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A New Type I Half Logistic Topp-Leone Kumaraswamy Distribution: Properties and Applications to Covid 19 data

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ABSTRACT

Various classical models have been generalized by many distributions theory to increase their flexibility and facilitate their use in a variety of contexts. Families of distributions are mostly used to generalize and extend the classical models. Using family of distributions, a new four-parameter distribution named type I half logistic Topp-Leone Kumaraswamy distribution, used in modelling real-life data sets, has been introduced. The new distribution is capable of modeling data with positively skewed and symmetric property. Important statistical properties of the proposed distribution such as the density function, hazard rate function, survival function, order statistics, probability weighted moments, moments, moment generating function. The maximum likelihood estimation of the unknown parameters of the distribution has been obtained. Two real data sets relating to COVID-19 cases were employed to illustrate the usefulness and fit of the new model. The results showed that type I half logistic Topp-Leone Kumaraswamy distribution provided better fits to the two data sets considered than the comparators used.

1. Introduction

High levels of precision in data analysis are necessary for modern studies to satisfy scientific standards. Unfortunately, traditional statistical models are typically insufficient to meet this need. Because of this, statisticians continue to find it extremely challenging to develop new, adaptable models that are appropriate for the given situation. Families of distributions with desirable attributes can be used to derive attractive models from a probabilistic perspective. Effective methods for adding shape parameters to well-known distributions can be used to define such families. These families are frequently distinguished by complex yet adaptable functions that contemporary programming software can manage. Among the high impacted families of distributions, some of the recent once are derived by Nasir et al., (2018), Hosseini et al., (2018), Aldahlan et al., (2019)

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Jamal et al., (2020), Ahmad et al., (2018), Ibrahim et al., (2020a), Ibrahim et al., (2020b), Adepoju et al., (2024c), Yahaya and Doguwa (2021), Bello et al., (2021), Sule et al., (2022), Isa et al., (2022), Adepoju et al., (2024d), Adepoju et al., (2024a) which also improved Inverse lomax distribution, Isa et al., (2023), Anabike et al., (2023), Adekunle et al., (2024) and Adepoju et al., (2024b).

A distribution with outcomes restricted to a particular range is described by the Kumaraswamy distribution (Kumaraswamy (1980)). The probability density function within this range is defined by two shape parameters. Because its probability density function (pdf) and cumulative distribution function (cdf) have simpler analytical expressions, it is comparable to the beta distribution and simpler to utilize. The cdf and pdf of Kumaraswamy distribution are given respectively as:

$$H(x;\varphi,\alpha)=1-\left[1-x^{\varphi}\right]^{\alpha},\ 0<x<1,\ \varphi,\alpha>0 \quad (1)$$

$$h(x;\varphi,\alpha)=\varphi\alpha x^{\varphi-1}\left[1-x^{\varphi}\right]^{\alpha-1},\ \ 0<x<1,\ \varphi,\alpha>0 \quad (2)$$

2. Methodology

2.1 Type I Half Logistic Topp-leone-G Family of Distribution

Adepoju et al., (2023) proposed a family of continuous distribution called type I half logistic-Topp-Leone-G to add flexibility and improve the fit of the standard distributions. The cdf and pdf of the family are given as:

$$F_{TIHLTL-G}\left(x;\zeta,\theta,\chi\right)=\frac{1-\left[1-\left[1-\left[1-H\left(x;\chi\right)\right]^{2}\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left[1-H\left(x;\chi\right)\right]^{2}\right]^{\theta}\right]^{\zeta}}, \quad (3)$$

$$f_{TIHLTL-G}\left(x;\zeta,\theta,\chi\right)=\frac{4\zeta\theta h\left(x;\chi\right)\left[1-H\left(x;\chi\right)\right]\left[1-\left[1-H\left(x;\chi\right)\right]^{2}\right]^{\theta-1}\left[1-\left[1-\left[1-H\left(x;\chi\right)\right]^{2}\right]^{\theta}\right]^{\zeta-1}}{\left[1+\left[1-\left[1-\left[1-H\left(x;\chi\right)\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{2}}, \quad (4)$$

Where $h(x;\xi)$ and $H(x;\xi)$ are the PDF and CDF of any baseline distribution and ξ is a vector parameter of the baseline distribution.

