IRAQI STATISTICIANS JOURNAL

https://isj.edu.iq/index.php/isj

ISSN: 3007-1658 (Online)

# Properties of New Extension of Exponentiated Exponential Distribution with Application

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ARTICLE INFO

*Article history:*

| | |
|---|---|
| Received | 12 April 2025 |
| Revised | 16 April 2025 |
| Accepted | 15 May 2025 |
| Available online | 08 June 2025 |

*Keywords:*

Exponentiated exponential distribution
Nadarajah-Haghighi
Order Statistics
MLE
[0, 1] Truncated

ABSTRACT

Lifetime data modeling is vital topic that has received extensive attention in the modern statistical literature. This study aims to present a new distribution resulting from the expansion of the Exponentiated Exponential distribution using the [0,1] Truncated Nadarajah – Haghighi – G family, and this distribution is named [0,1]TNHEE several mathematical properties of the new distribution are studied, including density function, cumulative distribution, survival function, hazard function, moments, moment generated function, entropies (Shannon Rényi, and Delta), Lorenz and Bonforoni function, residual function, in addition to rank statistic and maximum likelihood estimators. The distribution was applied to real world data representing the survival times of 72 guinea pigs infected with tuberculosis. The results showed that the new distribution outperformed several competing distribution, including [0,10TEEEE, BeEE, KuEE, EGEE, and WeEE using statistical comparison criteria such as AIC, BIC, CAIC, and HQIC, as well as the Kolmogorov – Smirnov test, which yielded a high p-value (0.603), confirming the goodness of fit. The results showed that the new distribution has high flexibility and excellent interpretability for real-world data, making it a effective candidate for medical and engineering applications.

## 1. Introduction

Recent years have seen increasing interest in developing new probability distribution that are more capable of accurately representing real-world data, particularly in the fields of survival and reliability analysis and medical data, Recent development in distribution theory have focused on improving the flexibility of statistical models to accommodate complex data patterns, such as skewed or single-peaked data, or data with variable risk patterns (such as basin-type or increasing-declining hazard function). A common approach to generating new distribution is to extend existing distributions by adding shape parameter or combining them with other families, such as

the MKi-G [1], GME family [2], MT-X [3], MOTL-G family [4], , logarithmic family [5], ITL-H family [6], SHE-G family [7], NOGEE−G family [8], WEE-X family [9], GOM-G family [10], OLG family [11], hybrid odd exponential-Φ [12], the CDF and pdf of [0,1] Truncated Nadarajah-Haghighi-G family which was introduced by [13] has a form:

$$F(x)_{[0,1]TNH-G}=\frac{1-e^{1-[1+\beta G(x,\xi)]^{\theta}}}{1-e^{1-[1+\beta]^{\theta}}} \quad (1)$$

$$f(x)_{[0,1]TNH-G}=\frac{\beta\theta e^{1-[1+\beta G(x,\xi)]^{\theta}}g(x,\xi)}{\left(1-e^{1-[1+\beta]^{\theta}}\right)[1+\beta G(x,\xi)]^{1-\theta}} \quad (2)$$

Where

$\beta,\theta>0\,,x\geq 0.$

Traditional statistical models, such as the Exponentiated Exponential Distribution, struggle to represent real-world data characterized by complex patterns, such as skewed or long-tailed distribution, or those

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https://doi.org/10.62933/yt0ycj49

containing non-typical hazard function, These models often lack sufficient flexibility to handle data variability, resulting in poor fit and reduced estimation accuracy. The need for more flexible and effective models to characterize life and survival data persists.

This study responds to the growing need to develop new distribution with more flexible mathematical properties, capable of representing experimental data more accurately, particularly in the fields of survival and reliability analysis. The use of truncated families such as the [0,1] Truncated Nadarajah-Haghighi-G opens the way to producing distribution models with greater formal flexibility, enabling researcher to represent diverse data types and improve statistical modelling performance.

This study aims to develop a new probability distribution called the [0,1] Truncate Nadarajah-Haghighi Exponentiated Exponential ([0,1] TNHEE) and to study its mathematical properties, such as moment, hazard function, statistical quantities, entropy function, and inequality. It also aims to evaluate the efficiency of this distribution by applying it to realistic data on the survival times of experimental animals, and comparing it with several alternative distribution using statistical criteria such as AIC, BIC, and K-S, to determine its suitability as an effective model for analysing life data.

## 2. [0,1] Truncated Nadarajah-Haghighi Exponentiated Exponential Distribution ([0,1]TNHEE)

The CDF of Exponentiated Exponential Distribution has a form $G(x) = \left(1 - e^{-\lambda x}\right)^{\alpha}$, and pdf with form $g(x) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}$ [8], [14] respectively. Then the CDF of [0,1] Truncated Nadarajah-Haghighi-Exponentiated Exponential ([0,1] TNHEE) distribution substitute in equation (1) , and equation (2) to get the CDF and pdf respectively by forms:

$$F(x)_{[0,1]TNHEE} = \frac{1-e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}}{1-e^{1-[1+\beta]^{\theta}}} \quad (3)$$

$$f(x)_{[0,1]TNHEE} = \frac{\alpha\beta\lambda\theta\, e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta-1} e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}}{1-e^{1-[1+\beta]^{\theta}}} \quad (4)$$

where $\beta, \theta, \alpha, \lambda > 0$ are shape parameters, and $, x \geq 0$.

