Dominating Sets and Domination Polynomial of k_r-gluingof Graphs

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Abstract

Let $G = (V, E)$ be a simple graph. Aset $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to at least one vertex in D. Let G be k_r-gluing of G_1 and $G_2$and denote by $C[G_1 \cup_r G_2]$ the family of all k_r-gluing of $G_1$and G_2. Let K_t be complete graph with order t and K_m be complete graph with order m and let G be k_r-gluing of K_t and K_m with order $n = m + t - r$. Let G_n^i be the family of dominating sets of G_n with cardinality i, and let $d(G_n, i) = |G_n^i|$. In this paper, we construct G_n^i, and obtain a recursive formula for $d(G_n, i)$. Using this recursive formula, we consider the polynomial $D(G_n, x) = \sum_{i=1}^{n} d(G_n, i)x_i$, which we call domination polynomial of k_r-gluing of graphs and obtain some properties of this polynomial
.

Keywords: Dominating sets, domination polynomial, k_r-gluing of graphs

الملخص:

إذا كان $G = (V,E)$ بيان بسيط . و كانت D مجموعة جزئية منVفأنDتسمى مجموعة مهيمنة للبيان G، إذا كان كل رأس في V-D يجاور على الأقل رأس واحد في D. وليكن G بيان مكون من بيانين تامين من الدرجتين t,m بواسطة بيان تام من الدرجة r ، لذلك تكون درجة البيان G هي n=m-t-r ولتكن $d(G_n, i)$ هي عدد كل المجموعات المهمنة التي عدد عناصرها i وان متعددة الحدود G_n تكون $D(G_n, x) = \sum_{i=1}^{n} d(G_n, i)x^i$. في هذا البحث وجدنا علاقة لإيجاد $d(G_n, i)$ وبعض الخواص فيها.واستخدامها في ايجاد متعددة الحدود المهيمنة $D(G_n, x) = \sum_{i=1}^{n} d(G_n, i)x^i$ للبيان المتكون من اتحاد بيانين تامين وبعض الخواص فيها

الكلمات مفتاحية: المجموعات المهينة، متعددات الحدود المهيمنة، البيانات المتحدة ببيان تام من الدرجة r

1 Introduction

Suppose G = (V, E) a simple graph. A set D subset of V is a dominating set of G, if each point in V-D is connects to at least one point in D. The domination number (G) is the minimum number of vertices that meets the definition of D. For a detailed treatment of this parameter, the reader is referred to [7]. It is known and generally accepted that the problem of identifying the dominant groups on an arbitrary graph is a difficult one (see [6]). Alikhani and Peng found the dominating set and domination polynomial of cycles, certain graph and non P4-free [1], [2], [3]. Dod, Kotek, Preen and Tittmann found Bipartition Polynomials, the Ising Model, and Domination in Graphs [4]. Kahat and Khalaf found the dominating set and domination polynomial of stars, wheels, complete graph with missing and k_r-gluing of Graphs see [8], [9], [10]. Kotek, JPreen and Tittmann found Domination Polynomials of Graph Products [11]. Let G_n be graph with order n and let the family of dominant sets G_n^i in a graph G_n have cardinality i and let $d(G_n, i) = |G_n^i|$. $D(G_n, x) = \sum_{i=r(G)}^{n} d(G_n, i)x_i$ is called domination polynomial of graph G [2]. Let G be k_r-gluing of G_n and G_n and denote by $C[G_1 \cup_r G_2]$thefamily of all k_r-gluing of G_1 and G_2[5]. Let K_t be complete graph with order t and K_m be complete graph with order m and let G be k_r-gluing of K_t and K_m with order $n = m + t - r$.
Use $\binom{n}{i}$ for the combination $n\ to\ i$.

2 Dominating sets of k_1-gluing of K_t and K_m

We shall investigate dominating sets of Let G_n be k_1-gluing of$K_t$andK_m. To prove our main results we need the following lemmas:

***Lemma* 1 [8].**
These properties apply to all graphs G.
(i) $|G_n^n|$ = 1 (ii) $|G_n^{n-}|$ = n (iii) $|G_n^i|$= 0 if $i > n$ (iv) $|G_n^0|$= 0

$\boldsymbol{Theorem}$ **1**.
Let K_t be complete graph with order t and K_m be complete graph with order m and let G_n be K_1- gluing of K_t and K_m with order $n = m + t - 1$, then $d(G_n, i)$=$\binom{n}{i} - \binom{m-1}{i} - \binom{t-1}{i}$, $\forall n, m, t \in Z^+$, and $i = 1, 2, \dots, n$.