2.2 The Type I Half Logistic Topp-Leone Kumaraswamy (TIHLTLKw) Distribution

To obtain the cdf of the TIHLTLKw distribution, equation (1) is inserted into equation (3) as

$$F_{TIHLTLKw}\left(x;\zeta,\theta,\varphi,\alpha\right)=\frac{1-\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}},0<x<1,\zeta,\theta,\varphi,\alpha>0 \quad , \quad (5)$$

the PDF corresponding to equation (3) is obtained as;

$$f_{TIHLTLKw}\left(x;\zeta,\theta,\varphi,\alpha\right)=\frac{4\zeta\theta\varphi\alpha x^{\varphi-1}\left[1-x^{\varphi}\right]^{\alpha-1}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}}{\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{2}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{-(\zeta-1)}} \quad (6)$$

Where $x\geq 0$, $\zeta,\theta,\alpha>0$ are the shape parameters and $\varphi>0$ is the scale parameter.

2.3 The New Type I Half Logistic Topp-Leone Kumaraswamy (TIHLTLKw) Distribution

To obtain the cdf of the TIHLTLKw distribution, equation (1) is inserted into equation (3) as

$$F_{TIHLTLKw}\left(x;\zeta,\theta,\varphi,\alpha\right)=\frac{1-\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}},0<x<1,\zeta,\theta,\varphi,\alpha>0 \quad , \quad (5)$$

Also, to obtain the pdf of the TIHLTLKw distribution, equation (2) is inserted into equation (4) as:

$$f_{TIHLTLKw}\left(x;\zeta,\theta,\varphi,\alpha\right)=\frac{4\zeta\theta\varphi\alpha x^{\varphi-1}\left[1-x^{\varphi}\right]^{\alpha-1}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}}{\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{2}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{-(\zeta-1)}}$$

(6)

Where $x \geq 0$, $\zeta, \theta, \alpha > 0$ are the shape parameters and $\varphi > 0$ is the scale parameter.

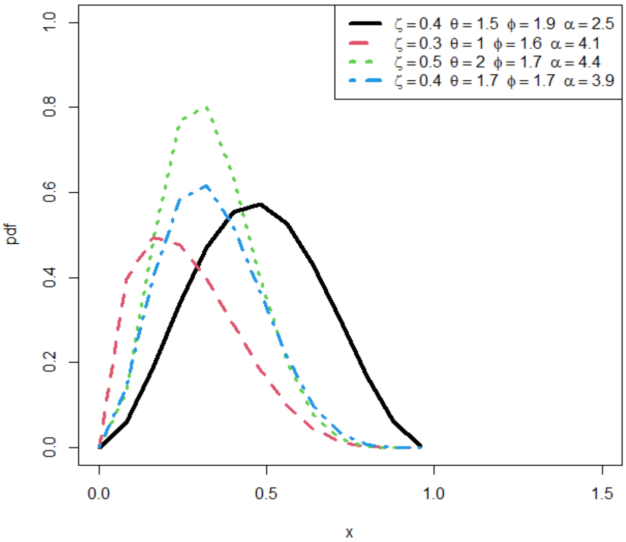


Figure 1. the pdf plot of the TIHLTLKw distribution with distinct values

3. Expansion of Density

In this section, the pdf and cdf of the TIHLTLKw model are expanded using binomial expansion to derive an explicit expression for the model which will be used in the derivation of some of the model's properties.

$$(1-x)^{p-1} = \sum_{i=0}^{\infty} (-1)^i \binom{p-1}{i} [x]^{2i} \tag{7}$$

Equation (6) can also be written as

$$f_{TIHLTLKw}(x;\zeta,\theta,\varphi,\alpha) = 4\zeta\theta\varphi\alpha x^{\varphi-1}\left[1-x^{\varphi}\right]^{\alpha-1}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}$$
$$\times\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta-1}\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{-2} \tag{8}$$

Now, using equation (7) on the last term in equation (8), we have

$$\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{-2} = \sum_{i=0}^{\infty} (-1)^i \binom{1+i}{i}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta i}$$

$$\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\zeta(i+1)-1}{j}\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta j}$$

$$\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta(j+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\theta(j+1)-1}{k}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2k}$$

$$\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2k+1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{2k+1}{m}\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]^{m}$$

$$\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]^{m}=\sum_{z=0}^{\infty}\binom{m}{z}\left[1-x^{\varphi}\right]^{\alpha z}$$

$$f_{TIHLTLKw}(x;\zeta,\theta,\varphi,\alpha)=4\zeta\theta\varphi\alpha x^{\varphi-1}\sum_{i,j,k,m,z=0}^{\infty}(-1)^{i+j+k+m+z}\binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\left[1-x^{\varphi}\right]^{\alpha(z+1)-1} \quad (9)$$

