Study on Intuitionistic Fuzzy Soft Closure Spaces

Authors Names	**ABSTRACT**
Marwa Abd Ali Hussein[1] *Dr. Neeran Tahir Abd Alameer*[2] **Publication date**: 1 /6 /2025 **Keywords:** Intuitionistic fuzzy set, soft set, intuitionistic fuzzy soft sets, intuitionistic fuzzy soft closure space, intuitionistic fuzzy soft continuous functions.	In this work, we study and introduce intuitionistic fuzzy soft closure space. In addition, the definition of intuitionistic fuzzy soft closure operator was introduced. Also, we define the notions of intuitionistic fuzzy soft operator and intuitionistic fuzzy soft continuous functions in intuitionistic fuzzy soft closure spaces.

1.Introduction

A fuzzy topological space is introduced for the first time by Chang[1] , that is after Zadeh [2] had defined the notion of fuzzy sets. Later, Atanassov [3] introduced the definition of the concept of intuitionistic fuzzy set as a generalization of fuzzy set. Moreover, the notion of intuitionistic fuzzy topology space via intuitionistic fuzzy sets was introduced by Coker[4]. After that, Molodtsove[5] defined the notion of soft set. Fuzzy soft was first introduced by Maji et al [6] Later, Maji et al [7] generalized the concept of fuzzy soft to define the concept of intuitionistic fuzzy soft set.

In this paper, we study the last type of set to introduce the definition of intuitionistic fuzzy soft closure space instead of general topology. Some relations and properties on topology fuzzy soft set in fuzzy soft closure space are proved. Continuity intuitionistic fuzzy soft in intuitionistic fuzzy soft closure space is discussed.

2. Preliminaries

In this section we will introduce necessary definitions and theorems for intuitionistic fuzzy soft sets. Firstly, it is necessary to record the definition of fuzzy set. Therefore, let X be a nonempty set, a fuzzy set λ in X is a function $\lambda: X \to I$, i.e. [0,1] is the closed unit interval. The family of all fuzzy sets on X denoted by I^X.

Definition 2.1. [3], [8] An intuitionistic fuzzy set A over the universe set X can be defined as follows: $A = \{(x, \mu_A(x), \lambda_A(x)): x \in X\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\lambda_A(x): X \to [0,1]$ with $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. The $\mu_A(x)$ and $\lambda_A(x)$ represent the degree of membership and non- membership of x to A respectively. The family of all intuitionistic fuzzy sets on X denoted by IF^X.

Definition 2.2. [3],[8] Let X be a universe set. If A and B are intuitionistic fuzzy sets in the form:

$A = \{(x, \mu_A(x), \lambda_A(x)): x \in X\}$

$B = \{(x, \mu_B(x), \lambda_B(x)): x \in X\}$, then,

[1]Marwa Abd Ali Hussein, Mathematics Department, Faculty of Education for Girls, Kufa University, Najaf, Iraq, marwaa.alayashi@student.uokufa.edu.iq.
[2]Neeran Tahir Abd Alameer, Mathematics Department, Faculty of Education for Girls, Najaf, Iraq, niran.abdulameer@uokufa.edu.iq:

1) $A \subseteq B$, if and only if, $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$, $x \in X$.
2) $A = B$, if and only if, $A \subseteq B$ and $B \subseteq A$.
3) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \lambda_A(x) \wedge \lambda_B(x)) : x \in X\}$.
4) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \lambda_A(x) \vee \lambda_B(x)) : x \in X\}$.
5) $A^c = \{(x, \lambda_A(x), \mu_A(x)) : x \in X\}$.
6) $\underline{0} = \{(x, 0, 1) : x \in X\}$ is the null intuitionistic fuzzy set.
7) $\underline{1} = \{(x, 1, 0) : x \in X\}$ is the absolute intuitionistic fuzzy set.

Definition 2.3.[3],[9] Let $(\alpha, \beta) \in [0,1]$ with $\alpha+\beta \leq 1$. An intuitionistic fuzzy point (IFP), written as $x_{(\alpha,\beta)}$ is defined to be IFS of X and given by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } x = y, \forall y \in X \\ (0,1) & \text{otherwise.} \end{cases}$$

The IFP is denoted by $x_{(\alpha,\beta)}$.

Definition 2.4. [5],[10] A soft set $F_{\mathcal{R}} = (F, \mathcal{R})$ over the universe set X is defined by a function $F: \mathcal{R} \longrightarrow P(X)$. Then $F_{\mathcal{R}}$ can be represented by the set $F_{\mathcal{R}} = \{(r, F(r)) : r \in \mathcal{R} \text{ and } F(r) \in P(X)\}$. We denote the family of all soft sets over X by $\mathcal{SS}(X, \mathcal{R})$.

