Some results of $p + \wp$ - valent Functions

Lecturer Dr. Lieth A. Majed

College of Science, University of Daylih

Lecturer Dr. Thamer khalil MS
College of Education, University of Al-Mustansirya,

المستخلص

في البحث نحن قدمنا وعرفنا صنف جزي جديد للدوال متعددة التكافوء $p + \wp$ مع المعاملات السالبة في قرص الوحدة $\Omega^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

حصلنا على بعض الخواص مثل متراجحة المعاملات، مبرهنة الانغلاق ،الوسط الوزني ومعامل التكا مل.

Abstract. In this present paper, we establish new subclass of $p + \wp$ - valent functions with negative coefficients in unit disk $\Omega^* = \{z \in \mathbb{C}: |z| < 1\}$. We obtain some properties, like, theorem of coefficient inequality, closure theorem ,weighted mean and integral operator.

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Introduction: This type of study was carry on by various different authors . In 2012 W. G. Atshan [3] study coefficient inequalities, closure theorems and obtain integral representation, convolution properties by using p-valent analytic functions . In 2013 S. Najafzadeh [7] use this to define a new class of meromorphically multivalent functions and give two useful subclasses of this class involving fixed points.

Consider Ĺ to be denoted category of functions take the following form:

$$f(z) = z^{p+\wp} + \sum_{k=2}^{\infty} a_k\, z^{k+p+\wp}, \qquad (1)$$

where $(0 \le \wp < 1 \; and \; p \in \mathbb{N})$

which are $p + \wp$- valent in $\Omega=\{z \in \mathbb{C}: |z| < 1\}$ the open unit disk and also analytic.

Let Ĺη indicate the subclass of Ĺ of functions write by:

$$f(z) = z^{p+\wp} - \sum_{k=2}^{\infty} a_k\, z^{k+p+\wp}, \qquad (a_k \ge 0, z \in \Omega). \qquad (2)$$

Note that the authors investigate some classes properties of analytic functions such as the form (1) in [5], We suppose $Ĺ(p, η)$ to be symbolize the subclass of $Ĺη$ including the functions f which satisfy:

$$\left|\frac{(p^2+\wp^2+2ηp-p-\wp)(p+\wp-2)z^{p+\wp-3}-f'''(z)}{f'''(z)\mu+\mu(p^2+\delta^2+\wp 2p-p-η)(p+\wp-2)z^{p+\wp-3}}\right|<\beta, \qquad (3)$$

where , $0<\beta\le p+\delta, 0\le η<1$.

The following interesting geometric properties of this function subclass were studied by several authors for another classes, like, [1], [3] and [4].

Theorem (1): Let $f \in Ĺ\delta$ .Then $f \in Ĺ(p,\delta)$ iff

$$\sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)a_k$$
$$\le 2\mu\beta(p+\wp)(p+\delta-1)(p+-2), \qquad (4)$$

where $0<\beta\le p+η, 0\le η<1$.

The result is sharp for the function

$$f(z)=z^{p+η}-\frac{2\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)}z^{k+p+\wp}, k\ge 2.$$

Proof: Assume (4) holds true and let $|z|=1$, then from (3), we obtain

$$|(p^2+\delta^2+2\wp p-p-\wp)(p+\wp-2)z^{p+\wp-3}-(f(z))'''|$$
$$-\beta\left|\mu\big(f(z)\big)'''+2\mu(p+\wp)(p+\wp-1)z^{p+\wp-3}\right|$$

$$=\left|\sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)\,a_k z^{k+p+\wp-3}\right|$$
$$-\beta\,|3\mu(p+\wp)(p+\wp-1)z^{p+η-3}| \quad (5)$$

$$\leq \sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)a_k - 2\beta\,\mu(p+\wp)(p+\wp-1)$$
$$+\sum_{k=2}^{\infty}\beta\mu(k+p+\wp)(k+p+\wp-1)\,a_k$$

$$=\sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\beta\mu+1)a_k - 2\beta\mu(p+\wp)(p+\wp-1)\leq 0,$$

by hypothesis.

Hence by maximum modulus principle, $f\in \acute{L}(p,\wp)$.