Equation (3) can also be written as:

$$\left[F_{TIHLTLKw}(x;\zeta,\theta,\varphi,\alpha)\right]^{h}=\underbrace{\left[1-\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{h}}_{A}\underbrace{\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{-h}}_{B}$$

$$A=\left[1-\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{h}=\sum_{w=0}^{h}(-1)^{w}\binom{h}{w}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta w}$$

$$B=\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta}\right]^{h}=\sum_{p=0}^{h}(-1)^{p}\binom{h+p-1}{p}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta p}$$

Combining A and B, we obtain

$$\left[F_{TIHLTLKw}(x;\zeta,\theta,\varphi,\alpha)\right]^{h}=\sum_{p,w=0}^{h}(-1)^{p+w}\binom{h}{w}\binom{h+p-1}{p}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta(w+p)}$$

$$\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]^{\zeta(w+p)}=\sum_{q=0}^{\infty}(-1)^{q}\binom{\zeta(p+w)}{q}\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta q}$$

$$\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta q}=\sum_{t=0}^{\infty}(-1)^{t}\binom{\theta q}{t}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2t}$$

$$\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^{2t}=\sum_{d=0}^{\infty}\left(-1\right)^{d}\binom{2t}{d}\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]^{d}$$

$$\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]^{d}=\sum_{g=0}^{\infty}\left(-1\right)^{g}\binom{d}{g}\left[1-x^{\varphi}\right]^{\alpha g}$$

$$\left[F_{TIHLTLKw}\left(x;\zeta,\theta,\varphi,\alpha\right)\right]^{h}=\sum_{q,t,d,g=0}^{\infty}\sum_{p,w=0}^{h}\left(-1\right)^{p+w+q+t+d+g}\binom{h}{w}\binom{h+p-1}{p}\binom{\zeta\left(p+w\right)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}\left[1-x^{\varphi}\right]^{\alpha g} \tag{10}$$

4. Statistical Properties

4.1 Probability Weighted Moments

$$\tau_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{0}^{1}x^{r}f(x)(F(x))^{s}dx$$

The Probability Weighted Moments (PWMs) of the TIHLTLKw distribution are derived by equations (9) and (10) and also replacing h with s in equation (10) as:

$$\tau_{r,s}=4\zeta\theta\varphi\alpha x^{\varphi-1}\sum_{i,j,k,m,z=0}^{\infty}\sum_{q,t,d,g=0}^{\infty}\sum_{p,w=0}^{s}\left(-1\right)^{p+w+q+t+d+g}\left(-1\right)^{i+j+k+m+z}$$
$$\times\binom{1+i}{i}\binom{\zeta\left(i+1\right)-1}{j}\binom{s}{w}\binom{s+p-1}{p}\binom{\zeta\left(p+w\right)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}$$
$$\times\binom{\theta\left(j+1\right)-1}{k}\binom{2k+1}{m}\binom{m}{z}\int_{0}^{1}x^{r}\left[1-x^{\varphi}\right]^{\alpha\left(z+g+1\right)-1}dx$$

Consider the integral part in equation (11)

Let

$$y=\left(1-x^{\varphi}\right)\Rightarrow x=\left[1-y\right]^{\frac{1}{\varphi}};\text{ when } x\Rightarrow 0, y\Rightarrow 1; \text{ when } x\Rightarrow 1, y\Rightarrow 0; dx=\frac{-dy}{\varphi x^{\varphi-1}}$$

Then

$$\tau_{r,s} = 4\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{i,j,k,m,z=0}^{\infty} \sum_{q,t,d,g=0}^{\infty} \sum_{p,w=0}^{s} (-1)^{p+w+q+t+d+g} (-1)^{i+j+k+m+z}$$
$$\binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{s}{w}\binom{s+p-1}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}$$
$$\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\int_0^1 [1-y]^{\frac{r}{\varphi}} y^{\alpha(z+g+1)-1} \frac{dy}{\varphi x^{\varphi-1}}$$

Where

$$\int_0^1 [1-y]^{\frac{r}{\varphi}} y^{\alpha(z+g+1)-1} dy = B\left(1+\frac{r}{\varphi}, \alpha(z+g+1)\right)$$