Proof.
Let K_1= {v}. Since every vertex $u_t \in K_t$ it is not adjacent with every other vertex $u_m \in K_m$such that $u_t \neq v \neq u_m$, then every subset of $K_t - v$ with cardinality i is not dominating sets of G_n, and every subset of $K_t - v$with cardinality i is not dominating sets of G_n therefore
$d(G_n, i)$=$\binom{n}{i} - \binom{m-1}{i} - \binom{t-1}{i}$.

$\boldsymbol{Theorem}$ **2**.
Let K_t be complete graph with order t and K_m be complete graph with order m and let G_n be K_r- gluing of K_t and K_m with order$n = m + t - 1$, then $d(G_n, i)$=$\binom{n}{i} - \binom{m-r}{i} - \binom{t-r}{i}$,$\forall n, m, r, t \in Z^+$, and $i = 1, 2, \dots, n$

Proof.
The proof is similar to the proof of (Theorem 1).

LetG_n be k_1-gluing ofK_2 and K_m with order $n = m + 1$. In Table 1, we obtain the coefficients of $D(G_n, x)$ for $2 \leq n \leq 15$ based on Theorem 1. Let $d(G_n, i)$= $|G_n^i|$. It is possible to get important relationships between numbers $d(G_n, i)$($1 \leq i \leq n$) in the table.

i			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n	m	t															
2	2	1	2	1													
3	2	2	1	3	1												
4	2	3	1	5	4	1											
5	2	4	1	7	9	5	1										
6	2	5	1	9	16	14	6	1									
7	2	6	1	11	25	30	20	7	1								
8	2	7	1	13	36	55	50	27	8	1							
9	2	8	1	15	49	91	105	77	35	9	1						
10	2	9	1	17	64	140	196	182	112	44	10	1					
11	2	10	1	19	81	204	236	378	294	156	54	11	1				
12	2	11	1	21	100	285	440	614	672	450	210	65	12	1			
13	2	12	1	23	121	385	725	1054	1286	1122	660	275	77	13	1		
14	2	13	1	25	144	506	1110	1779	2340	2408	1782	935	352	90	14	1	
15	2	14	1	27	169	650	1616	2889	4119	4748	4190	2717	1287	442	104	15	1

Table 1. $d(G_n, i)$ The number of dominating sets of G_n with cardinality i sech that G_n be $k_1 - gluing$ of K_2 and K_m

$d(G_n, i)$ has some properties as we prove the following theorem

***Theorem* 3.**

For every $n \in Z^+$, The following properties achieved of $d(G_n, i)$:

(i) $d(G_n, 1) = 1\ \forall n \geq 3$.

(ii) $d(G_n, 2) = d(G_{n-1}, 2) + 2$.

(iii) $d(G_n, i) = d(G_{n-1}, i) + d(G_{n-1}, i - 1)\ \forall i \geq 2$

(iv) $d(G_n, n - 1) = n$.

(v) $d(G_n, n) = 1$.

(vi) $\gamma(G_n) = 1\ \forall n \geq 3$

(vii) $d(G_n, i) = d(K_n, i) for\ n = 2$

***Proof*.**

Let G_n be k_3-gluing of K_2 and K_m with order $n = m + 1$, then

(i) By Theorem 1 d(G_n, 1)=$\binom{n}{1} - \binom{m-1}{1} - \binom{1}{1}$, since $n = m + 1$ hence $m = n - 1$, therefore $d(G_n, 1)$=$\binom{n}{1} - \binom{n-2}{1} - \binom{1}{1}$= 1

(ii) By Theorem 1 $d(G_n, 2) = \binom{n}{2} - \binom{n-1}{2} - \binom{1}{2}$= $\binom{n}{2} - \binom{n-2}{2}$= $2n - 3$, and $d(G_{n-1}, 2)$ +2=$\binom{n-1}{2} - \binom{n-2}{2} - \binom{0}{2} + 2$ =$2n - 3$, then $d(G_n, 2) = d(G_{n-1}, 2) + 2$.