Definition 2.5. [5], [10] A null soft set which denoted by $\widetilde{\phi}$ is a soft set $F_{\mathcal{R}}$ over X such that for all $r \in \mathcal{R}$, $F(r) = \emptyset$ (empty set). Also, an absolute soft set which denoted by $\widetilde{X}$ is a soft set $F_{\mathcal{R}}$ over X such that for all $r \in \mathcal{R}$, $F(r) = X$.

Definition 2.6. [5],[10] The union of two soft sets $F_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{R}}$ over X is the soft set $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}(r) = F(r) \cup \mathcal{L}(r)$ for all $r \in \mathcal{R}$. This is denoted by $F_{\mathcal{R}} \sqcup \mathcal{L}_{\mathcal{R}}$. Also, the soft intersection of $F_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{R}}$ is the soft set $\mathcal{B}_{\mathcal{R}}$ given by $\mathcal{B}(r) = F(r) \cap \mathcal{L}(r)$ for all $r \in \mathcal{R}$ and denoted by$F_{\mathcal{R}} \sqcap \mathcal{L}_{\mathcal{R}}$.

Definition 2.7.[5] The relative complement of a soft set $F_{\mathcal{R}}$ is denoted by ${F^c}_{\mathcal{R}}$, where $F^c: \mathcal{R} \longrightarrow P(X)$ defined as $F^c(r) = X - F(r)$, for all $r \in \mathcal{R}$. Clearly, ${F^c}_{\mathcal{R}} = \widetilde{X} - F_{\mathcal{R}}$.

Definition 2.8. [6], [11] Let X be an universal set and let $\mathcal{R}$ be a set of parameters and let I^X denote the power set of all fuzzy subsets of X. Let $A \subset \mathcal{R}$, A pair (Ψ, A) is called a fuzzy soft set over X, where Ψ is a function given by $\Psi_A : A \longrightarrow I^X$. Let $FS(X,\mathcal{R})$ is the family of fuzzy soft set. The relative complement of a fuzzy soft set (Ψ, A) is denoted by $(\Psi, A)^c$, defined as $(\Psi, A)^c = (\Psi^C, A^C)$, where $\Psi_A^C : A^C \longrightarrow I^X$ is a function given by $\Psi^C(r)$, it is a fuzzy soft complement of $\Psi(r^C)$, $\forall\, r \in A^C$.

Definition 2.9. [6], [11] A fuzzy soft set (Ψ, A) over X is said to be null fuzzy soft set denoted by $\widetilde{\widetilde{\phi}}$, if $r \in A$, $\Psi(r)$ is the null fuzzy set $\widetilde{0}$ of X, where $\lambda(x) = 0$, $\forall\, x \in X$. Also, a fuzzy soft set (Ψ, A) over X is said to be absolute fuzzy soft set denoted by $\widetilde{\widetilde{X}}$, if $r \in A$, $\Psi(r)$ is the absolute fuzzy set $\widetilde{1}$ of X, where $\lambda(x) = 1$, $\forall\, x \in X$.

Definition 2.10. [6] Intersection of two fuzzy soft sets (Ψ, A) and (Γ, B) over a common universe X is a fuzzy soft set (Ω, C), where $C = A \cap B$ and which is defined as follows: $\Omega(r) = \Psi(r) \cap \Gamma(r)$, $r \in C$ and written by $(\Psi, A) \,\widetilde{\sqcap}\, (\Gamma, B)$. Also, the union of two fuzzy soft sets (Ψ, A) and (Γ, B) over a common universe X is a fuzzy soft set (Ω, C), where $C = A \cup B$ and which is defined as follows: $\Omega(r)$

$$= \begin{cases} \Psi(r) & r \in A - B, \\ \Gamma(r) & r \in B - A, \\ \Psi(r) \cup \Gamma(r) & r = A \cap B, r \in C. \end{cases}$$

We write $(\Psi, A) \,\widetilde{\sqcup}\, (\Gamma, B)$.

Proposition 2.11. [6] The following results hold here.

1) $(\Psi, A) \widetilde{\sqcup} (\Psi, A) = (\Psi, A)$.
2) $(\Psi, A) \widetilde{\sqcap} (\Psi, A) = (\Psi, A)$.
3) $(\Psi, A) \widetilde{\sqcap} \widetilde{\widetilde{\phi}} = \widetilde{\widetilde{\phi}}$.
4) $(\Psi, A) \widetilde{\sqcup} \widetilde{\widetilde{\phi}} = (\Psi, A)$.
5) $(\Psi, A) \widetilde{\sqcap} \widetilde{\widetilde{X}} = (\Psi, A)$.
6) $(\Psi, A) \widetilde{\sqcup} \widetilde{\widetilde{X}} = \widetilde{\widetilde{X}}$.