Therefore

$$\tau_{r,s} = 4\zeta\theta\alpha \sum_{i,j,k,m,z=0}^{\infty} \sum_{q,t,d,g=0}^{\infty} \sum_{p,w=0}^{s} (-1)^{p+w+q+t+d+g} (-1)^{i+j+k+m+z}$$
$$\times\binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{s}{w}\binom{s+p-1}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g} \quad (12)$$
$$\times\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z} B\left(1+\frac{r}{\varphi}, \alpha(z+g+1)\right)$$

4.2 Moments

$$E(X^r) = \int_0^1 x^r f(x) dx \quad (13)$$

$$E(X^r) = 4\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{i,j,k,m,z=0}^{\infty} (-1)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\int_0^1 x^r \left[1-x^{\varphi}\right]^{\alpha(z+1)-1} dx \quad (14)$$

Consider the integral part in equation (14)

$$\int_0^1 x^r \left[1-x^{\varphi}\right]^{\alpha(z+1)-1} dx$$

Let $y = (1- x^j) Þ \; x = [1- y]^{\frac{1}{}}$; when $x Þ \; 0, y Þ \; 1$; when $y = (1- x^j) Þ \; x = [1- y]^{\frac{1}{}}$;

Then

$$E(X^r) = 4\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{i,j,k,m,z=0}^{\infty} (-1)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta(i+1)-1}{j}$$
$$\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\int_0^1 [1-y]^{\frac{r}{\varphi}} [y]^{\alpha(z+1)-1} \frac{dy}{\varphi x^{\varphi-1}}$$

where $\int_{0}^{1} \left[1-y\right]^{\frac{r}{\varphi}} y^{\alpha(z+1)-1} dy = B\left(1+\frac{r}{\varphi}, \alpha\left(z+1\right)\right)$

Therefore

$$E(X^r) = 4\zeta\theta\alpha \sum_{i,j,k,m,z=0}^{\infty} \left(-1\right)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta\left(i+1\right)-1}{j}\binom{\theta\left(j+1\right)-1}{k}\binom{2k+1}{m}\binom{m}{z} B\left(1+\frac{r}{\varphi}, \alpha\left(z+1\right)\right)$$

4.3 *Mean*

To obtain the mean of TIHLTLKw distribution, we set $r=1$ in equation (15) as

$$E(X) = 4\zeta\theta\alpha \sum_{i,j,k,m,z=0}^{\infty} \left(-1\right)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta\left(i+1\right)-1}{j}\binom{\theta\left(j+1\right)-1}{k}\binom{2k+1}{m}\binom{m}{z} B\left(\frac{\varphi+1}{\varphi}, \alpha\left(z+1\right)\right) \quad (16)$$

4.4 *Moment generating function (mgf)*

The mgf is given as:

$$M_x(t) = \int_0^1 e^{tx} f(x) dx \quad (17)$$

$$M_x(t) = 4\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{i,j,k,m,z=0}^{\infty} \left(-1\right)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta\left(i+1\right)-1}{j}\binom{\theta\left(j+1\right)-1}{k}\binom{2k+1}{m}\binom{m}{z} \int_0^1 e^{tx}\left[1-x^{\varphi}\right]^{\alpha(z+1)-1} dx \quad (18)$$

By expanding $e^{tx} = \sum_{b=0}^{\infty}\frac{t^b x^b}{b!}$ and following the process for deriving moments as outlined above,

$$M_x\left(t\right) = 4\zeta\theta\alpha \frac{t^b}{b!}\sum_{b=0}^{\infty}\sum_{i,j,k,m,z=0}^{\infty} \left(-1\right)^{i+j+k+m+z} \binom{1+i}{i}\binom{\zeta\left(i+1\right)-1}{j}\binom{\theta\left(j+1\right)-1}{k}\binom{2k+1}{m}\binom{m}{z} B\left(1+\frac{b}{\varphi}, \alpha\left(z+1\right)\right) \quad (19)$$

4.5 *Reliability Function*

$$R\left(x;\zeta,\theta,\varphi,\alpha\right) = \frac{2\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta}\right]^{\zeta}} \quad (20)$$

4.6 *Hazard Function*

$$T\left(x;\zeta,\theta,\varphi,\alpha\right) = \frac{4\theta\lambda\varphi\alpha x^{\varphi-1}\left[1-x^{\varphi}\right]^{\alpha-1}\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta-1}\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta}\right]^{\zeta-1}}{2\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta}\right]^{\zeta}\left[1+\left[1-\left[1-\left[1-\left[1-\left[1-x^{\varphi}\right]^{\alpha}\right]\right]^2\right]^{\theta}\right]^{\zeta}\right]} \quad (21)$$

