A is invariant mod μ.

Definition (2.8) [4]:

Let (X, μ, T) be a nonsingular dynamical, system and let d a μ −compatible metric on X. We say that a pair (x, y) is a Li-Yorke pair if

$\lim_{n\to\infty} inf\, d(T^n x, T^n y) = 0$ and $\lim_{n\to\infty} sup\, d(T^n x, T^n y) > 0$.

We say (X, μ, T) is Li-Yorke measurably sensitive for the metric d if the set of Li-Yorke pairs $(x, y) \in X \times X$ has full measure. We say it is Li- Yorke measurably sensitive if it is Li-Yorke M-sensitive for all μ −compatible metrics.

Definition (2.9) [3]:

We say a metric d, on X is μ −compatible if assigns a positive measure to non-empty d-balls

Remark (2.10)[4]:

The system is $W-$ measurably sensitive if it is W- measurably sensitive with respect to each μ-compatible metric d.

Lemma (2.11) [3]:

Let (X, S) be a standard Bore space with nonatomic measure μ. Let $U \subset X$ be a Borel subset of full measure and let d be a μ-compatible metric defined on U. Then the metric d can be extended to a μ-compatible metric d_1 on all of X in such that d and d_1 agree on a set of full measure.

3.Li-yorke measurable sensitive

Definition (3.1):

A nonsingular dynamical system (X, μ, S, φ) is standard Boral space (X, S) with, σ-finte nonatomic measure μ and a nonsingular endomorphism $\varphi: Z \times X \to X$

[i.e., for all $A \in S, \varphi(-n, A) \in S$ and $\mu A = 0$ if and only if μ $(\varphi(-n, A))$ =0].

φ be conservative if ∀A of positive measure $\exists n > 0$ such that μ $(\varphi(-n, A) \cap A) > 0$

Definition (3.2):

We say a nonsingular, dynamical system (X, μ, φ) is measurable sensitive if [every

isomorphism mod 0 dynamical system(X_1, μ_1, φ_1) and μ_1-compatible metric d on X_1 there exists a $\delta > 0$ such that for all $x \in X_1$and $\varepsilon > 0$ threr is an $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x): d(\varphi_1(n, x), \varphi_1(n, y)) > \delta\} > 0)].$$

Definition (3.3):

Let (X, μ, S, φ) be nonsingular, dynamical system and d a μ-compatible metric

on X. We say the system is W-measurably sensitive with respect to d if there is a

$\delta > 0$ Such that for each $x \in X$

$\lim_{n\to\infty} Sup \, d(\varphi(n, x), \varphi(n, y)) > \delta$ for a.e $y \in X$.

Theorem(3.4):

Let (X, μ, S, φ)be a measurable sensitive dynamical system . If (X, μ, S, φ) is ergodic, then it is also conservative.

Proof:

Assume that (X, μ, S, φ) is a measurable sensitive and ergodic dynamical system. We need to show that for any set $A \in S \; with \; \mu(A) > 0$, there exists an integer $n > 0$ such that $\mu(\varphi(-n, A) \cap A) > 0$. Suppose, to the contrary, that there exists a set $A \in S$ with $\mu(A) > 0$ such that for all integers $n > 0$ $\mu(\varphi(-n, A) \cap A) = 0$.

Since (X, μ, S, φ) is ergodic, by the Poincare recurrence theorem , for almost every point $x \in A$, there exists a subsequence of positive integers $\{n_k\}$ such that

$$\varphi(n_k, x) \to x \;\; as \; k \to \infty.$$

Consider the sequence of sets $B_k = (\varphi_{-n_k}(A) \cap A)$. Since $\mu(\varphi(-n, A) \cap A) = 0$ for all $n > 0$, we have $\mu(B_k) = (\varphi(-n_k, A) \cap A) = 0$ for all k.

Now, let`s define the set $B = \cup_k B_k$. Since μ is a σ −finite measure, we have

$$\mu(B) \leq \sum_k \mu(B_k) = 0.$$

However, since for almost every point $x \in A$. Therefore, $\mu(A \cap B) > 0$, which contradicts the fact that $\mu(B) = 0$.Therefore, for any $A \in S$ with $\mu(A) > 0$, there exists an integer $n > 0$ such that $\mu(\varphi(-n, A) \cap A) > 0$. This proves that $(X, \mu, S, \varphi)$is conservative. Thus, we have shown that a measurable sensitive and ergodic dynamical system is conservative.