Conversely, Let $f\in \acute{L}(p,\wp)$ Then

$$\left|\frac{(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)z^{p+\wp-3}-f(z)'''}{\mu f(z)'''+\mu(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)z^{p+\wp-3}}\right|<\beta,(z\in U).$$

That is

$$\left|\frac{\sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)(p+\wp-2)a_k z^{k+p+\wp-3}}{\mu(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)z^{p+\wp-2}-\sum_{k=2}^{\infty}\mu(k+p+\wp)(k+p+\wp-1)(k+p+\eta-2)a_k z^{k+p+\wp-3}}\right| < \beta,(6)$$

Since $Re(z)\leq |z|$ for all z $(z\in U)$, we get

$$Re\left\{\frac{\sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)(p+\wp-2)a_k z^{k+p+\wp-3}}{\mu(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)z^{p+\eta-2}-\sum_{k=2}^{\infty}\mu(k+p+\wp)(k+p+\wp-1)(p+\delta-2)a_k z^{k+p+\wp-3}}\right\} \leq \beta,(7)$$

We choose the value of the real axis on z and so that $(f(z))'''$ is real.

$$\sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)(p+\wp-2)a_k z^{k+p+\wp-3}$$

$$\leq 2\mu\beta(p+\wp)(p+\wp-1)(p+\wp-2)z^{p+\wp-3} - \sum_{k=2}^{\infty} \gamma\mu(k+p+\wp)(k+p+\wp-1)(p+\wp-2)a_k z^{k+p+\wp-3}.$$

Letting z → 1^-, through real values,

$$\sum_{k=2}^{\infty}(k+p+\wp)(k+p+\wp-1)(p+\wp-2)a_k$$

$$\leq 2\mu\beta(p+\wp)(p+\wp-1)(p+\wp-2) - \sum_{k=2}^{\infty} \beta\mu(k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)a_k,$$

we obtain inequality (4).

Finally, sharpness follows, if we take

$$f(z) = z^{p+\wp} - \frac{2\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)\big)(\beta\mu+1)} z^{k+p+\wp}, k \geq 2. \ (8)$$

Corollary (1): Let $f \in \acute{L}(p,\wp)$. Then

$$a_k \leq \frac{2\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)\big)(\beta\mu+1)}, \geq 2. \quad (9)$$

In the following theorem, we obtain closure theorem of the class $\acute{L}(p,\wp)$

Theorem (2): Let the functions f_i defined by

$$f_i(z) = z^{p+\wp} - \sum_{k=2}^{\infty} a_k \ z^{k+p+\wp}, \big(a_{k,i} \geq 0, p \in \mathbb{N}, i = 1,2,\dots.\ell\big) \quad (10)$$

is in the class $\acute{L}(p,\eta)$ for every i= 1,2,…,ℓ. Then the function M_1 defined by

$$M_1(z) = z^{p+\wp} - \sum_{k=2}^{\infty} a_k\, w_k\, z^{k+p+\wp}, (w_k \geq 0),$$

also belongs to the $\acute{L}(p,\delta)$ where

$$w_{k=}\frac{1}{\ell}\sum_{i=1}^{\ell} a_{k,i}\ ,\quad k = 2,3 \ldots .$$

Proof: Since $f_i \in \acute{L}(p,\wp)$, we note that

$$\sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)a_{k,i}$$
$$\leq 2\mu\beta(p+\wp)(p+\wp-1)(p+\wp-2),$$

for every i=1,2,…,ℓ. Hence

$$\sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)w_k$$
$$= \sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)\quad (\frac{1}{\ell}\sum_{i=1}^{\ell} a_{k,i})$$
$$= \frac{1}{\ell}\sum_{i=1}^{\ell}\quad \left(\sum_{k=2}^{\infty}\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)a_k\, a_{k,i}\right)$$
$$\leq 2\mu\beta(p+\wp)(p+\wp-1)(p+\wp-2).$$

Therefore by Theorem1, obtain $h_1 \in \acute{L}(p,\eta)$

In the following theorem, we obtain weighted mean is in the class $\acute{L}(p,\wp)$

Definition (2)[6]: Let f_1 and f_2 belong to the class of $\acute{L}(p,\wp)$. Then the weighted mean w_j of f_1 and f_2 is given by:

$$w_j(z) = \frac{1}{2}\ [\ (1-j)f_1\ (z) +\ (1+j)f_2(z)],\quad 0 < j < 1.$$

Theorem (2): Let f_1 and f_2 be in the class $\acute{L}(p,\wp)$ Then the weighted mean w_j of f_1 and f_2 is also in the class $\acute{L}(p,\wp)$

Proof: By definition (1), we have

$$w_j(z) = \frac{1}{2}\ [\ (1-j)f_1\ (z) +\ (1+j)f_2(z)], \qquad (11)$$

$$= \frac{1}{2}\left[(1-j)\left(z^{p+\wp} - \sum_{k=2}^{\infty} a_{k,1}\ z^{k+p+\wp}\right) + (1+j)\left(z^{p+\wp} - \sum_{k=2}^{\infty} a_{k,2}\ z^{k+p+\wp}\right)\right]$$

$$= z^{p+\wp} - \sum_{k=2}^{\infty}\frac{1}{2}\ \left[(1-j)a_{k,1} + (1+j)a_{k,2}\ \right]z^{k+p+\wp}.$$