The pdf and cdf function charts of the [0, 1] TNHEE distribution are shown in Figures 1 and 2. Figure 1 displays decreasing curves, right-skewed, unimodal curves, and a reverse J forms by using different values of parameters and it shows more flexible with many shapes, the software used for drawing is R programming language.

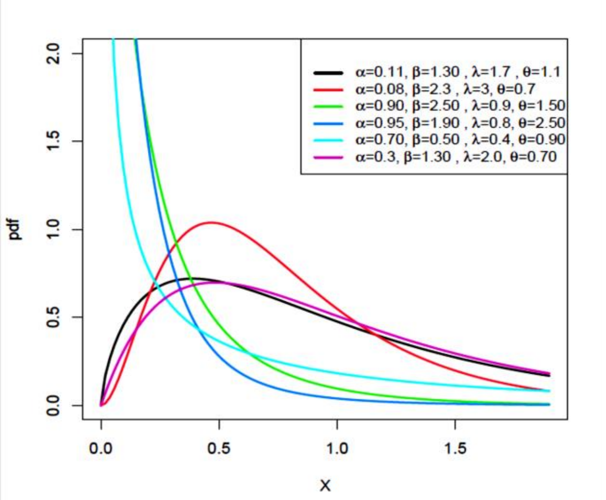


**Figure 1.**: different plot for the [0, 1] TNHEE pdf at Various Parameter Value

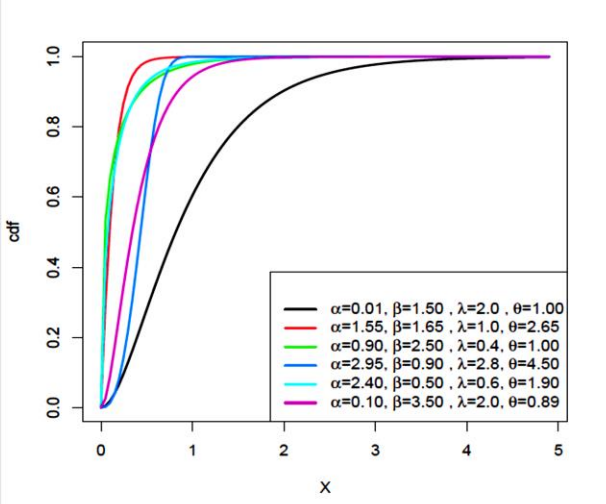


**Figure 2.**: different plots for the [0,1] TNHEE cdf at Various Parameter Values.

The survival (reliability), hazard rate, reversed hazard rate, cumulative hazard rate, and odd functions of [0,1] Truncated Nadarajah-Haghighi Exponentiated Exponential distribution [15], [16].

$$S(x)=\frac{e^{1-\left(1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{\theta}}-e^{1-(1+\beta)^{\theta}}}{1-e^{1-(1+\beta)^{\theta}}} \tag{5}$$

$$h(x)=\frac{\theta\beta\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta-1}e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}-e^{1-(1+\beta)^{\theta}}} \tag{6}$$

$$r(x)=\frac{\theta\beta\left(1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{\theta-1}e^{1-\left(1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{\theta}}\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{1-e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}} \tag{7}$$

$$H(x)=-\ln\left\{\frac{e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}-e^{1-(1+\beta)^{\theta}}}{1-e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}}\right\} \tag{8}$$

$$O(x)=\frac{1-e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}}{e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}-e^{1-(1+\beta)^{\theta}}} \tag{9}$$

Figures 3. and Figure4. show examples of the [0, 1] TNHEE distribution's survival and hazard function graphs. Depending on the parameter settings, the hazard rate shape of the [0, 1] TNHEE distribution could be bathtub or decreasing, rising decreasing increasing, the software used for drawing is R programming language.