4.7 *Quantile Function*

$$x = Q(u) = \left[1-\left[1-\left[1-\left[1-\left[1-\left(\frac{1-u}{u+1}\right)^{\frac{1}{\ }}\right]^{\frac{1}{\ }}\right]^{\frac{1}{\ }}\right]^{\frac{1}{\ }}\right]^{\frac{1}{\ }}\right] \tag{22}$$

5. Order Statistics

$$f_{r:n}(x;\zeta,\theta,\varphi,\alpha) = \frac{f(x)}{B(r,n-r+1)} \sum_{v=0}^{n-r} (-1)^{v} \binom{n-r}{v} \left[F(x)\right]^{v+r-1} \tag{23}$$

The pdf of the rth order statistic is derived by substituting equations (9) and (10) into equation (23) and also replacing h with v + r - 1 in equation (10), we obtain the following expression.

$$f_{r:n}(x;\zeta,\theta,\varphi,\alpha) = \frac{4\zeta\theta\varphi\alpha x^{\varphi-1}}{B(r,n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j,k,m,z=0}^{\infty} \sum_{q,t,d,g=0}^{\infty} \sum_{p,w=0}^{v+r-1} (-1)^{i+j+k+m+z+v} (-1)^{p+w+q+t+d+g}$$
$$\times \binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\binom{v+r-1}{w}\binom{v+r+p-2}{p} \tag{24}$$
$$\times \binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}\binom{n-r}{v}\left[1-x^{\varphi}\right]^{(\alpha(z+g+1)-1)+v+r-1}$$

The pdf of the minimum order statistic is obtained by setting r = 1 in equation (24), resulting in the following expression.

$$f_{1:n}(x;\zeta,\theta,\varphi,\alpha) = 4n\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{v=0}^{n-r} \sum_{i,j,k,m,z=0}^{\infty} \sum_{q,t,d,g=0}^{\infty} \sum_{p,w=0}^{v} (-1)^{i+j+k+m+z+v} (-1)^{p+w+q+t+d+g}$$
$$\times \binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\binom{v}{w}\binom{v+p-1}{p} \tag{25}$$
$$\times \binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}\binom{n-1}{v}\left[1-x^{\varphi}\right]^{(\alpha(z+g+1)-1)+v}$$

Similarly, the pdf of the maximum order statistic is derived by setting r = n in equation (24), resulting in the following expression.

$$f_{n:n}(x;\zeta,\theta,\varphi,\alpha) = 4n\zeta\theta\varphi\alpha x^{\varphi-1} \sum_{v=0}^{n-r} \sum_{i,j,k,m,z=0}^{\infty} \sum_{q,t,d,g=0}^{\infty} \sum_{p,w=0}^{v+n-1} (-1)^{i+j+k+m+z+v} (-1)^{p+w+q+t+d+g}$$
$$\times \binom{1+i}{i}\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{m}{z}\binom{v+n-1}{w} \tag{26}$$
$$\times \binom{v+n+p-2}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{d}{g}\left[1-x^{\varphi}\right]^{(\alpha(z+g+1)-1)+v+n-1}$$

6. Maximum Likelihood Estimation

Let $x_1, x_2, \dots x_n$ be random sample of size n from the TIHLTLKw$(\zeta,\theta,\varphi,\alpha)$ distribution. Then the sample log-likelihood function of the TIHLTLKw $(\zeta,\theta,\varphi,\alpha)$ distribution is obtained as

$$\begin{aligned}\log(L) &= n\log(4)+n\log(\zeta)+n\log(\theta)+n\log(\varphi)+n\log(\alpha)+(\varphi-1)\sum_{i=1}^{n}\log(x_i)+(\alpha-1)\sum_{i=1}^{n}\log\left[1-x_i^{\varphi}\right] \\ &+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]+(\theta-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right] \\ &+(\zeta-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right] \\ &-2\sum_{i=1}^{n}\log\left[1+\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}\right]\end{aligned} \tag{27}$$