(iii) By Theorem 1 Let ω=$\frac{(n-1)(n-2).....(n-i+1)}{i!}$ $d(G_n, i) = \binom{n}{i} - \binom{n-2}{i}$= $\frac{n(n-1).....(n-i+1)(n-i)!}{i!(n-i)!} - \frac{(n-2)(n-3).....(n-i-1)(n-2-i)!}{i!(n-2-i)!}$= $\omega n - \omega \frac{(n-i)(n-i-1)}{(n-1)}$, and $d(G_{n-1},i)+d(G_{n-1},i-1)=\binom{n-1}{i}-\binom{n-3}{i}+\binom{n-1}{i-1}-\binom{n-3}{i-1}=$ $\omega(n-i)-\omega\frac{(n-i)(n-i-1)(n-i-2)}{(n-1)(n-2)}+\omega i-\omega\frac{(n-i)(n-i-1)}{(n-1)(n-2)}i=$ $\omega\left[\frac{n(n-1)-(n-i)(n-i-1)}{(n-1)}\right]=\omega n-\omega\frac{(n-i)(n-i-1)}{(n-1)}$, then $d(G_n,i) = d(G_{n-1},i) + d(G_{n-1},i-1)$

(iv) By Theorem 1 $d(G_n,n-1)=\binom{n}{n-1}-\binom{n-2}{n-1}=\binom{n}{n-1}$=n

(v) By Theorem 1we have$d(G_n,n)=\binom{n}{n}-\binom{n-2}{n}$= $\binom{n}{n}=1.$

(vi) Since K_1={v} is dominating set of (G_n), then$\gamma(G_n) = 1.$

(vii) $d(G_2,i)=\binom{2}{i}-\binom{0}{i}=\binom{2}{i}=d(K_2,i)$by Lemma 1 (iii)■

Let G_n be k_1-gluing ofK_3 and K_m with order $n=m+2$. Obtain the coefficients of $D(G_n,x)$ for $1\le n\le 12$ in Table 2 based on Theorem 2. Let $d(G_n,i)$= $|G_n^i|$. It is possible to get important relationships between numbers $d(G_n,i)$($1\le i\le n$) in the table.

i			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n	m	t															
3	3	1	3	3	1												
4	3	2	1	5	4	1											
5	3	3	1	8	10	5	1										
6	3	4	1	11	19	15	6	1									
7	3	5	1	14	31	34	21	7	1								
8	3	6	1	17	46	65	55	28	8	1							
9	3	7	1	20	64	111	120	83	36	9	1						
10	3	8	1	23	85	175	231	203	119	45	10	1					
11	3	9	1	26	109	260	406	434	322	164	55	11	1				
12	3	10	1	29	136	369	666	840	756	486	219	66	12	1			
13	3	11	1	32	166	505	1035	1054	1596	1242	705	285	78	13	1		
14	3	12	1	35	199	671	1540	2089	3650	2838	1947	990	363	91	14	1	
15	3	13	1	38	235	870	1616	3629	5739	6488	4785	2937	1353	454	105	15	1

Table 2. d(G_n,i) The number of dominating sets of G_n with cardinality i sech that G_n be k_1- *gluing* of K_3 and K_m

The following theorem, obtain some results of $d(G_n,i)$:G_n be k_1-gluing of K_3 and K_m

***Theorem* 4.**

The following results achieved of $d(G_n,i)$, for all $n \in Z^+$:

(i) $d(G_n,1)=1\ \forall n\ge 4.$

(ii) $d(G_n,2)=\binom{n}{2}-\binom{m-1}{2}-1$

(iii) $d(G_n,2) = d(G_{n-1},2) + 3$.
(iv) $d(G_n,3) = d(G_{n-1},3) + d(G_{n-1},2) + 1$.
(v)$d(G_n,i) = d(G_{n-1},i) + d(G_{n-1},i-1) \forall i \geq 3$.
(vi)$d(G_n,n-1) = n$
(vii)$d(G_n,n) = 1$
(viii)$\gamma(G_n) = 1 \forall n \geq 7$
(ix)$d(G_3,i) = d(K_3,i)$