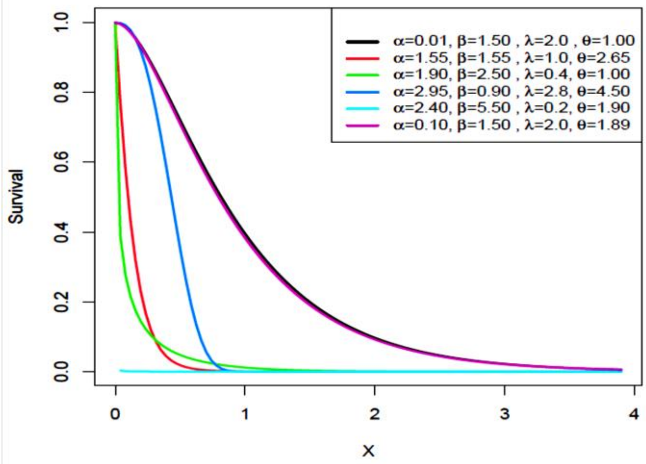


**Figure 3.**: Plot for the SF of the [0, 1] TNHEE distribution

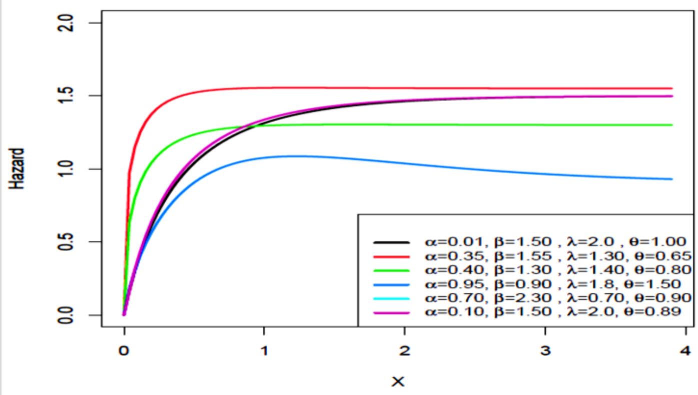


**Figure 4.**: Plot for the H of the [0,1] TNHEE distribution

## 3. Mathematical Properties

Here, derive some mathematical properties of the [0,1] TNHEE distribution. These properties include (i) quantile function (QF), (ii) expansion for the pdf, (iii) $r^{th}$ non-central moment, (iv) moment generating function(MGF), (v) Incomplete Moments, (vi) Order Statistics, (vii) Inequality Measure.

*3.1. Quantile Function:*

The quantitative function is the inverse of the cumulative distribution function and one way to get the probability function. Without torque or a significant deviation number, it is used to compute the median, skewness, and oblations of distributions. To investigate simulations, data must also be generated [17], [18], [19].

The quantile function of the [0,1] TNHEE family of distributions can be obtained by inverting $u = F_{TNHEE}(x)$ given in (4) as follows:

$$u = \frac{1 - e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}}{1 - e^{1-[1+\beta]^{\theta}}}, 0 < u < 1$$

Which implies that

$$x = \left(\frac{1}{\beta}\left(1 - ln\left\{1 - u\left[1 - e^{1-[1+\beta]^{\theta}}\right]\right\}^{\frac{1}{\theta}}\right)\right) \quad (10)$$

*3.2. Expansion for the pdf*

In this part, will expand the pdf of the distribution in order to study some of the statistical properties of this distribution. By use the expansion of the exponential function and generalized binomial theorem formula [18], [20], [21] to get a pdf by form:

$$f(x)_{[0,1]TNHEE} = \Omega_{j,k,m,n}\, e^{-\lambda x(n+1)} \quad (11)$$

where

$$\Omega_{j,k,m,n} = \frac{\alpha\lambda\theta}{1-e^{1-[1+\beta]^{\theta}}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{j}{k}\binom{\theta(k+1)-1}{m}\binom{\alpha(m+1)-1}{n} \frac{(-1)^{k+n}}{j!}\, \beta^{m+1}$$

*3.3. Moment*

The $r^{th}$ moment of a random variable $X$ defined as [22], [23]:

$$\mu_r = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

Where $f(x)$ is given in (11)

$$\mu_r = \Omega_{j,k,m,n} \int_0^{\infty} x^r\, e^{-\lambda x(n+1)}\, dx$$

Let t= $\lambda x(n+1)$ we can $\frac{t}{\lambda(n+1)} = x \Longrightarrow \frac{dt}{\lambda(n+1)} = dx$

By derive the equation we get:

$$\int_0^{\infty} \left(\frac{t}{\lambda(n+1)}\right)^r e^{-t}\, \frac{1}{\lambda(n+1)} dt = \left(\frac{1}{\lambda(n+1)}\right)^{r+1} \Gamma(r+1)$$

So The $r^{th}$ moment is

$$\mu_r = \Omega_{j,k,m,n} \left(\frac{1}{\lambda(n+1)}\right)^{r+1} \Gamma(r+1) \quad (12)$$

Table 1 in the study presents the values of the values of the first four moments, variance, skewness, and kurtosis of the [0,10 TNHEE distribution at different values of the parameters to illustrate the flexibility of the distribution with different parameter values. It is also explored how these parameters affect the statistical properties of the distribution.

**Table. 1** Some values of first four moments, variance, skewness, and kurtosis for [0, 1] TNHEE

| $\beta$ | $\theta$ | $\alpha$ | $\lambda$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $Var(X)$ | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.4 | 1.2 | 1.3 | 1.1 | 1.985121 | 4.624668 | 12.40814 | 37.70018 | 0.683963 | 1.24763 | 1.762716 |
| | | | 1.2 | 1.862946 | 3.975513 | 9.576761 | 25.69633 | 0.504945 | 1.208172 | 1.625866 |
| | | 1.5 | 1.3 | 1.972624 | 4.373004 | 10.7688 | 29.14297 | 0.481759 | 1.177599 | 1.523962 |
| | | | 1.4 | 1.871664 | 3.876852 | 8.804346 | 21.73056 | 0.373726 | 1.153396 | 1.445814 |
| | 1.5 | 1.7 | 1.5 | 1.986465 | 4.194397 | 9.358978 | 21.95364 | 0.248354 | 1.089492 | 1.247865 |
| | | | 1.6 | 1.899532 | 3.80834 | 8.019649 | 17.66091 | 0.200118 | 1.079075 | 1.217704 |
| | | 1.9 | 1.7 | 1.949842 | 3.989136 | 8.528279 | 18.982 | 0.187252 | 1.070393 | 1.192846 |
| | | | 1.8 | 1.876392 | 3.675888 | 7.492177 | 15.83754 | 0.155041 | 1.063077 | 1.172096 |