Differentiating equation (27) with respect to $\zeta,\alpha,\varphi,\theta$ and setting the result to zero, we obtain:

$$\begin{aligned}\frac{\partial L}{\partial \zeta} &= \frac{n}{\zeta}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right] \\ &-2\sum_{i=1}^{n}\frac{\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}\log\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]}{1+\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}}\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= \frac{n}{\theta}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]+(\zeta-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\log\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]}{1-\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}} \\ &+2\sum_{i=1}^{n}\frac{\zeta\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta-1}\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\log\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]}{1+\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}}\end{aligned} \tag{29}$$

$$\frac{\partial L}{\partial \varphi}=\frac{n}{\varphi}+\sum_{i=1}^{n}\log(x_i)-(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\varphi}\log x_i}{\left[1-x_i^{\varphi}\right]}-\sum_{i=1}^{n}\frac{\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\alpha\left[1-x_i^{\varphi}\right]^{\alpha-1}x_i^{\varphi}\log x_i}{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]}$$

$$+(\theta-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\alpha\left[1-x_i^{\varphi}\right]^{\alpha-1}x_i^{\varphi}\log x_i}{\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]}$$

$$-(\zeta-1)\sum_{i=1}^{n}\frac{\theta\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\alpha\left[1-x_i^{\varphi}\right]^{\alpha-1}x_i^{\varphi}\log x_i}{\left[1-\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]}$$

$$+2\sum_{i=1}^{n}\frac{\zeta\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta-1}\theta\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\alpha\left[1-x_i^{\varphi}\right]^{\alpha-1}x_i^{\varphi}\log x_i}{\left[1+\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}\right]} \quad (30)$$

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-x_i^{\varphi}\right]+\sum_{i=1}^{n}\frac{\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\left[1-x_i^{\varphi}\right]^{\alpha}\log\left[1-x_i^{\varphi}\right]}{\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]}$$

$$-(\theta-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\left[1-x_i^{\varphi}\right]^{\alpha}\log\left[1-x_i^{\varphi}\right]}{\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]}$$

$$+(\zeta-1)\sum_{i=1}^{n}\frac{\theta\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\left[1-x_i^{\varphi}\right]^{\alpha}\log\left[1-x_i^{\varphi}\right]}{\left[1-\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta}\right]}$$

$$-2\sum_{i=1}^{n}\frac{\zeta\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta-1}\theta\left[1-\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]^{2}\right]^{\theta-1}2\left[1-\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\right]\left[1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]\left[1-x_i^{\varphi}\right]^{\alpha}\log\left[1-x_i^{\varphi}\right]}{\left[1+\left[1-\left[1-\left[1-1-\left[1-x_i^{\varphi}\right]^{\alpha}\right]^{2}\right]^{\theta}\right]^{\zeta}\right]} \quad (31)$$

These equations [(28), (29), (30), (31)] are nonlinear and cannot be solved analytically. Therefore, statistical software such as *R* with iterative numerical techniques is required to obtain the value of the unknown parameters.

7. Simulation study

This section addresses a numerical analysis to evaluate the performance of MLE for TIHLTLKw distribution.

Table 1: MLEs, biases and RMSE for some values of the parameters of TIHLTLKw distribution

		(2.16,0.66,1.59,2.11)			(2.32,0.68,2.55,2.4)		
n	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
20	ζ	2.1710	0.0110	1.6865	2.4207	0.1007	1.7805
	α	1.0235	0.3635	1.4439	0.8953	0.2153	1.3478
	θ	1.6993	0.1093	2.0067	2.6753	0.1253	1.8349
	φ	2.3499	0.2399	1.6330	2.6437	0.2437	2.7739
50	ζ	2.1671	0.0071	1.0797	2.3346	0.0146	1.0917
	α	1.0043	0.3443	0.8547	0.7859	0.1059	0.9477
	θ	1.6528	0.0628	1.2066	2.5734	0.0234	1.0125
	φ	2.2458	0.1358	1.0962	2.5460	0.1460	2.3243
100	ζ	2.1664	0.0064	0.8104	2.3249	0.0049	0.8369
	α	0.7554	0.0954	0.6222	0.6891	0.0091	0.7138
	θ	1.6890	0.0990	0.7206	2.5656	0.0156	0.7127
	φ	2.1349	0.0249	1.0424	2.4449	0.0449	1.9036
250	ζ	2.1637	0.0037	0.7068	2.3238	0.0038	0.7509
	α	0.6682	0.0082	0.4710	0.6852	0.0052	0.6212
	θ	1.5977	0.0077	0.5020	2.5529	0.0029	0.5392
	φ	2.1154	0.0054	0.9480	2.4387	0.0387	1.6219
500	ζ	2.1634	0.0034	0.6561	2.3216	0.0016	0.7050
	α	0.6613	0.0013	0.4358	0.6833	0.0033	0.5538
	θ	1.5946	0.0046	0.4487	2.5531	0.0031	0.3690
	φ	2.1133	0.0033	0.8245	2.4160	0.0160	1.3241
1000	ζ	2.1612	0.0012	0.6468	2.3200	0.0000	0.6525
	α	0.6608	0.0008	0.4292	0.6811	0.0011	0.5173
	θ	1.5914	0.0014	0.4497	2.5504	0.0004	0.2846
	φ	2.1110	0.0010	0.7142	2.4012	0.0012	1.1043