Proposition(3.5):

Suppose φ is a nonsingular transformation, if for almost every pair $(x, y) \in X \times X$ there exists $n \geq 0$ such that $d(\varphi(n, x), \varphi(n, y)) \geq \beta$, there for almost every pair $(x, y) \in X \times X$ we have

$$\lim_{n\to\infty} Sup\, d\big(\varphi(n, x), \varphi(n, y)\big) \geq \beta.$$

Proof:

Assume that there is $\beta > 0$ such that $x \in X$ and for a.e $y \in X$ and for every natural number m.

Define set $W\ (m, x)$ ={$y \in X : \exists n > m$, $d(\varphi(n + m, x), \varphi(n, y)) \geq \beta$}.

We at the present show that for every m and X the set $W\ (m, X)$ has full measure. Consider the point $\varphi(n, x)$ .Using our hypothesis for almost every $y \in X$, there exists n such that

$$d\ (\varphi(n, x), \varphi(n, y)) \geq \beta.$$

In other word the set $Z(m, X)$ ={$y \in X : \exists n > 0, d(\varphi(n + m, x), \varphi(n, y)) > \beta$}

has full measure.

Note that $W\ (m, x) = \varphi\big(-m, Z(m, X)\big)$.

Because φ be anon singular transformation, $W\ (m, X)$ must as well have full measure.

Finally, if$W_x = \cap_{m=0}^{\infty} W(m, X)$.Cleary W_x has full measure .Furthermore, for all $y \in W_x$,there are infinitely many values of n such that d($\varphi(n, x)$,$\varphi(n, y)$)> β.

So

$$\lim_{n\to\infty} Sup\ \mathrm{d}\big(\varphi(n,x),\varphi(n,y)\big) \geq \beta, \text{ for almost } y \in X.$$

Proposition(3.6):

Suppose φ is a non-singular transformation. If for almost every pair $(x,y) \in X \times X$ there exists $n \geq 0$ such that d $(\varphi(n,x)$, $\varphi(n,y)) \leq \beta$, then for almost every pair $(x,y) \in X \times X$we have

$$\lim_{n\to\infty} inf\ d\ (\varphi(n,x), \varphi(n,y)) \leq \beta.$$

Proof:

Assume that there is $\beta > 0$ such that $\forall x \in X$, for a.e $y \in X$, $\exists\ n$ such that $d\ (\varphi(n,x),\varphi(n,y)) \leq \beta$. For each natural number $\mathbb{N}$ and $x \in X$ describe

$$W\ (N,x) = \{y \in X :\ \exists n > N\ , d(\varphi(n,x),\varphi(n,y)) \leq \beta\}$$

We now proof that for all$\mathbb{N}$ and x the set $W\ (\mathbb{N},x)$ has full measure. Consider the point $\varphi(n,x)$. Using our assumption, for almost every $y \in X$, there exists n such that $d\ (\varphi(n,x),\varphi(n,y)) \leq \beta$. Then The set

$$Z\ (\mathbb{N},x)\ = \{y \in X, \exists n >\ 0: d\ (\varphi(n+\mathbb{N},x),\varphi(n,y)) < \beta\}$$

has full measure. Not that $W\ (N,x)$ =$\varphi(-N, Z(N,X)$). As φ is anon singular transformation, $W\ (N,x)$ have to full measure.

Now, let $Wx = \cap\ W\ (N,x)$.Then Wx has full measure. Furthermore, $\forall\ y \in Wx$, there are infinity a lot of values of n, therefore

$d\ (\varphi(n,x),\varphi(n,y))\ \leq \beta$. So $\lim_{n\to\infty} sup\ d\ (\varphi(n,x),\varphi(n,y)) \leq \beta$.

Proposition (3.7):

Suppose non-singular dynamical system (X, μ, φ) is Li-Yorke M-sensitive. Then any isomorphic system (Y, V, ρ) is also Li-Yorke M-sensitive.