The first-order moments (mean, variance, and highest moments) increase as $\alpha$ and $\beta$ increase, indicating that the distribution becomes more dispersed and extends to the right tail. The variance reflects how dispersed the data are. The larger the parameters, the greater the variance, meaning that the distribution widens. For example, the variance increases from 0.48 to more than 1.9 as the parameters change. Skewness indicate that skewness of the distribution: positive = skewed to the right. Values range from 1.06 to 1.24, indicating that most cases have a moderate positive shew. This is suitable for modelling life-time data, which are typically skewed to the right. Kurtosis: values less than 3 indicate excessive kurtosis. This means the distribution has heavier tails, an important property when modelling data with rare events.

*3.4. Moment Generating Functions(MGF):*

The MGF of a random variable $X$ is defined as follow [24], [25]:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x)\, dx$$

Where $f(x)$ is defined in (11) we get

$$M_X(t) = \sum_{r=0}^{\infty} \frac{\Omega_{j,k,m,n}\, t^r\, \Gamma(r+1)}{r!\left(\lambda(n+1)\right)^{r+1}} \quad (13)$$

$M_X(t)$ can be expressed in the terms of quantile function as:

$$M_X(t) = \Omega_{j,k,m} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 e^{tQ_G(x)} u^k \, du \ ,0 < u < 1$$

*3.5.Incomplete Moments:*

The incomplete moments of a random variable $X$ defined by [26], [27], :

$M_r(y) = \int_{-\infty}^{y} x^r \, f(x) \, dx$

Where $f(x)$ is defined in (11) we have gotten

$M_r(y) = \Omega_{j,k,m,n} \int_0^y x^r \, e^{-\lambda x(n+1)} \, dx$

By use the expansion of the exponential function we get:

$M_r(y) = \Omega_{j,k,m,n} \int_0^y x^r \left( \sum_{\aleph=0}^{\infty} \frac{(-\lambda x(n+1))^{\aleph}}{\aleph!} \right) dx$

$M_r(y) = \Omega_{j,k,m,n} \frac{\sum_{\aleph=0}^{\infty} (-\lambda(n+1))^{\aleph}}{\aleph!} \int_0^y x^r \, x^{\aleph} \, dx$

$M_r(y) = \Omega_{j,k,m,n} \frac{\sum_{\aleph=0}^{\infty} (-\lambda(n+1))^{\aleph}}{\aleph!} \int_0^y x^{r+\aleph} \, dx$

$M_r(y) = \Omega_{j,k,m,n} \frac{\sum_{\aleph=0}^{\infty} (-\lambda(n+1))^{\aleph} (r+\aleph+1)}{\aleph! (y^{r+\aleph+1})}$ (14)

*3.6.Order statistic:*

From the smallest value $X_1; n$ to the largest value $X_n: n$, the sample of size n of symmetrically distributed independent random variables $(X_1, X_2, \dots, X_n)$ from the [0,1] TNHEE distribution is arranged by ordered statistics $X_s: n$. The probability density function and cumulative distribution function are determined by equations (7) and (8), respectively. Order Statistics are specified by [28], [29]:

$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!} (F(x))^{p-1} (1 - F(x))^{n-p} \, f(x)$

Expanding $(F(x))^{p-1}$ in the definition of $f_{p:n}(x, \Phi)$ using binomial series expansion yields

$(F(x))^{p-1} = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} (1 - F(x))^i$

Substituting it back into the expression of $f_{p:n}(x, \Phi)$ we get:

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!} \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \, (1 - F(x))^{i+n-p} \, f(x)$$

$= \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \, (S(x))^{i+n-p} \, f(x)$

Where $S(x)$ is the survival function of our family. Then

$$(S(x))^{i+n-p} = \left( \frac{e^{1-\left(1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{\theta}} - e^{1-(1+\beta)^{\theta}}}{1 - e^{1-(1+\beta)^{\theta}}} \right)^{i+n-p}$$

$$= \frac{1}{\left(1 - e^{1-(1+\beta)^{\theta}}\right)^{i+n-p}} \times \left( e^{1-\left(1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{\theta}} - e^{1-(1+\beta)^{\theta}} \right)^{i+n-p}$$

Employing a similar concept of expansion, the density function of [0,1]TNHEE distribution, a mixture representation of the PDF the $p^{th}$order statistic is defined as

$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!} \Upsilon_{i,k,m,j,l}(x) \left( \left(1 - e^{-\lambda x}\right)^{\alpha} \right)^j \left[ \alpha \lambda e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha - 1} \right]$