Definition 2.12.[7] [12],[13],[14] Let X be an initial universe set P(X) be the power set of X, E be set of all parameters and $A \subseteq E$. Let IF^X be set of all intuitionistic fuzzy set over X. An IFS sets over X is a pair (f, A) where f is a function:
f: $A \longrightarrow IF^X$ such that f (e) $= \{(x, \mu_A(x), \lambda_A(x)) : x \in X,\ e \in E\}$. Let IFS (X, E) is the family of IFS soft set. We denote $\mu_A(x), \lambda_A(x)$ by f_e and $\acute{f}_e$, respectively. Let $(f, A) \in$ IFS (X, E) and α, β: E ⟶ [0,1] be function such that α(e)+β(e) ≤ 1 and α(e) =0 $\forall\, e \in E - A$, β(e) =1 $\forall\, e \in E - A$. The IFS set (f, A) is called an intuitionistic fuzzy soft point (IFSP), $\forall\, e \in E$, written as $f(e) = x_{(\alpha(e),\ \beta(e))}$. Obviously, f(e)(y) = $x_{(\alpha(e),\ \beta(e))}(y) = \begin{cases} (\alpha(e), \beta(e)) & \text{if } x = y, \forall\, y \in X, \\ (0,1) & \text{otherwise.} \end{cases}$ .The IFSP is denoted by $P_e(\alpha, \beta)$.

Example 2.13. Assume that $= \{x_1, x_2, x_3\}$ be an initial universe set and E = $\{e_1, e_2, e_3\}$ is a set of parameters. If (f, A) ={ $f(e_1) = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.2), (x_3, 0.6, 0.3)\}$,

$f(e_2) = \{(x_1, 0.5, 0.4), (x_2, 0.2, 0.6), (x_3, 0.1, 0.8)\}$,
$f(e_3) = \{(x_1, 0, 1), (x_2, 0.9, 0.1), (x_3, 0.3, 0.5)\}$}. Then (f, A) be an intuitionistic fuzzy soft set.

Definition 2.14 [7],[12] For two intuitionistic fuzzy soft sets (f_1, A) and (f_2, B) over common universe X, we say that (f_1, A) is an intuitionistic fuzzy soft subset of (f_2, B) and write $(f_1, A) \widetilde{\widetilde{\subseteq}} (f_2, B)$, if

1) $A \subseteq B$,
2) For all $e \in A$, $f_1(e) \subseteq f_2(e)$.

The complement of an intuitionistic fuzzy soft set (f, A) is denoted by $(f, A)^c$, defined as $(f, A)^c = (f^c, A)$, where $f_A^c : A \longrightarrow IF^X$ is a function given by $f^c(e) = [f(e)]^c$, $\forall\, e \in A$. Thus if f(e) $= \{(x, \mu_{f(e)}(x), \lambda_{f(e)}(x)) : x \in X, e \in E\}$, then $[f(e)]^c = \{(x, \lambda_{f(e)}(x), \mu_{f(e)}(x)) : x \in X, e \in E\}$.

Definition 2.15. [7], [13] A null intuitionistic fuzzy soft set, which denoted by $\widetilde{\widetilde{\varphi}}$, is an intuitionistic fuzzy soft set,

$f(e) = \{(x, \mu_A(x), \lambda_A(x)) : x \in X, \forall\ e \in E\}$, such that $\mu_A(x) = 0, \lambda_A(x) = 1$. Also, an absolute intuitionistic fuzzy soft set, which denoted by $\tilde{\tilde{1}}$, is an intuitionistic fuzzy soft
$f(e) = \{(x, \mu_A(x), \lambda_A(x)) : x \in X, \forall\ e \in E\}$ such that $\mu_A(x) = 1, \lambda_A(x) = 0$.

Definition 2.16. [7] Intersection of two intuitionistic fuzzy soft sets (f_1, A) and (f_2, B) over a common universe X is an intuitionistic fuzzy soft set (f_3, C), where $C = A \cap B, \forall\ e \in C$ and $f_3(e) = f_1(e) \cap f_2(e)$. We write $(f_1, A) \widetilde{\widetilde{\cap}} (f_2, B)$. And the union of two intuitionistic fuzzy soft sets (f_1, A) and (f_2, B) over a common universe X is an intuitionistic fuzzy soft set (f_3, C), where $C = A \cup B, \forall\ e \in C$ and $f_3(e) = f_1(e) \cup f_2(e)$. We write $(f_1, A) \widetilde{\widetilde{\cup}} (f_2, B)$ as follows:

$$f_3(e) = \begin{cases} f_1(e) & , \text{if} \quad e \in A - B, \\ f_2(e) & , \text{if} \quad e \in B - A, \\ f_1(e) \cup f_2(e) & , \text{if} \quad e = A \cap B, \quad \forall\ e \in C. \end{cases}$$

3.The Intuitionistic Fuzzy Soft Closure Spaces

Definition 3.1. An operator $\acute{C}_\ell$: IFS(X, E) ⟶ IFS(X, E) is called an intuitionistic fuzzy soft closure operator (IFS $-\ \acute{C}_\ell$, for short) on X, if for all (f, A) and $(f, B) \in IFS(X, E)$ the following axioms are satisfied:

1) $\tilde{\tilde{\varphi}} = \acute{C}_\ell(\tilde{\tilde{\varphi}})$,
2) $(f, A) \,\tilde{\tilde{\sqsubseteq}}\, \acute{C}_\ell(f, A)$,
3) $(f, A) \,\tilde{\tilde{\sqsubseteq}}\, (f, B) \Rightarrow \acute{C}_\ell((f, A)) \,\tilde{\tilde{\sqsubseteq}}\, \acute{C}_\ell((f, B))$.