Proof:

Suppose non singular dynamical system$(X, \mu, \varphi)$is Li-Yorke measurabl$e$ sensitiv$e$. To proof then any isomorphic system $(Y, V, \rho)$is also Li-York$e$ M-sensitiv$e$. Suppose (Y, V, ρ) is not Li-York$e$ M-Sensitiv$e$. Then $\exists$ a V-Compatibl$e$ metric d_y on Y for which (Y, V, ρ) is not Li-York$e$ M-Sensitive. Since the system is isomorphic, ther$e$ ar$e$ Bor$e$l s$e$ts $U \subseteq\ X$ and $V \subseteq Y$ of full measure and a bijection

$\pi: U \to V$ Such that $\pi o \varphi = \rho o \pi$. Define a μ-compatible metric d_U on U by d_U (x, y)=$d_y(\pi(x), \pi(y))$. By lemma 2.11 extends d_U to a μ-compatible metric d_X on X which agree with d_U on a set $X_0 \subset U \times U$ of full measur$e$ in $X \times X$.

By hypotheces is, φ is Li-Yorke measurable sensitive, so the set $L \subset X \times X$ of Li-York$e$ pairs has full measure. It follows that for any n, there exists $(u, v) \in X_0 \cap L$ such that

$$(x, y) \in A = \cap \{\varphi(-n) \times \varphi(-n)(u, v): (u, v) \in X_\circ \cap L\} = \cap \{\varphi(-n, u), \varphi(-n, v): (u, v) \in X_\circ \cap L\}.$$

Now for all n there exists $(u, v) \in X_\circ \cap L$ suchthat$(x, y)(\varphi(-n, u), \varphi(-n, v))$

this implies that

$$x = \varphi(-n, u), y = (-n, v).$$

Then

$$u = \varphi(n, x), v = \varphi(n, y)$$

implies that

$$\big(\varphi(n, x), \varphi(n, y)\big) \in X_\circ \cap L \subseteq U \times U.$$

Since $\pi: U \to V$, we have

$\pi \times \pi: U \times U \to V \times V$ we get $\pi\big(\varphi(n, x)\big), \pi\big(\varphi(n, y)\big) \in V$.

Since $A \subset U \times U$ and$(\pi \times \pi)(A) \subset V \times V \subset Y \times Y$, the set A has full measure.

Since $(x, y) \in A$ then

$$(\pi \times \pi)(x, y) \in (\pi \times \pi)(A)$$

and

$$\big(\pi(x), \pi(y)\big) \in (\pi \times \pi)(A) \subset Y \times Y$$

therefore

$$(\pi(x), \pi(y)) \in Y.$$

We get

$$\rho\big(n, \pi(x)\big) = \pi\big(\varphi(n, x)\big).$$

Hence,

$d_y(\rho(n,\pi(x)),\rho(n,\pi(y))) = d_Y(\pi(\varphi(n,x)),\pi(\varphi(n,y)))$

$= d_X(\varphi(n,x),\varphi(n,y))$.

It follows, that all pairs in $(\pi \times \pi)(A)$ are Li-Yorke for d_y , a contradiction. Then any isomorphic system (Y, V, ρ) is also Li-Yorke M-sensitive.

4. Constructing 1-Lipshitz metrics

Remark(4.1):

We shall use the term 1-Lipshitz to denote metrics that satisfy the inequality $d(\varphi(n,x),\varphi(n,y)) \leq d(x,y)$, $\forall\, x, y \in X, n \in Z$.

Definition(4.2):

Let (X, μ, φ) be a non-singular dynamical system. And let d be a metric on X, $\forall x, y \in X$. define,

$$d_\varphi(x,y) = sup_{n\geq 0} d(\varphi(n,x),\varphi(n,y)).$$

Lemma (4.3):

d_φ is a metric on X (measurable and bounded). Moreover, it is a 1-Lipshitz metric.

Proof:

To show that d_φ is a metric on X, we need to verify the following properties:

1) Non-negativity: $d_\varphi(x,y) \geq 0$ for all $x, y \in X$and $d_\varphi(x,y) = 0$ if and only if $x = y$.
2) Symmetry : $d_\varphi(x,y) = d_\varphi(y,x)$for all $x, y \in X$.
3) Triangle inequality: $d_\varphi(x,y) \leq d_\varphi(x,y) + d_\varphi(y,z)$for all $x, y, z \in X$.

First, note that $d_\varphi(x,y)$ is non-negative since it is the supremum of the a set of non-negative values. Furthermore, $d_\varphi(x,y) = 0$ if and only if

$$d(\varphi(n,x),\varphi(n,y)) = 0 \text{ for all } n \geq 0,$$

Which implies that $\varphi(n,x) = \varphi(n,y)$for all $n \geq 0$, and hence $x = y$ by the non-singularity of the system.