$f_{X_{p:n}}(x) = \frac{\alpha.\lambda.n!}{(p-1)!(n-p)!} \Upsilon_{i,k,m,j,l}(x) \left(1 - e^{-\lambda x}\right)^{\alpha j + \alpha - 1} e^{-\lambda x}$

$f_{X_{p:n}}(x) = \frac{\alpha.\lambda.n!}{(p-1)!(n-p)!} \Upsilon_{i,k,m,j,l}(x) \sum_h^{\infty} \binom{\alpha j + \alpha - 1}{h} (-1)^h \, e^{-h\lambda x} \, e^{-\lambda x}$

$f_{X_{p:n}}(x) = \frac{\alpha.\lambda.n!}{(p-1)!(n-p)!} \Upsilon_{i,k,m,j,l,h}(x) \, e^{-\lambda x(h+1)}$ (15)

where

$$\Upsilon_{i,k,m,j,l,h} = \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{\infty}\binom{p-1}{i}\binom{\theta l+\theta m+\theta-1}{j}\binom{i+n+p}{k}\binom{\alpha j+\alpha-1}{h} \times \frac{\alpha\lambda\theta(\beta)^{j+1}(-1)^{i+k+m+l+h}e^{k+1}(k)^m e^{\left(1-[1+\beta]^{\theta}\right)^{i+n-p-k}}}{(m!)(l!)\left(1-e^{1-[1+\beta]^{\theta}}\right)^{i+n-p+1}}$$

*3.7.Moments of Order statistic:*

The $E\left(X_{p:n}^r\right)$ of random variable $X$ is defined as follows [30], [31]:

$$E\left(X_{p:n}^r\right) = \int_{-\infty}^{\infty} x^r f_{X_{p:n}}(x)\,dx$$

Now when $f_{X_{p:n}}(x)$ as in (15) we get that

$$E\left(X_{p:n}^r\right) = \frac{n!}{(p-1)!\,(n-p)!}\Upsilon_{i,k,m,j,l,h}\int_0^{\infty} x^r e^{-\lambda x(h+1)}\,dx$$

Let $t = \lambda x(h+1)\ then\ \frac{t}{\lambda(h+1)} = x \Longrightarrow \frac{dt}{\lambda(h+1)} = dx$

Now by derive the equation we get:

$$E\left(X_{p:n}^r\right) = \frac{1}{\left(\lambda(h+1)\right)^{r+1}}\Gamma(r+1)$$

So we have

$$E\left(X_{p:n}^r\right) = \frac{n!\,\Gamma(r+1)}{(p-1)!(n-p)!\,\left(\lambda(h+1)\right)^{r+1}}\Upsilon_{i,k,m,j,l,h} \qquad (16)$$

*3.8.Inequality Measure:*

Lorenz and Bonferroni curves are used in econometrics, insurance, and reliability to analyses inequality measurements like poverty and income.

i. **Lorenz curve:**

The Lorenz curve of a random variable X is defined as [13], [32]:

$$L_F = \frac{1}{\mu}\int_{-\infty}^{y} x\,f(x)\,dx$$

So that where $f(x)$ defined in (11) we get the Lorenz curve of [0,1] TNHEE family of distributions is:

$$L_F = \frac{1}{\mu}\int_{-\infty}^{y} x\,f(x)\,dx$$

$$L_F = \frac{1}{\mu}\Omega_{j,k,m,n}\int_0^{y} x\,e^{-\lambda x(n+1)}\,dx$$

So the Lorenz curve is

$$L_F = \frac{-\Omega_{j,k,m,n}\left(\lambda(n+1)ye^{-\lambda y(n+1)} - e^{-\lambda y(n+1)} + 1\right)}{\mu\lambda^2(n+1)^2} \qquad (17)$$

ii. **Bonferroni Curve**

the Bonferroni Curve of a random variable X defined as $B_F(y) = \frac{L_F(y)}{F(y)}$ now where $L_F$ as in (17) and $F(y)$ was defined in (7) with respect to $y$ so that we get:

$$B_F(y) = \frac{\Omega_{j,k,m,n}\left(1-e^{1-[1+\beta]^{\theta}}\right)\left(-\lambda(n+1)ye^{-\lambda y(n+1)} - e^{-\lambda y(n+1)} + 1\right)}{\mu\lambda^2(n+1)^2\left(1-e^{1-\left[1+\beta\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{\theta}}\right)} \qquad (18)$$

iii. **Mean Residual Life**

The mean residual life of a random variable X is defined as:

$$\bar{M} = E(X-y|X>y)$$

Thus

$$\bar{M} = \frac{1}{F(y)}\left(\mu - \int_{-\infty}^{y} x\,f(x)\,dx\right) - y$$

Now substituted $f(x)$ as mixture density defined in (7) we get:

$$\bar{M} = \frac{1}{F(y)}\left(\mu - \Omega_{j,k,m,n}\int_0^{y} x\ e^{-\lambda x(n+1)}\,dx\right) - y$$