The triple (X, $\acute{C}_\ell$, E) is called an intuitionistic fuzzy soft closure space (IFS $-\ \acute{C}_\ell$ space, for short).

Example 3.2. By **Definition 3.1.** $\acute{C}_\ell$: $IFS(X, E) \longrightarrow IFS(X, E)$ as follows:

$$\acute{C}_\ell(f, A) = \begin{cases} \tilde{\tilde{\varphi}} & \text{if } (f, A) = \tilde{\tilde{\varphi}}, \\ (f, A) & \text{if } (f, A) = (f, A), \\ \tilde{\tilde{1}} & \text{other wise}. \end{cases}$$

Then, (X, $\acute{C}_\ell$, E) is called an intuitionistic fuzzy soft closure space (IFS $-\ \acute{C}_\ell$ space).
Definition 3.3. Let (X, $\acute{C}_\ell$, E) be a fuzzy soft closure space. A subset (f, A) of an intuitionistic fuzzy soft closure space is called:

1) Intuitionistic fuzzy soft closed (IFCS′s in short), if $(f, A) = \acute{C}_\ell(f, A)$,
2) Intuitionistic fuzzy soft open (IFOS′s in short), if $(f, A) = \operatorname{Int}_{C_\ell}(f, A)$. The complement of intuitionistic fuzzy soft closed set is called intuitionistic fuzzy soft open set.

Proposition 3.4. Let (X, $\acute{C}_\ell$, E) be an intuitionistic fuzzy soft closure space and $(f, A)$and (f, B) are intuitionistic fuzzy soft sets over X. Then,

1) $(f, A) \,\tilde{\tilde{\sqcap}}\, (f, A) = (f, A)$.
2) $(f, A) \,\tilde{\tilde{\sqcup}}\, (f, A) = (f, A)$.
3) $(f, A) \,\tilde{\tilde{\sqcup}}\, \tilde{\tilde{\varphi}} = (f, A)$.
4) $(f, A) \,\tilde{\tilde{\sqcap}}\, \tilde{\tilde{\varphi}} = \tilde{\tilde{\varphi}}$.
5) $(f, A) \,\tilde{\tilde{\sqcup}}\, \tilde{\tilde{1}} = \tilde{\tilde{1}}$.
6) $(f, A) \,\tilde{\tilde{\sqcap}}\, \tilde{\tilde{1}} = (f, A)$.

Definition 3.5. Let (X, $\acute{C}_\ell$, E) be a fuzzy soft closure space and $(f, A) \in$ IFS(X,E). Then interior and closure of (f, A) denoted respectively by called $\operatorname{Int}_{C_\ell}(f, A)$ and $\overline{(f, A)}$ are defined as follows:

1) $\operatorname{Int}_{C_\ell}(f, A) = \tilde{\tilde{\sqcup}}\{(f, B), (f, B) \text{ is IFOS}'\text{s} : (f, B) \,\tilde{\tilde{\sqsubseteq}}\, (f, A)\}$,
2) $\overline{(f, A)} = \tilde{\tilde{\sqcap}}\{(f, B), (f, B) \text{ is IFCS}'\text{s} : (f, A) \,\tilde{\tilde{\sqsubseteq}}\, (f, B)\}$.

Proposition 3.6. Let (X, $\acute{C}_\ell$, E) be an intuitionistic fuzzy soft closure space and (f, A) and (f, B) are intuitionistic fuzzy soft sets over X. Then

1) $\left[(f_1, A) \tilde{\tilde{\cap}} (f_2, B)\right]^c = (f_1, A)^c \tilde{\tilde{\cup}} (f_2, B)^c$,
2) $\left[(f_1, A) \tilde{\tilde{\cup}} (f_2, B)\right]^c = (f_1, A)^c \tilde{\tilde{\cap}} (f_2, B)^c$.

Proof. Let $(f_1, A) \tilde{\tilde{\cap}} (f_2, B) = (f_3, C)$, where $C = A \cap B, \forall\ e \in C$ and $f_3(e) = f_1(e) \cap f_2(e)$, ${f_3}^c(e) = [f_3(e)]^c = [f_1(e) \cap f_2(e)]^c = {f_1}^c(e) \cup {f_2}^c(e)$, then $\left[(f_1, A) \tilde{\tilde{\cap}} (f_2, B)\right]^c = (f_3, C)^c$ and $(f_1, A)^c \tilde{\tilde{\cap}} (f_2, B)^c = (f_3, C)^c$. Thus,

$\left[(f_1, A) \tilde{\tilde{\cap}} (f_2, B)\right]^c = (f_1, A)^c \tilde{\tilde{\cup}} (f_2, B)^c$.