To proof symmetry, observe that

$$d_\varphi(x,y) = sup_{n\geq 0}\,\{d\big(\varphi(n,x),\varphi(n,y)\big) = sup_{n\geq 0}\,\{d(\varphi\big(-n,\varphi(n,x)\big),\varphi\big(-n,\varphi(n,y)\big)$$
$$= sup_{m\leq 0}\big\{d\big(\varphi(m,x),\varphi(m,y)\big)\big\} = d_\varphi(y,x),$$

Where we used the fact that φ is invertible and preserves the metric .

To establish the triangle inequality, not that for any $n \geq 0$, we have

$$d\big(\varphi(n,y),\varphi(n,z)\big) \leq d\big(\varphi(n,x),\varphi(n,y)\big) + d\big(\varphi(n,y),\varphi(n,z)\big)$$

by the triangle inequality for d. Taking the supremum over all n, we get

$$d_\varphi(x,z) \leq d_\varphi(x,y) + d_\varphi(y,z).$$

Finally, to show that d_φ is 1-Lipschitz metric note that for any $x, y \in X$, we have

$d_\varphi\big(\varphi(t,x),\varphi(t,y)\big) \leq d_\varphi(x,y)$for all $t \in Z$, by the definition of d_φ, and hence

$$d\big(\varphi(t,x),\varphi(t,y)\big) \leq d_\varphi(x,y)\text{for all } t \in Z.$$

This implies that

$$d(x,y) \leq d_\varphi(x,y)\text{forall } x,y \in X.$$

Which implies that d_φ is a 1-Lipschitz metric.

Lemma (4.4):

Let $(X$, μ,$\varphi)$ be a non-singular dynamical system, and d be a metric on X. If d is 1-Lipshitz then,

$$D^d \ \geq D^d \circ \varphi \quad on\ X.$$

Proof:

Let $\varphi^* d$ mean the metric $\varphi^* d(x,y)$ =$d(\varphi(x),\varphi(y))$. First, we observe

$$\varphi\big(-n, B(\varphi(x),\varepsilon)\big) = \big\{y \in X: d\big(\varphi(x),\varphi(y)\big) \leq \varepsilon\big\} = B^{\varphi * d}(x,\varepsilon).$$

Since φ is non-singular, $\mu(B^{\varphi * d}(x,\varepsilon) = 0 \iff \mu\,(B^d(\varphi(x),\varepsilon)) = 0$.

It follows that

$$D^{\varphi * d}(x) = D^d\,(\varphi(x)) \text{ for all } x \in X.$$

Since φ is 1-Lipshitz, $d\,(x,y) \geq d\,\big(\varphi(x),\varphi(y)\big)$, which implies

$D^d(x) \geq D^{\varphi * d}(x)$for all x.Completing the proof.

Lemma (4.5):

Let (X, μ, φ) be a non-singular dynamical system that is conservative and ergodic. Allow$d$ to aμ −compatible metric on X.Let's assume that φ it's W-measurable sensitive to d.Then, if X_1 is a positively invariant measurable set of full measure(i.e.$X_1 \subset \varphi(-n, X_1)$) andμ $(X - X_1) = 0$), thend_φ is a μ-compatible metric for the system $(X_{1,}\mu, \varphi)$, where μ and $\varphi$are the restrictions to X_1 of the original measure and transformation, respectively.

Proof:

Let d be a μ −compatible metric on X, and let X_1 be a positively invariant measurable set of full measure with respect to μ such that $X - X_1$ has measure zero. We want to show that d_φ is a μ-compatible metric on X_1.

First, we show that d_φ is a μ-measurable. Let $x, y \in X_1$and let $\varepsilon > 0$.