Proof.
The proof is similar to the proof of (Theorem 2). Let G_n bek_1-gluing ofK_3 and K_m with order $n = m + 3 - 1 = m + 2$, then
(i)By Theorem 1 $d(G_n,1)$=$\binom{n}{1} - \binom{m-1}{1} - \binom{2}{1}$, since n = m + 2 then m = n - 2, therefore$d(G_n,1)$=$\binom{n}{1} - \binom{n-3}{1} - \binom{2}{1}$ = n - n + 3 -2 = 1
(ii) By Theorem 1 $d(G_n,2)$=$\binom{n}{2} - \binom{m-1}{2} - \binom{2}{2} = \binom{n}{2} - \binom{m-1}{2} - 1$
(iii) By Theorem 1 $d(G_n,2)$=$\binom{n}{2} - \binom{m-1}{2} - 1$=3n-7, and$d(G_{n-1},2) +$ 3=$\binom{n-1}{2} - \binom{m-2}{2} - 1 + 3$=$\binom{n-1}{2} - \binom{n-4}{2} + 2 = 3n - 9 + 2 = 3n - 7$, then $d(G_n,2) = d(G_{n-1},2) + 3$
(iv)- (ix) The proof is the same way in the (Theorem 3) in (ii)- (vii)■

By the following theorem, we prove some properties of$d(G_n,i)$:G_n be k_1-gluing ofK_t and $K_m$such that (t) is constant $\forall\ 1 \leq m \leq n - t + r$.

***Theorem* 5**.
For all $n \in Z^+$, $d(G_n,i)$ has the following properties :
(i) $d(G_n,1) = 1 \forall n \geq t$.
(ii) $d(G_n,i) = d(G_{n-1},i) + d(G_{n-1},i-1) + \binom{t-1}{i-1} \forall n > t$.
(iii) $d(G_n,i) = d(G_{n-1},i) + d(G_{n-1},i-1) \quad \forall i > t$.
(iv)$(G_n,n-1) = n$
(v)$d(G_n,n) = 1$.
(vi) $\gamma(G_n) = 1 \ \forall n > t$
(vii)$d(G_n,i) = d(K_n,i) \forall n = t$

***Proof*.**
The proof is similar to the proof of (Theorem 3) and (Theorem 4)

3 Domination Polynomial of K_1-gluing of Graphs

In this section we introduce and investigate the domination polynomial of K_1-gluing of $K_m$and $K_t$such that (t) is constant $\forall\ 1 \leq m \leq n - t + 1$.

Definition.
Let G_n^i be the family of dominating sets of a graph G_n(K_1-gluing of $K_m$and K_t) When cardinality is considered i, and let $d(G_n, i) = |G_n^i|$, and since $\gamma(G_n) = 1$. Then $D(G_n, x)$of G_n (domination polynomial) is defend as

$$D(G_n, x) = \sum_{i=1}^{n} d(G_n, i)x^i\ \forall n > t$$

In the following corollary, we obtain some properties of (G_n, x) : G_n be K_1-gluing of $K_m$and $K_t$such that (t) is constant $\forall\ 1 \leq m \leq n - t + 1$.

Corollary 1.
The following properties hold for all $D(G_n, x) \forall\ n > t$

(i) $D(G_n, x)=\sum_{i=1}^{n}\binom{n}{i}x^i - \sum_{i=1}^{m-1}\binom{m-1}{i}x^i - \sum_{i=1}^{t-1}\binom{t-1}{i}x^i$

(ii) $D(G_n, x) = D(G_{n-1}, x)+ xD(G_{n-1}, x)+\sum_{i=1}^{t-1}\binom{t-1}{i}x^{i+1}$