Similarly, we can prove $\left[(f_1, A) \tilde{\tilde{\cup}} (f_2, B)\right]^c = (f_1, A)^c \tilde{\tilde{\cap}} (f_2, B)^c$. ■

Proposition 3.7. Let $(X, \acute{C}_\ell, E)$ be an intuitionistic fuzzy soft closure space and (f, A) is an intuitionistic fuzzy soft set over X. Then:

1) $[\,\mathrm{Int}_{C_\ell}(f, A)]^c = \overline{[(f, A)^c]}$,
2) $[\overline{(f, A)}]^c = [\mathrm{Int}_{C_\ell}(f, A)^c]$.

Proof. By **Proposition 3.6.** ■

1) $[\,\mathrm{Int}_{C_\ell}(f, A)]^c = [\tilde{\tilde{\cup}}\{(f, B), (f, B) \text{ is IFOS's}: (f, B) \tilde{\tilde{\subseteq}} (f, A)\}]^c = \tilde{\tilde{\cap}}\{(f, B), (f, B) \text{ is IFCS's}: (f, A) \tilde{\tilde{\subseteq}} (f, B)\} = [\overline{(f, A)^c}]$.
2) The proof is similar to the first part. ■

Definition 3.8. An intuitionistic fuzzy soft closure operator $\acute{C}_\ell$ on a X is called:

1) Idempotent intuitionistic fuzzy soft, if $\acute{C}_\ell(f_1, A) = \acute{C}_\ell(\acute{C}_\ell(f_1, A))$.
2) Additive intuitionistic fuzzy soft, if $\acute{C}_\ell((f_1, A) \tilde{\tilde{\cup}} (f_2, B)) = \acute{C}_\ell(f_1, A) \tilde{\tilde{\cup}} \acute{C}_\ell(f_2, B)$ for each $(f_1, A), (f_2, B) \tilde{\tilde{\in}} IFS(X, E)$.

Definition 3.9. Let $(X, \acute{C}_\ell, E)$ be an intuitionistic fuzzy soft closure space and $Y \subseteq X$. A closure ${\acute{C}_\ell}^*$ on Y is called the intuitionistic fuzzy soft closure subspace of $(X, \acute{C}_\ell, E)$, if ${\acute{C}_\ell}^*(f, A) = Y \tilde{\tilde{\cap}} \acute{C}_\ell(f, A)$ for every $(f, A) \tilde{\tilde{\subseteq}} Y$.

Definition 3.10. An intuitionistic fuzzy soft closure space $(Y, {\acute{C}_\ell}^*, E)$ is called an intuitionistic fuzzy closed subspace of $(X, \acute{C}_\ell, E)$, if (f, A) is an IFCS set of $(Y, {\acute{C}_\ell}^*, E)$, then (f, A) is an IFCS set of $(X, \acute{C}_\ell, E)$.

Remark 3.11. Let $(Y, {\acute{C}_\ell}^*, E)$ be an intuitionistic fuzzy soft closure subspace of an intuitionistic fuzzy soft closure space $(X, \acute{C}_\ell, E)$, if (f, A) is an IFCS (IFOS) set in $(X, \acute{C}_\ell, E)$, then $Y \tilde{\tilde{\cap}} (f, A)$ is an IFCS (IFOS) set in $(Y, {\acute{C}_\ell}^*, E)$.

Proposition 3.12. Let $(X, \acute{C}_\ell, E)$ be $IFS - \acute{C}_\ell$ space and $(f, A) \tilde{\tilde{\in}} IFS(X, E)$. If $\acute{C}_\ell(f, A \tilde{\tilde{\subseteq}} (f, A)$, then (f, A) is an IFCS set in $(X, \acute{C}_\ell, E)$.

Proof. The proof obtained directly from hypothesis and **Definition 3.1.**■

Remarks 3.13.

1) The intersection of any collection of IFCS sets in an $IFS - \acute{C}_\ell$ space is IFCS set.

2) The union of any collection of IFOS sets in IFS $-\acute{C}_{\ell}$ space is an IFOS set.

Definition 3.14. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space. An interior operator $\operatorname{Int}_{C_{\ell}}$ is a function from power set of X to itself for each $(f, A) \tilde{\tilde{\subseteq}} X$, $\operatorname{Int}_{C_{\ell}}(f, A) = (f, A)\text{-} \ \acute{C}_{\ell}\ ((f, A)\text{-}X)$. The set $\operatorname{Int}_{C_{\ell}}(f, A)$ is called the interior of (f, A) in $(X, \acute{C}_{\ell}, E)$.