Since φ is conservative, there exists N such that

$$\mu(\varphi(-N, A) \cap A > 0$$

for any set A of positive measure. Thus, we can find sets A and B of positive measure such that $x, y \in A$ and

$$d\big(\varphi(n,x), \varphi(n,y)\big) > d_\varphi(x,y) - \varepsilon \ \ for\ all\ n \le N \ \ and\ \varphi(n,x), \varphi(n,y) \in B.$$

Then,

$$\begin{aligned} d_\varphi(x,y) &\le d\big(\varphi(N,x), \varphi(N,y)\big) + d_\varphi\big(\varphi(N,x), \varphi(N,y)\big) \\ &\le d\big(\varphi(N,x), \varphi(N,y)\big) + d\big(\varphi(N+1,x), \varphi(N+1,y)\big) + \cdots + d_\varphi\big(\varphi(0,x), \varphi(0,y)\big). \end{aligned}$$

Where the last inequality follows from the definition of d_φ. There fore, we have

$$\begin{aligned} &\mu\big(\{z \in X_1 : d_\varphi(z,y) > d_\varphi(x,y) - \varepsilon\}\big) \\ &\qquad \le \mu\big(\{z \in X_1 : d\big(\varphi(n,z), \varphi(n,y)\big) > \varepsilon, n \le N, \varphi(n,z), \varphi(n,y) \in B\}\big), \end{aligned}$$

Which is measurable since B has positive measure.

Next, we show that d_φ is bounded. Let $x, y \in X_1$, and let M be such that

$$d\big(\varphi(n,x), \varphi(n,y)\big) \le M \text{ for all } n \in Z.$$

Then,

$$d_\varphi(x,y) \leq d\big(\varphi(0,x),\varphi(0,y)\big) + d\big(\varphi(1,x),\varphi(1,y)\big) + \cdots \leq M(1 + M + M^2 + \cdots)$$

which is a convergent geometric series since $M > 0$. Thus,d_φ is bounded.

Finally, we show that d_φ is 1-Lipschitz. Let $x, y, z \in X_1$, then

$$d_\varphi(x,z) \leq d(\varphi(n,x),\varphi(n,z) + d_\varphi\big(\varphi(n,x),\varphi(n,z)\big)$$

$$\leq d\big(\varphi(n,x),\varphi(n,y)\big) + d\big(\varphi(n,y),\varphi(n,z)\big) + d_\varphi(x,y) + d_\varphi(y,z) \text{ for all } n \in Z,$$

By the triangle inequality and the definition of d_φ. Dividing both sides by $|n|$ and taking the limit as $|n| \to \infty$, we get

$$d_\varphi(x,z) \leq 2d_\varphi(x,y) + 2d_\varphi(y,z).$$

Therefore, d_φ is 1-Lipschitz. Thus, we have shown that d_φ is a μ-compatible metric on X_1.

Lemma (4.6):

Let (X, μ, φ) be anon-singular dynamical system that is conservative and ergodic. The $\mu - a.e.$ point of X is transitive. If d is a μ-compatible metric on X.

Proof:

Let x be a $\mu - a.e$ point of X. We want to show that x is transitive. Suppose not. Then there exist open set U and V such that $x \in U$ and $\varphi(n,x) \notin V$ for all $n \in Z$.

Let $A = U^c$and $B = V$. Then, A and B are both closed and φ −invarint. Moreover, since $x \in U$ and x is $\mu - a.e$ we have $\mu(A) = 0$.

Since φ is conservative , there exists $m > 0$ such that $\mu(-m, A) \cap A) > 0$.

Since A is closed and φ −invarint, we have $\varphi(-m, A) \subset A$. Therefore, $\mu\big(\varphi(-m,A)\big) > 0$, which contradicts $\mu(A) = 0$.Thus, x must be transitive.

Now, we will show that d is transitive. Let $x, y, z \in X$. Since x is transitive, there exists an integer n such that $\varphi(n,x)$ is arbitrarily close to y. Similarly, since y is transitive, there exists an integer m such that $\varphi(m,y)$ is arbitrarily close to z. Then, for any $\varepsilon > 0$, there exist integer n and m such that $d(\varphi(m,z) < \varepsilon/2$.

Let $k = n + m$. Then, we have

$$d(\varphi(k,x),z) \leq d\big(\varphi(k,x),\varphi(m,y)\big) + d(\varphi(m,y),z) \leq d(\varphi(n,x),y) + d(\varphi(m,y),z) < \varepsilon.$$

This, d is transitive, and we have shown that if (X, μ, φ) is non-singlar dynmical system that is conservative and ergodic, and d is a μ −compatible metric on X, then d is transitive.

Proposition (4.7):

Let (X, d) be a metric space, and let the 1-Lipschitz transformation be the$\varphi : X \times Z \to X$. It is a uniformly rigid minimal isometry. If φ is transitive.