Table 1 displays the values of biases, estimated values and RMSEs. It is noticed from the table that the RMESs approach zero and the estimates tend to the true parameter values as the sample size increases. This is an indication that the maximum likelihood estimates are efficient and consistent.

8. Applications

The fit of TIHLTLKw distribution is tested with applications to real-life data sets to assess its flexibility and robustness. The fit of the TIHLTLKw model is compared with some existing distributions having Kumaraswamy distribution as their baseline. The comparators are:

- Kumaraswamy-Kumaraswamy (Kw-Kw) distribution (El-Sherpieny *et al.*, 2014).

$$f(x;\lambda,\beta,\alpha,\theta)=\lambda\beta\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1}\left(1-\left(1-x^{\alpha}\right)^{\theta}\right)^{\lambda-1}\left(1-\left(1-\left(1-x^{\alpha}\right)^{\theta}\right)^{\lambda}\right)^{\beta-1}$$

- Weibull-Kumaraswamy (Wkw) distribution (Aminu *et al.*, 2018)

$$f(x;\lambda,\beta,\alpha,\theta)=\lambda\theta\alpha\beta\frac{x^{\lambda-1}\left(1-x^{\lambda}\right)^{\theta-1}}{\left[1-\left(1-x^{\lambda}\right)^{\theta}\right]}\left[-\log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right]^{\beta-1}\exp\left(-\alpha\left(-\log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right)^{\beta}\right)$$

- Type II Half Logistic Kumaraswamy distribution (ZeinEldin *et al.*, 2020)

$$f(x;\lambda,\beta,\alpha)=\frac{2\lambda\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\lambda-1}}{\left(1+\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\lambda}\right)^{2}}$$

- Exponentiated Kumaraswamy distribution (Lemonte *et al.*, 2013)

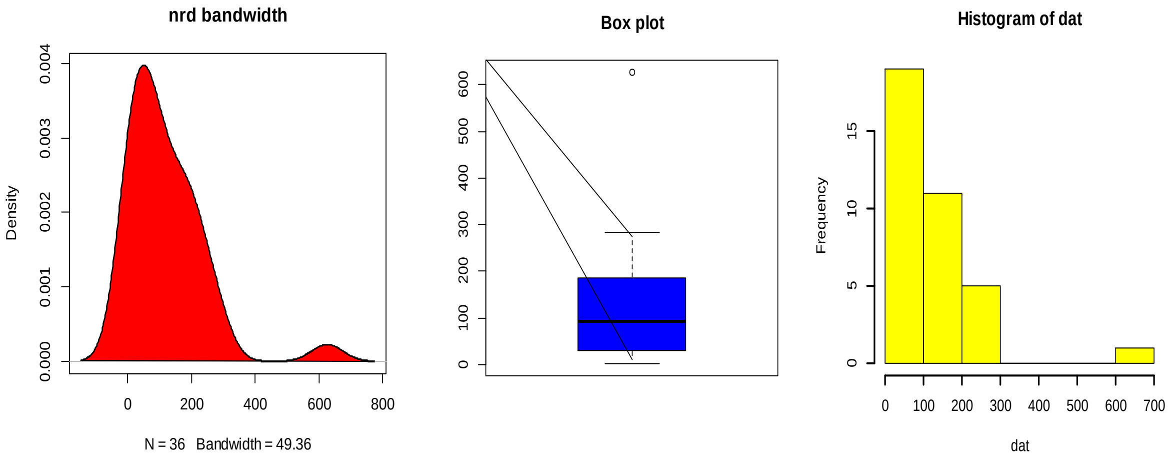
$$f(x;\beta,\alpha,\theta)=\alpha\beta\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\theta-1}$$

- Kumaraswamy distribution (Kumaraswamy, 1980)

$$f(x;\alpha,\theta)=\alpha\theta x^{\alpha-1}\left[1-x^{\alpha}\right]^{\theta-1}$$

The first data set as listed below represents the daily recovered cases of COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Bello, et al., (2021):

2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36, 102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.