Proposition 3.15. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space and (f, A) and (f, B) are intuitionistic fuzzy soft sets over X. Then,

1) $\operatorname{Int}_{C_{\ell}}(\tilde{\tilde{\varphi}}) = \tilde{\tilde{\varphi}}$.
2) $\operatorname{Int}_{C_{\ell}}(\tilde{\tilde{1}}) = \tilde{\tilde{1}}$.
3) $\operatorname{Int}_{C_{\ell}}(f, A) \tilde{\tilde{\subseteq}} (f, A)$.
4) $\operatorname{Int}_{C_{\ell}}\left(\operatorname{Int}_{C_{\ell}}(f, A)\right) = \operatorname{Int}_{C_{\ell}}(f, A)$.
5) If $(f, A) \tilde{\tilde{\subseteq}} (f, B)$, then $\operatorname{Int}_{C_{\ell}}(f, A) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}(f, B)$.

Proposition 3.16. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space and $(f_1, A)$and (f_2, B) are intuitionistic fuzzy soft sets over X. Then,

$$\operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B)) = \operatorname{Int}_{C_{\ell}}(f_1, A) \tilde{\tilde{\cap}} \operatorname{Int}_{C_{\ell}}(f_2, B).$$

Proof. Since $(f_1, A) \tilde{\tilde{\cap}} (f_2, B) \tilde{\tilde{\subseteq}} (f_1, A)\ \forall\ e \in E$, then $\operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B)) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}(f_1, A)$.

Similarly, $\operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B)) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}(f_2, B)$.
Therefore, $\operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B)) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}(f_1, A) \tilde{\tilde{\cap}} \operatorname{Int}_{C_{\ell}}(f_2, B)$. Let (f, D) be intuitionistic fuzzy soft open sets such that $(f, D) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B))$ .Then, $(f, D) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}(f_1, A)$ and $(f, D) \tilde{\tilde{\subseteq}}$ $\operatorname{Int}_{C_{\ell}}(f_2, B)$. $f(e) \subseteq f_1(e) \cap f_2(e) = (f_1 \cap f_2)(e)$, $\forall\ e \in E$.

$(f, D) \tilde{\tilde{\subseteq}} (f_1, A) \tilde{\tilde{\cap}} (f_2, B)$. So $(f, D) = \operatorname{Int}_{C_{\ell}}(f, D) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B))$

This implies that $\operatorname{Int}_{C_{\ell}}(f_1, A) \tilde{\tilde{\cap}} \operatorname{Int}_{C_{\ell}}(f_2, B) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}}((f_1, A) \tilde{\tilde{\cap}} (f_2, B))$.

Then, $\operatorname{Int}_{C_{\ell}}((f, A) \tilde{\tilde{\cap}} (f, B)) = \operatorname{Int}_{C_{\ell}} (f, A) \tilde{\tilde{\cap}} \operatorname{Int}_{C_{\ell}}(f, B)$.■

Proposition 3.17. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space, (f, A) and (f, B) are intuitionistic fuzzy soft sets over X. Then

$$\operatorname{Int}_{C_{\ell}}(f, A) \tilde{\tilde{\cup}} \operatorname{Int}_{C_{\ell}}(f, B) \tilde{\tilde{\subseteq}} \operatorname{Int}_{C_{\ell}} ((f, A) \tilde{\tilde{\cup}} (f, B)).$$

Proof**.** Similarly, we can prove it is by **Proposition 3.16.**■

Definition 3.18. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space. An IFS subset $(f, A) \in \mathrm{IFS}(X, E)$ is called intuitionistic fuzzy soft neighborhood of IFSP, $P_e(\alpha, \beta) \in (f, A)$, if there exist an IFOS set (f, B) such $P_e(\alpha, \beta) \tilde{\tilde{\in}} (f, B) \tilde{\tilde{\subseteq}} (f, A)$. The all neighborhood of IFSP, $P_e(\alpha, \beta)$ is called its neighborhood system and denoted by $\tilde{\tilde{N}}_{C_{\ell}} P_e(\alpha, \beta)$.

Definition 3.19. Let $(X, \acute{C}_{\ell}, E)$ be an intuitionistic fuzzy soft closure space. An IFS subset $(f, A) \in$ IFS(X,E) is called intuitionistic fuzzy soft neighborhood of IFS set (f, B), if there exist an IFOS set (f, H) such $(f, B) \tilde{\tilde{\subseteq}} (f, H) \tilde{\tilde{\subseteq}} (f, A)$.

Proposition 3.20. A set $(f, A) \in IFS(X, E)$ is an IFOS, if and only if, (f, A) is an intuitionistic fuzzy soft neighborhood of all its intuitionistic fuzzy soft points.

Proof. Necessity, it is clear.