Proof:

To prove the proposition, we need to show that:

(1) φ is an isometry, i.e., $d(\varphi(x,n), \varphi(y,n)) = d(x,y)$, for all $x, y \in X$ and $n \in Z$.

(2) φ is minimal , i.e., for any $x \in X$, the orbit $\{\varphi(x,n): n \in Z\}$ is dense in X.

(3) φ is uniformly rigiod, i.e., there exists a constant $\varepsilon > 0$ such that for any $x, y \in X$ and $n \in Z$, if $d(\varphi(x,n), \varphi(y,n)) < \varepsilon$, then $d(x,y) < \varepsilon$.

Proof (1):

Since φ is 1-Lipschitz, we have

$d(\varphi(n,x), \varphi(n,y)) \leq d((n,x),(n,y)) = d(x,y)$, for all $x, y \in X$ and $n \in Z$.

On the other hand, for any $\varepsilon > 0$, there exists a k such that $\frac{1}{k} < \varepsilon$, and then

$d(x,y) = d(\varphi(x,k), \varphi(y,k)) \leq kd(\varphi(n,x), \varphi(n,y)$for all $n \in Z$.

Letting $n \to \pm\infty$, we obtain $d(x,y) = 0$, which implies $x = y$. Therefore, φ is an isometry.

Proof (2):

Let $x \in X$ and $\varepsilon > 0$. Since φ is transitive, there exists $y \in X$ and $n \in Z$ such that

$$d(\varphi(n,x), y) < \varepsilon/2.$$

Since φ is an isometry, we have

$$d(\varphi(x, n+k), \varphi(y,k)) = d(x,y) \text{for all } k \in Z.$$

Therefore, for any $z \in X$, there exists $k \in Z$ such that

$$d(\varphi(y,k), z) < \frac{\varepsilon}{2}, \text{and then } d(\varphi(x, n+k), z) < \varepsilon.$$

This shows that the orbit $\{\varphi(n,x): n \in Z\}$ is dense in X, and hence φ is minimal.

Proof (3):

Assume, for contradiction, that φ is not uniformly rigid. Then there exist $x, y \in X$.

And a sequence of integers $\{n_k\}$ such that $d\big(\varphi(n_k, x), \varphi(n_k, y)\big) < \frac{1}{k}$ for all $k \in \mathbb{N}$.

But $d(x, y) = 1$. By passing to a subsequence if necessary, we may assume that $\lim_{k\to\infty} n_k = \infty$. Let $z_k = \varphi(n_k, x)$and $w_k = \varphi(n_k, y)$for all $k \in \mathbb{N}$.

Since φ is an isometry, we have

$$d(z_k, w_k) < \frac{1}{k} \text{for all } k \in \mathbb{N}.$$

And hence the sequence $\{z_k\}$ and $\{w_k\}$ are Cauchy. Let z and w be their respective limits. Then $d(z, w) = 0$, which implies that $z = w$ by the fact that φ is an isometry. Therefore, $x = y$, which contradicts the assumption that $d(x, y) = 1$. This prove that φ is uniformly rigid.

Remark (4.8):

Let $C_d: X \to X$ be the continuous maps on the space X, with the metric

$$d(S, S_1) = sup_{x\in X}\{d(Sx, S_1x)\}.$$

We also define a subset

$$\beta = \{S \in C_d(X, X): S \circ \varphi = \varphi \circ S\}.$$

This is a sub-semigroup of $C_d(X, X)$ under composition.

Theorem (4.9):

Let φ be a transitive and 1-Lipshitz transform and (X, d) be a metric space. The evaluation map ev_x: $\beta \to X$ defined by $S \mapsto Sx$ is an isometry for each $x \in X$. The space β is also the closure of the sequence $\{id, \varphi, \varphi \circ \varphi, \ldots.\}$in $C_d(X, X)$. The evaluation mappings ev_xan invertible isometry. If the metric space (X, d) is also complete. The semigroup β is then a group, so $\varphi \in \beta$ must be invertible.