So we get

$$\bar{M} = \frac{1-e^{1-[1+\beta]^{\theta}}}{1-e^{1-\left[1+\beta\left(1-e^{-\lambda y}\right)^{\alpha}\right]^{\theta}}} \left(\mu - \Omega_{j,k,m,n} \frac{-\lambda y(n+1)e^{-\lambda y(n+1)} - e^{\lambda y(n+1)} + 1}{\lambda^{2}(n+1)^{2}}\right) - y \qquad (18)$$

## iv. MAXIMUM LIKELIHOOD ESTIMATION

In order to estimate distribution parameters, one of the best techniques is the Maximum Likelihood methodology. In this portion of the research, we will extract the partial derivative of the Maximum Likelihood function [33], [34], [35] of the distribution in order to estimate the parameters of the new distribution [0,1] TNHEE.

$$L(\theta,\beta,\lambda,\alpha) = \prod_{i=1}^{n} \frac{\alpha\beta\lambda\theta\, e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta-1} e^{1-\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta}}}{1-e^{1-[1+\beta]^{\theta}}}$$

$$= \frac{(\alpha\beta\lambda\theta)^{n} e^{-\sum_{i=1}^{n}\lambda x_i} \prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\alpha-1} \prod_{i=1}^{n}\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta-1} e^{\sum_{i=1}^{n} 1-\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta}}}{\prod_{i=1}^{n}\left(1-e^{1-[1+\beta]^{\theta}}\right)^{n}}$$

Now the log-likelihood function given as:

$$l = n\log\alpha + n\log\beta + n\log\lambda + n\log\theta - \lambda\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{n}\left(1-\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta}\right) + (\theta-1)\sum_{i=1}^{n}\log\left(1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right) - n\log\left(1-e^{1-[1+\beta]^{\theta}}\right)$$

Now we find the partial derivative of each parameter

$$\frac{\partial(\mathrm{L})}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{\infty}\log\left(1+\beta\left[1-e^{-\lambda x_i}\right]^{\alpha}\right) - \sum_{i=1}^{n}\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta}.\log\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right] + \frac{n\,[1+\beta]^{\theta}\log(1+\beta)\, e^{1-[1+\beta]^{\theta}}}{1-e^{1-[1+\beta]^{\theta}}} \qquad (19)$$

$$\frac{\partial(\mathrm{L})}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{\infty}\beta\theta\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta-1}\log\left(1-e^{-\lambda x_i}\right)\left(1-e^{-\lambda x_i}\right)^{\alpha} + \sum_{i=1}^{\infty}\frac{\beta(\theta-1)\log\left(1-e^{-\lambda x_i}\right)\left(1-e^{-\lambda x_i}\right)^{\alpha}}{1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}} \qquad (20)$$

$$\frac{\partial(\mathrm{L})}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{\infty}\theta\left(1-e^{-\lambda x_i}\right)^{\alpha}\left[1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\theta-1} + \sum_{i=1}^{n}\frac{(\theta-1)\left(1-e^{-\lambda x_i}\right)^{\alpha}}{1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}} + \frac{n.\theta e^{1-[1+\beta]^{\theta}}[1+\beta]^{\theta-1}}{1-e^{1-[1+\beta]^{\theta}}} \qquad (21)$$

$$\frac{\partial(\mathrm{L})}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{(\alpha-1)x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)} + \sum_{i=1}^{n}\frac{\alpha\beta(\theta-1)x_i\, e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}}{1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}} + \alpha\beta\theta\sum_{i=1}^{n} x_i e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\left(1+\beta\left(1-e^{-\lambda x_i}\right)^{\alpha}\right)^{\theta-1} \qquad (22)$$

The Maximum Likelihood Estimation is obtained by equating equation numbers (19), (20), (21), (22) with zero. The aforementioned equations have a convoluted form and are difficult to solve algebraically. As a result, we use numerical methods to solve them. For most applications, the R programme is a suitable option.

## 5. Application

The dataset used in this study includes the survival times of 72 guinea pigs infected with virulent tuberculosis bacilli. Survival times are measured in days. The original observation and reporting of this data set was done by Bjerkedal also which was used in [11], [36]

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0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

In order to analyze this data and compare it with other distributions, we use the R programming language.

**Table 2**. Data descriptive

| **Vars** | **N** | **mean** | **sd** | **Median** | **skew** | **Kurtosis** |
|---|---|---|---|---|---|---|
| $x_1$ | 72 | 1.77 | 1.04 | 1.5 | 1.3 | 1.83 |

From Table 2, it is clear that the number of observations,72, represents a sufficient number for a preliminary statistical analysis, but it is considered relatively average. It also indicates that the expected value for survival times is approximately 1.77 days. This reflects the center of gravity of the distribution, but it is affected by extreme values (especially with skewness). The standard deviation indicates a moderate amount of variation within the sample, and skewness indicates a clear positive skew. Most of the data is concentrated at low values, while there are some high values (right tail). This is very common in life time data, as most individuals live short lives, and a few live longer. Kurtosis (1.83) indicates that the distribution is less “steep” than the normal distribution (which has kurtosis = 3). This relatively low kurtosis indicates a slightly flattened distribution (platykurtic), meaning less clustered around the center and more spread out.