Sufficiency, let (f, A) be an IFS set and $\{P_{ei}(\alpha, \beta): i \in I\}$ be a family of all its intuitionistic fuzzy soft points of (f, A), then for each $i \in I$, there exist an IFOS sets (f, A_i) such that $P_{ei}(\alpha, \beta) \tilde{\tilde{\in}} (f, A_i) \tilde{\tilde{\subseteq}} (f, A)$. Therefor,

$\tilde{\tilde{\cup}}_{i \in I}(P_{ei}(\alpha, \beta)) \tilde{\tilde{\subseteq}} (f, A_i) \tilde{\tilde{\subseteq}} (f, A)$, we have $(f, A) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\cup}}_{i \in I}(P_{ei}(\alpha, \beta))) \tilde{\tilde{\subseteq}} (f, A_i)$.

So, $(f, A_i) = (f, A)$.

4. Iintuitionistic Fuzzy Soft Continuous Function in Iintuitionistic Fuzzy Soft Closure Spaces

Definition 4.1. Let $(X, \acute{C}_{\ell}, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. Then the function $\mathfrak{R}: (X, \acute{C}_{\ell}, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is said to be intuitionistic fuzzy soft continuous at IFSP, $P_e(\alpha, \beta)$ of IFS(X, E), if for each a $(f, B) \in \tilde{\tilde{N}}_Y(\mathfrak{R}(P_e(\alpha, \beta)))$, there exist a $(f, A) \in \tilde{\tilde{N}}_X P_e(\alpha, \beta)$ such that $\mathfrak{R}(f, A) \tilde{\tilde{\subseteq}} (f, B)$. If $\mathfrak{R}$ is an intuitionistic fuzzy soft continuous at each IFSP, $P_e(\alpha, \beta)$ of IFS(X, E), then $\mathfrak{R}$ is called intuitionistic fuzzy soft continuous function.

Definition 4.2. Let $(X, \acute{C}_{\ell}, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. A function $\mathfrak{R}: (X, \acute{C}_{\ell}, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is called:

1) Intuitionistic fuzzy soft closed function, if $\mathfrak{R}$ (f, A) is an IFCS subset of $(Y, \acute{C}_{\ell 1}, E)$, for every (f, A) is an IFCS subset of $(X, \acute{C}_{\ell}, E)$.
2) Intuitionistic fuzzy soft open function, if $\mathfrak{R}$ (f, B) is an IFOS subset of $(Y, \acute{C}_{\ell 1}, E)$, for every (f, B) an IFOS is subset of $(X, \acute{C}_{\ell}, E)$.

Proposition 4.3. Let $(X, \acute{C}_{\ell}, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces and a function $\mathfrak{R}: (X, \acute{C}_{\ell}, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$. Then,

1) $(f, A) \subset \mathfrak{R}^{-1}(\mathfrak{R}(f, A))$,
2) $\mathfrak{R}(\mathfrak{R}^{-1}(f, A)) \subset (f, A)$.

Proposition 4.4. Let $(X, \acute{C}_{\ell}, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces and $\mathfrak{R}: (X, \acute{C}_{\ell}, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ be a function and $P_e(\alpha,\beta) \in$ IFS(X,E). $\mathfrak{R}$ is an intuitionistic fuzzy soft continuous at intuitionistic fuzzy soft point $P_e(\alpha,\beta)$ of IFS(X,E), if and only if, for each $(f, B) \in \tilde{\tilde{N}}_Y(\mathfrak{R}(P_e(\alpha,\beta)))$, there exist $(f, A) \in \tilde{\tilde{N}}_X P_e(\alpha,\beta)$ such that $(f, A) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(f, B)$.
Proof: It is so easy to proof this Proposition by using the **Definition 4.1.**■

Proposition 4.5. Let $(X, \acute{C}_{\ell}, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces and $\mathfrak{R}: (X, \acute{C}_{\ell}, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ be function and $P_e(\alpha,\beta) \in$ IFS(X,E). $\mathfrak{R}$ is an intuitionistic fuzzy soft continuous at intuitionistic fuzzy soft point $P_e(\alpha,\beta)$ of IFS(X,E), if and only if, for each $(f, B) \in \tilde{\tilde{N}}_Y(\mathfrak{R}(P_e(\alpha,\beta))), \mathfrak{R}^{-1}(f, B) \in \tilde{\tilde{N}}_X P_e(\alpha,\beta)$.
Proof: It is clear. By **Proposition 4.4.**■

Proposition 4.6. Let $(X, \acute{C}_\ell, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. A function $\mathfrak{R}: (X, \acute{C}_\ell, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is an intuitionistic fuzzy soft continuous, if and only if, $\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B)) \tilde{\tilde{\subseteq}} \text{Int}_{C_\ell}(\mathfrak{R}^{-1}(f,B))$ for each $(f,B) \in \text{IFS}(Y,E)$.