Proof:

Let $x \in X$ be fix a point and allow S and $S_1 \in \beta$. Now, we need evidence that the map ev_x is isometric. Considering S and S_1 both trip with φ, and φ is 1-Lipshitz, for all m,

$$d\ \Big(S\big(\varphi(m, x)\big), S_1\big(\varphi(m, x)\big)\Big) \leq d(Sx, S_1x).$$

Given that S and $S_1$are both continuous and $S, S_1 \in \beta$ and

$$\beta = \{S \in C_d(X, X): S \circ \varphi = \varphi \circ S\}$$

And $C_d(X, X)$ is continuous map, the set $\{\varphi(m, x)\}$ is densefor all$y \in X$,

$$d(Sy, S_1y) \leq d(Sx, S_1x)$$

and there four ev_x is an isometry,since

$$d_X(S, S_1) = sup_{y \in X} d_X(Sy, S_1y) = d_X(Sx, S_1x) d_X(ev_xS, ev_xS_1).$$

The subset β is closed in $C_d(X, X)$. Fix a few $S \in \beta$ and $x \in X$.since x is transitive point and φ is minimum, there is sequence $\{n_j\}$ such that $\lim_{j\to\infty} \varphi(n_j, x) = Sx$. To put it another way,$\lim_{j\to\infty} ev_x \varphi(n_j, x) = Sx$ in X. This means that $\lim_{j\to\infty} \varphi(n_j) = S$ in $C_d(X, X)$, because ev_x is an isometry. Assuming that the space (X, d) is complete, the space $C_d(X, X)$is also complete. We proof that $ev_x$is surjective for all $x \in X$.

Choose $y \in X$.There is a sequence of n_j such that $\varphi(n_j, x) \to y$.The sequence $ev_x(\varphi(n_j))$ in particular is Cauchy. The sequence $\varphi(n_j)$ is Cauchy in $C_d(X, X)$because ev_x is an isometry. since β is closed, it has a limit $S \in \beta$, since $ev_xS = y$ then ev_x is surjective.

Let $S \in \beta$ be arbitrary. Because the map $ev_x$is surjective then,

$$S_1(Sx) = ev_xSS_1 = x.$$

Given that ev_x is injective and $ev_xSS_1 = (S_1S)x$, S,$S_1$is the identity, and $S_1 = S^{-1}$. Thus, every maps in β are invertible.

Theorem(4.10):

Let (X, μ, φ) be a conservative and ergodic nonsingular dynamical system. Then φ is sensitive to W-measurable or φ is isomorphic mod 0 to minimally invertible uniformly rigid isometry in a polished space.

Proof:

Let (X, μ, φ) be a conservative and ergodic non-singular dynamical system. To proof φ is either W-measurably sensitive or φ is isomorphic mod 0 to invertible minimal uniformly rigid isometry on a polish space. Assume that φ is not W-measurably sensitive. Then, by Lemma (4.5), there exists a positive invariant set X_1 of full measure such that d_φ is μ-compatible for the system (X_1, μ_1, φ_1), where μ_1 is restriction of μ on X_1 and φ_1 the restriction of φ to X_1. By lemma(4.6), φ_1 is transitive with appreciate to d_φ. Since φ_1 is 1-Lipshitz with appreciate to $d_\varphi$by proportion(4.7), φ_1 is a uniformly rigid minimal isometry on (X_1,d_φ).

Now let (X_2,d_2) be the topological completion of the metric space (X_1,d_φ). Since d_φ is separable, so d_2 is also separable, then (X_2,d_2) is polish space. We extend the measure μ_1 to X_2 by defining a set $S \subset X_2$ to be measurable if $S \cap X_1$ is measurable with $\mu_2(S) = \mu_1(S \cap X)$. Since φ_1 is an isometry, it's continuous on(X_1,d_φ), so there's a unique way to extend it to a continuous transform φ_2 on(X_2,d_2). So that φ_2 must also be an isometry with respect to d_2.According to theorem(4.9),it is invertible. Then the dynamical system(X_2, μ_2, φ_2), is measurably isomorphic to (X, μ, φ).

Proposition (4.11):

Let (X, μ, φ) be a conservative ergodic and non-singular dynamical system. If it is Li-Yorke measurable sensitive, then it is W-measurable sensitive.

Proof:

Let (X, μ, φ) be a conservative ergodic and non-singular dynamical system. Suppose it is Li-Yorke measurable sensitive. To proof it is W-measurable sensitive.

Weshow the contra positive. If φ is not W-measurably sensitive, then by theorem(4.10), it is isomorphic mod 0 to an isometry. But then the isomorphic system is both Li-Yorke measurable sensitive and an isometry for a μ-compatible metric, which is impossible. Then it is W-measurably sensitive.

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