The second data set as listed below represents the daily confirmed cases of COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Bello, et al., (2021):

108, 102, 133, 170, 121, 99, 236, 178, 250, 161, 258, 172, 407, 577, 210, 243, 281, 186, 254, 336, 342,269, 520, 414, 463, 514, 427, 796, 555, 742, 642, 785, 783, 605, 751, 806.

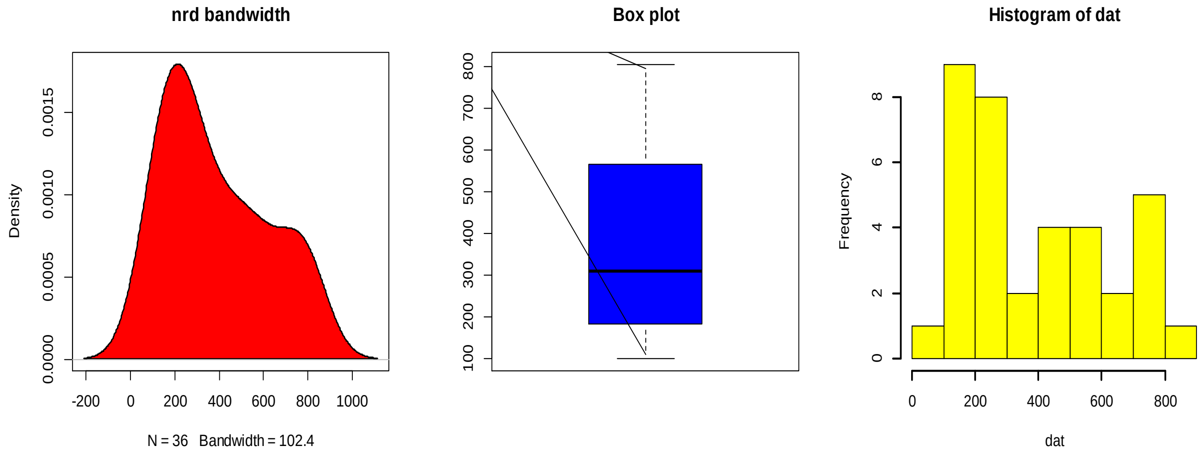


Table 2: The models' MLEs and performance requirements based on data set 1

Model	z	α	j	θ	LL	AIC
TIHLTLKw	5.5234	8.1539	0.3300	7.6858	- 78.2356	164.4713
Kw-Kw	1.0000	5.2140	0.5318	1.4560	- 98.7281	205.4562
Wkw	0.5144	0.2368	10.3984	2.4068	-230.6396	469.2792
TIIHLKw	0.6723	2.5184	5.2032	-	-274.1351	554.2702
EKw	-	0.4861	1.0000	6.1035	-121.5162	249.0324
Kw	-	0.8260	0.2566	-	-248.0185	500.0369

Table 3: The models' MLEs and performance requirements based on data set 2

Model	z	α	j	θ	LL	AIC
TIHLTLKw	6.5948	3.7771	0.4193	9.2558	- 154.1285	316.257
Kw-Kw	3.2361	0.1134	5.6043	2.2687	- 205.4299	418.8599
Wkw	0.3709	2.1396	6.0167	6.3317	-246.5476	501.0951
TIIHLKw	4. 1832	3. 0523	0.5013	-	-390.0542	786.1084
EKw	-	5.4613	1.6289	0. 1641	-188.4981	382.9963
Kw	-	9.0925	8.9295	-	454.279	912.558

Tables 2 and 3 outline the results of the mle of the parameters of the TIHLTLKw distribution together with the comparator distributions. Based on the goodness of fit statistic AIC, the new model recorded the lowest AIC value suggesting that the TIHLTLKw distribution best fits the two data sets.

9. Conclusion

This research article proposed and studied a new continuous probability distribution called the type I half logistic Topp-Leone Kumaraswamy distribution. The new model was derived from the type I half logistic Topp-Leone-G family of distribution using Kumaraswamy as the baseline distribution. The properties of the new model such as probability-weighted moments, moments, quantile function, moment generating function, reliability function, hazard function, and order statistics were examined as statistical components of the newly proposed model. The parameters of the model are estimated using the method of maximum likelihood technique. Simulation was conducted and the results of the new distribution's performance were carried out to assess the efficiency of the estimation method used. Two real data sets were applied to ascertain the importance and flexibility of the new distribution. The results reveal that the new model appears to be superior to the existing models considered.

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