Since f_1 and f_2 are in the class $\acute{L}(p,\wp)$ so by Theorem (1), we get

$$\sum_{k=2}^{\infty}\Big(\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)\Big)\,(\beta\mu+1)a_{k,1}$$
$$\leq \mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2),$$

and

$$\sum_{k=2}^{\infty}\Big(\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)\Big)\,(\beta\mu+1)a_{k,2}$$
$$\leq \mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2).$$

Hence

$$\sum_{k=2}^{\infty}\Big(\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)\Big)\,(\beta\mu+1)]\frac{1}{2}\Big[\ (1-j)a_{k,1}$$
$$+ (1+j)a_{k,2}\Big]$$

$$= \frac{1}{2}\ (1-j)\sum_{k=2}^{\infty}\Big(\big((k+p+\delta)(k+p+\delta-1)(p+\delta-2)\big)(\gamma\mu+1)\Big)\,(\beta\mu+1)a_{k,1}$$

$$+ \frac{1}{2}\ (1+j)\sum_{k=2}^{\infty}\Big(\big((k+p+\wp)(k+p+\wp-1)(p+\wp-2)\big)(\gamma\mu+1)\Big)\,(\beta\mu+1)a_{k,1}$$

$$\leq \frac{1}{2}(1-j)\mu\gamma(p^2+\wp^2+2\wp p-p-\delta)(p+\wp-2)+\frac{1}{2}\ (1+j)\mu\gamma(p^2+\wp^2+2\wp p$$
$$-p-\wp)(p+\wp-2)=\mu\beta(p+\wp)(p+\wp-1)(p+\wp-2).$$

Therefore, $w_j \in \acute{\text{L}}(p,\wp)$.

The proof is complete.

In the following theorem, we obtain integral operator [2] is in the class $\acute{\text{L}}(p,\wp)$

Theorem (3): Let $f(z) \in \acute{\text{L}}(p,\wp)$ then the integral operator

$$F_\iota(z)=(1-\iota)z^{p+\wp}+\iota(p+\wp)\int_0^z \frac{f(c)}{c}\,ds\ (\iota\geq 0, z\in U),$$

is also in $\acute{\text{L}}(p,\wp)$ if $0\leq \iota\leq \frac{2+p+\wp}{p+\wp}$.

Proof: If

$$f(z)=z^{p+\wp}-\sum_{k=2}^{\infty} a_k\ z^{k+p+\wp},$$

then

$$F_\iota(z)=(1-\iota)z^{p+\wp}+\iota(p+\wp)\int_0^z\left(\frac{c^{p+\wp}-\sum_{k=2}^{\infty} a_k\ c^{k+p+\wp}}{c}\right)ds$$

$$=(1-\iota)z^{p+\wp}+\iota(p+\wp)\left[\frac{z^{p+\wp}}{(p+\wp)}-\sum_{k=2}^{\infty}\frac{a_k z^{k+p+\wp}}{k+p+\wp}\right]$$

$$=z^{p+\wp}-\sum_{k=2}^{\infty}\frac{\iota(p+\wp)}{k+p+\wp}a_k\ z^{k+p+\wp}$$

$$=z^{p+\eta}-\sum_{k=2}^{\infty} O_k\ z^{k+p+\wp}$$

Where $O_k=\frac{\iota(p+\wp)}{k+p+\wp}a_k$. But

$$\sum_{k=2}^{\infty} \frac{3\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)\big)(\beta\mu+1)} g_k$$

$$=\sum_{k=2}^{\infty} \frac{2\mu\beta(p^2+\eta^2+2\eta p-p-\eta)(p+\eta-2)}{\big((k+p+\eta)(k+p+\eta-1)(k+p+\eta-2)\big)(\beta\mu+1)(} \frac{\iota(p+\eta)}{k+p+\eta} a_k$$

$$\leq \sum_{k=2}^{\infty} \frac{3\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)\big)(\beta\mu+1)} \frac{\iota(p+\delta)}{2+p+\wp} a_k$$

where

$$\frac{\iota(p+\eta)}{2+p+\eta} \leq 1$$

$$\leq \sum_{k=2}^{\infty} \frac{\mu\beta(p^2+\wp^2+2\wp p-p-\wp)(p+\wp-2)}{\big((k+p+\wp)(k+p+\wp-1)(k+p+\wp-2)\big)(\beta\mu+1)} a_k$$

(by (4))

$$\leq\ 2\mu\beta(p^2+\delta^2+2\delta p-p-\delta)(p+\delta-2).$$

$F_\iota(z) \in \acute{L}(p,\wp)$,

So the proof is complete.

Conclusion: We obtain the properties theorem of coefficient inequality, closure theorem ,weighted mean and integral operator.

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