In order to determines the efficiency of the proposed distribution and compare it with other distributions, a number of statistical criteria and measures were used, as follows: $AIC = -2\sum_{j=1}^{T}\log f_m(x_j/\hat{\theta}_m) + 2k$ [37]

$CAIC = AIC + \frac{2k(k+1)}{n-k-1}$ [38], [39]

$BIC = -2l(\hat{\theta}) + k\, log(n)$ [40], [41]

$HQIC = 2k\, ln[ln(n)] - 2l(\hat{\theta})$ [40]

$K - S = \sup_{n}|F_n(x) - F(x)|$

$$A = \sqrt{-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)H} \quad (28)$$

$$H = [\ln F(x_i) + (1 - \ln F(x_{n+1-i}))]$$

$$W = \sqrt{\frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i) - \frac{2i-1}{2n}\right)^2}$$

$$p - value = P(T \geq T_{obs}|H_0)$$

where $k$ is number of parameters, $n$ is number of sample size, $l(\hat{\theta})$ the likelihood function, $F_n(x) = \frac{1}{N}\sum_{i=1}^{N} I(X_i \leq x)$ Empirical cumulative distribution function (for a sample), $I(*)$ indicator function, $F(x_i)$ theoretical value distribution function for $x_i$ which represent the values arranged in ascending order.

While the comparative distributions have CDF functions defined as follows:

$$\text{KuEE} = 1 - \left(1 - \left(\left(1 - e^{-\lambda x}\right)^{\alpha}\right)^{\beta}\right)^{\theta}$$

$$[0,1]\text{TEEEE} = \frac{\left(1 - e^{-\beta\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)}\right)^{\theta-1}}{(1 - e^{-\beta})^{\theta}}$$

$$\text{BeEE} = p\beta\left(\left(1 - e^{-\lambda x}\right)^{\alpha}, shape1 = \beta, shape2 = \theta\right)$$

$$\text{WeEE} = 1 - e^{\frac{1}{\theta\beta}\left(-\ln\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)^{\beta}}$$

$$\text{EGEE} = \left(1 - \left(\left(1 - e^{-\lambda x}\right)^{\alpha}\right)^{\beta}\right)^{\theta}$$

$$\text{LGamEE} = p\beta\left(-\theta\log\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right), shape1 = \beta\right)$$

**Table 3**. Comparison of the results of data fitting between the [0,1] TNHEE distributions and other distributions.

| Dist. | MLEs | -2L | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|---|
| **[0,1]TNHEE** | $\hat{\beta}$**:** 2.1820533<br>$\hat{\theta}$**:** 0.7690040<br>$\hat{\alpha}$**:** 3.1122842<br>$\hat{\lambda}$**:** 0.9391701 | 94.07655 | 196.1533 | 196.7503 | 205.26 | 199.7787 |
| **[0,1]TEEEE** | $\hat{\beta}$ **:** 2.6273416<br>$\hat{\theta}$**:** 0.5548838<br>$\hat{\alpha}$**:** 5.5957002<br>$\hat{\lambda}$**:** 0.7442931 | 94.12027 | 196.2405 | 196.8376 | 205.3472 | 199.8659 |
| **BeEE** | $\hat{\beta}$**:** 1.7194697<br>$\hat{\theta}$**:** 1.5601909<br>$\hat{\alpha}$**:** 1.8665930<br>$\hat{\lambda}$**:** 0.7951174 | 94.36767 | 196.7363 | 197.3334 | 205.843 | 200.3617 |
| **KuEE** | $\hat{\beta}$**:** 1.7465373<br>$\hat{\theta}$**:** 1.6586717<br>$\hat{\alpha}$**:** 1.7465373<br>$\hat{\lambda}$**:** 0.7742843 | 94.31733 | 196.6347 | 197.2317 | 205.7414 | 200.2601 |
| **EGEE** | $\hat{\beta}$**:** 1.7156044<br>$\hat{\theta}$**:** 1.6552797<br>$\hat{\alpha}$**:** 1.9215021<br>$\hat{\lambda}$**:** 0.7121551 | 94.32519 | 196.6512 | 197.2482 | 205.7579 | 200.2766 |
| **WeEE** | $\hat{\beta}$**:** 1.44048293<br>$\hat{\theta}$**:** 0.0986681<br>$\hat{\alpha}$**:** 1.28504912<br>$\hat{\lambda}$**:** 0.08744960 | 95.70837 | 199.426 | 200.023 | 208.5326 | 203.0514 |
| **LGamEE** | $\hat{\beta}$**:** 0.7645026<br>$\hat{\theta}$**:** 1.5062510<br>$\hat{\alpha}$**:** 3.9774719<br>$\hat{\lambda}$**:** 0.8059463 | 94.30158 | 196.6032 | 197.2002 | 205.7098 | 200.2285 |
| **EE** | $\hat{\alpha}$**:** 3.565090<br>$\hat{\lambda}$**:** 1.119862 | 94.49579 | 196.9916 | 197.1655 | 205.5449 | 200.8043 |