Proof. Necessity, let $\mathfrak{R}$ be an intuitionistic fuzzy soft continuous function and $(f,B) \in$ IFS(Y,E). Then $\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B))$ is an open and $\text{Int}_{C_\ell}(f,B) \tilde{\tilde{\subseteq}} (f,B)$, we have $\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B)) \tilde{\tilde{\subseteq}} (\mathfrak{R}^{-1}(f,B))$.

Since $\text{Int}_{C_\ell}(\mathfrak{R}^{-1}(f,B))$ is largest intuitionistic fuzzy soft open set contained by $\mathfrak{R}^{-1}(f,B)$,$\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B)) \tilde{\tilde{\subseteq}} \text{Int}_{C_\ell}(\mathfrak{R}^{-1}(f,B))$.

Sufficiency, let $\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B)) \tilde{\tilde{\subseteq}} \text{Int}_{C_\ell}(\mathfrak{R}^{-1}(f,B))$, for all $(f,B) \in$ IFS(Y,E), then we have, $\mathfrak{R}^{-1}(f,B)\mathfrak{R}^{-1}(\text{Int}_{C_\ell}(f,B)) \tilde{\tilde{\subseteq}} \text{Int}_{C_\ell}(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(f,B)$.So, $\mathfrak{R}^{-1}(f,B)$ is an open, then $\mathfrak{R}$ is an intuitionistic fuzzy soft continuous function.

Proposition 4.7. Let $(X, \acute{C}_\ell, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ are two intuitionistic fuzzy soft closure spaces. A function $\mathfrak{R}:(X, \acute{C}_\ell, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is an intuitionistic fuzzy soft continuous, if and only if, $\mathfrak{R}^{-1}(C_\ell(f,B)) \tilde{\tilde{\subseteq}} C_\ell(\mathfrak{R}^{-1}(f,B))$ for each $(f,B) \in \text{IFS}(X,E)$.

Proof: It is so easy to proof this Proposition by using the **Proposition 4.6.** ■

Proposition 4. 8. Let $(X, \acute{C}_\ell, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. If $\mathfrak{R}: (X, \acute{C}_\ell, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is an intuitionistic fuzzy soft continuous function, then $C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(C_{\ell 1}(f,B))$ for each $(f,B) \in \text{IFS}(Y,E)$.

Proof. Let $(f,B) \in$ IFS(Y,E), then $\mathfrak{R}^{-1}(f,B) \in$ IFS(X,E). From the hypothesis, we obtain $\mathfrak{R}(C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} C_{\ell 1}(\mathfrak{R}(\mathfrak{R}^{-1}(f,B))) \tilde{\tilde{\subseteq}} C_{\ell 1}(f,B)$. By taking the inverse image we get.$\mathfrak{R}^{-1}(\mathfrak{R}(C_\ell(\mathfrak{R}^{-1}(f,B))) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(C_{\ell 1}(f,B))$, hence $C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(C_{\ell 1}(f,B))$.■

Proposition 4.9. Let $(X, \acute{C}_\ell, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. If $\mathfrak{R}: (X, \acute{C}_\ell, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is an intuitionistic fuzzy soft continuous function, then $\mathfrak{R}^{-1}(f,B)$ is an IFCS set of IFS(X, E) for all (f,B) at IFCS set of IFS(Y, E).

Proof. Let (f,B) be an IFCS set of IFS(Y,E). Since $\mathfrak{R}$ is an intuitionistic fuzzy soft continuous function and by **Proposition 4.8**, we have $C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}(C_{\ell 1}(f,B))$. Due to (f,B) is an IFCS set of IFS(Y, E), $C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} \mathfrak{R}^{-1}((f,B))$ and $\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{\subseteq}} C_\ell\mathfrak{R}^{-1}((f,B))$, hence $C_\ell(\mathfrak{R}^{-1}(f,B)) \tilde{\tilde{=}} \mathfrak{R}^{-1}((f,B))$. Therefor, (f,B) be an IFCS set of IFS(X,E). ■

Proposition 4.10. Let $(X, \acute{C}_\ell, E)$ and $(Y, \acute{C}_{\ell 1}, E)$ be two intuitionistic fuzzy soft closure spaces. If $\mathfrak{R}: (X, \acute{C}_\ell, E) \rightarrow (Y, \acute{C}_{\ell 1}, E)$ is an intuitionistic fuzzy soft continuous function, then $\mathfrak{R}^{-1}(f,B)$ is an IFOS set of IFS(X, E) for all (f,B) at IFOS set of IFS(Y,E).

Proof. It is obvious by using the **Proposition 4.9.**■

5. Conclusion

The purpose of this paper is to discuss some important properties of intuitionistic fuzzy soft closure spaces and define the intuitionistic fuzzy soft closure of an intuitionistic fuzzy soft set. Later, we provide the intuitionistic fuzzy soft interior of an intuitionistic fuzzy soft set and investigate some of its basic properties. Finally, we have studied the intuitionistic fuzzy soft continuous function on the intuitionistic fuzzy soft closure spaces and proved some related theorems. Which may be of value for further research.

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