Proof.
(i) From Theorem 1 and definition in above , we get
$D(G_n, x) =\sum_{i=1}^{n} d(G_n, i)x^i = \sum_{i=1}^{n}\ [\binom{n}{i} - \binom{m-1}{i} - \binom{t-1}{i}]x^i=\sum_{i=1}^{n}\binom{n}{i}x^i - \sum_{i=1}^{n}\binom{m-1}{i}x^i - \sum_{i=1}^{n}\binom{t-1}{i}x^i=\sum_{i=1}^{n}\binom{n}{i}x^i - \sum_{i=1}^{m-1}\binom{m-1}{i}x^i - \sum_{i=1}^{t-1}\binom{t-1}{i}x^i$(by Lemma1) $\binom{n}{i}$ = 0 if i > n
(ii) From definition of the domination polynomial and Theorem 5, we have
$D(G_n, x) =\sum_{i=1}^{n} d(G_n, i)x^i=\sum_{i=1}^{n}[d(G_{n-1}, i) + d(G_{n-1}, i-1) + \binom{t-1}{i-1}x^i = \sum_{i=1}^{n} d(G_{n-1}, i)x^i + \sum_{i=1}^{n} d(G_{n-1}, i-1)x^i + \sum_{i=1}^{n}\binom{t-1}{i-1}x^i$, we have $d(G_n, i)$ = 0 if $i > n\ or\ i = 0$ (Lemma1), then
$\sum_{i=1}^{n} d(G_{n-1}, i)x^i=\sum_{i=1}^{n-1} d(G_{n-1}, i)x^i = D(G_{n-1}, x)$

and$\sum_{i=1}^{n} d(G_{n-1}, i-1)x^{i} = x\sum_{i=1}^{n} d(G_{n-1}, i-1)x^{i-1} =$
$x[\sum_{i=1}^{n-1} d(G_{n-1}, i)x^{i}]$=$x\, D(G_{n-1}, x)$ and $\sum_{i=1}^{n}\binom{t-1}{i-1}x^{i} = \sum_{i=1}^{n}\binom{t-1}{i}x^{i+1}$
then$D(G_n, x)$ = $D(G_{n-1}, x)$+ $xD(G_{n-1}, x)$+$\sum_{i=1}^{t-1}\binom{t-1}{i}x^{i+1}$.■

Example 1

The following properties hold forallD(G_n, x):G_n be k_1-gluing of tow complete graphsK_m and K_2, $\forall\, n > 2$by Corollary 1

(i) $D(G_n, x)$=$\sum_{i=1}^{n}\binom{n}{i}x^{i} - \sum_{i=1}^{n-3}\binom{n-3}{i}x^{i} - x$

(ii) $D(G_n, x)$ = $D(G_{n-1}, x)$+ $xD(G_{n-1}, x)$+x^2

Example 2.

The following properties hold forallD(G_n, x):G_n be k_1-gluing of tow complete graphs$K_m$and K_3, $\forall\, n > 3$by Corollary 1

(i) $D(G_n, x)$=$\sum_{i=1}^{n}\binom{n}{i}x^{i} - \sum_{i=1}^{n-3}\binom{n-3}{i}x^{i} - 2x$-$x^2$

(ii) $D(G_n, x)$ = $D(G_{n-1}, x)$+ $xD(G_{n-1}, x)$+$2x^2 + x^3$

Example 3 .

Let G_7 be K_1-gluing of two complete graphs K_5 and K_3 with order 7, we can get on $D(G_7, x)$without the table. We have
$D(G_7, x)$= $\sum_{i=1}^{7}\binom{7}{i}x^{i} - \sum_{i=1}^{4}\binom{4}{i}x^{i} - \sum_{i=1}^{2}\binom{2}{i}x^{i}$= $(7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7) - (4x + 6x^2 + 4x^3 + x^4) - (2x + x^2) = x + 14x^2 + 31x^3 + 34x^4$ + $21x^5 + 7x^6 + x^7$(by Corollary 1).
(see Fig-1).

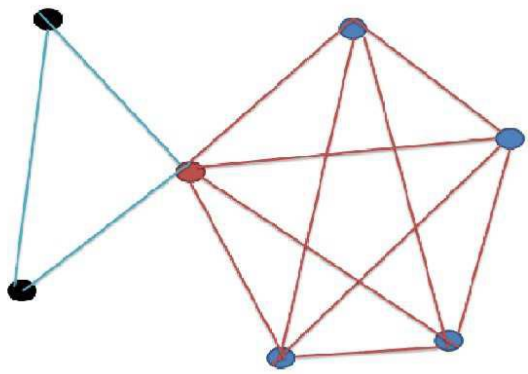

Fig-1:
G_7beK_1-gluing of two complete graphs K_5 and K_3

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