**Table 4**. The statistically criteria and the value of estimation parameters for [0,1] TNHEE distribution with other distributions which compare with its data

| Dist. | W | A | K-S | p-value |
|---|---|---|---|---|
| **[0,1]TNHEE** | 0.07222499 | 0.4471378 | 0.09007385 | 0.6031515 |
| **[0,1]TEEEE** | 0.0٨634171 | 0.4٦93228 | 0.١٢774409 | 0.٥٨63761 |
| **BeEE** | 0.08454242 | 0.5223707 | 0.09289601 | 0.563375 |
| **KuEE** | 0.08694654 | 0.5314 | 0.09188029 | 0.5776112 |
| **EGEE** | 0.08312028 | 0.5141324 | 0.09232983 | 0.571298 |
| **WeEE** | 0.1450977 | 0.850256 | 0.1122102 | 0.324884 |
| **LGamEE** | 23.13348 | 143.2302 | 0.9993798 | 0.573476 |
| **EE** | 0.07963858 | 0.504796 | 0.09612848 | 0.5188593 |

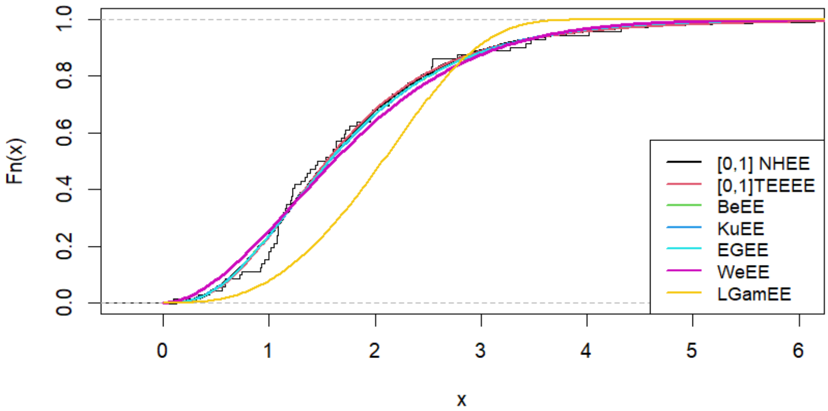


**Figure 5**. The cdf plot for the [0,1] TNHEE distributions and other distributions when applying the failure times of 72 components data

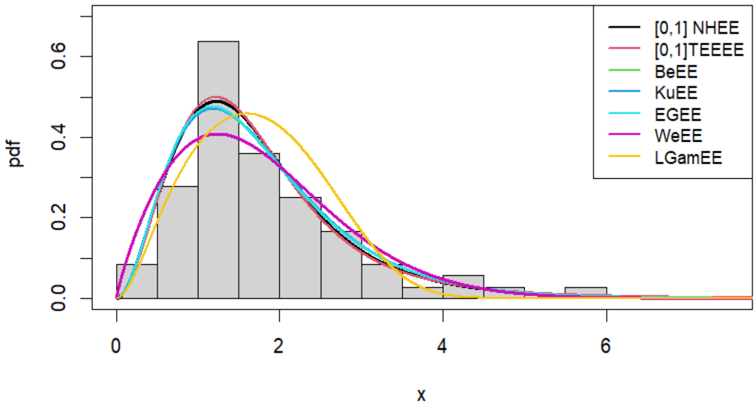


**Figure 6.** The pdf plot for the [0,1] TNHEE distributions and other distributions when applying the failure times of 72 components data.

The [0,1] TNHEE distribution was compared with several other distributions (such as [0,1] TEEEE, BeEE, KuEE, EGEE, WeEE, LGamEE, EE) using statistical criteria such as AIC (Akaike Information Inventory), BIC (Bayes Information Inventory), CAIC, and HQIC, with [0,1] TNHEE scoring the lowest values, indicating its superior fit to the data. The Kolmogorov-Smirnov (K-S) test yielded a high p-value (0.603), confirming that there was no statistically significant difference between the data and the proposed distribution. The distribution parameters were estimated using the maximum likelihood (MLE) method, and the results demonstrated high estimation efficiency with statistically significant parameter values. The density curves (PDF) and cumulative distribution curves (CDF) of the proposed distribution showed a strong fit with the actual data compared to other distributions.

## 6. CONCLUSION

The [0,1] TNHEE distribution provides a sophisticated mathematical framework that extends the exponential distribution, adding additional shape parameters that enhance its flexibility in data modelling. Its derived mathematical properties (such as moments, quantile function, and survival function) make it a powerful tool in statistical analysis, particularly in the areas of survival and reliability analysis. When applied to laboratory animal survival data, the [0,1] TNHEE outperformed competing distributions based on information criteria (AIC, BIC) and goodness-of-fit (K-S) tests. The results confirmed the distribution's efficiency in modelling real-world data, making it a suitable choice for practical applications in medical science and engineering. The distribution can be studied in other contexts, such as the analysis of engineering failure data or complex medical data, and the development of Bayesian estimators or advanced simulation methods is proposed to improve the accuracy of parameter estimation in small samples.

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