Classical and Bayes Estimators for Exponential Distribution With Comparison of Different Priors & the El-Sayyad's loss function

المقدرات الكلاسيكية والبيزية للتوزيع الاسي مع مقارنه لدوال اولية مختلفة باستعمال دالة خسارة El-Sayyad

Assistant prof. Dr. Jinan Abbas Al-obedy
Technical College of Management-Baghdad

الخلاصة

في هذا البحث ،استعملنا طرائق مختلفة لتقدير معلمة القياس للتوزيع الاسي كمقدر الإمكان الأعظم ومقدر العزوم ومقدر بيز في ستة انواع مختلفة عندما يكون التوزيع الاولي لمعلمة القياس : توزيع لاف (Levy) وتوزيع كامبل من النوع الثاني وتوزيع معكوس مربع كاي وتوزيع معكوس كاما وتوزيع غير الملائم (Improper) وتوزيع Non−informative وفقا لدالة خسارة EL-Sayyad. استعمل اسلوب المحاكاة في مقارنة اداء كل مقدر، بافتراض عدة حالات لمعلمة القياس للتوزيع الاسي استعملت لتوليد البيانات ولاحجام مختلفة من العينات (صغيرة، متوسطة، كبيرة). وقد اظهرت نتائج المحاكاة بان طريقة بيز الافضل عندما يكون التوزيع الاولي لـ التوزيع غير الملائم (Improper) عند قيم معلمتي التوزيع الاولي, (a=9, b=1) ولقيم معلمتي دالة خسارة EL-Sayyad عندما تكون القيمة الحقيقة لوتوزيع Non-informative عند قيمة معلمة التوزيع الاولي, (c=8) ولقيم معلمتي دالة خسارة EL−Sayyad عندما تكون القيمة الحقيقة لـ.وكذلك توزيع Non-informative عند قيمة معلمة التوزيع الاولي, (c=8) ولقيم معلمتي دالة خسارة EL-Sayyad عندما تكون القيمة الحقيقة لـ ، وفقا لمقياس اقل قيمة متوسط مربع الاخطاء (MSE) لكل احجام العينات ، مقارنة بنفس القيم المستحصلة بطريقتي MLE . وME

مفاتيح الكلمات : التوزيع الاسي، طريقة الإمكان الأعظم،طريقة العزوم ،طريقة بيز، التوزيعات الأولية : توزيع لافي, (Levy) توزيع كامبل من النوع الثاني ،توزيع معكوس مربع كاي ،توزيع معكوس كاما ، توزيع غير الملائم ,(Improper) توزيع .Non-informative دالة , EL-Sayyad متوسط مربعات الاخطاء(MSE) .

Abstract

In this study, different estimators were used for estimating scale parameter for Exponential distribution, such as maximum likelihood estimator, moment estimator and the Bayes estimator, in six types when the prior distribution for the scale parameter is: Levy distribution, Gumbel type-II distribution, Inverse Chi-square distribution, Inverted Gamma distribution, improper distribution, Non-informative distribution. Under El-Sayyad's loss function .we used simulation technique, to compare the performance for each estimator, several cases from Exponential distribution for data generating, for different sample sizes (small, medium, and large). Simulation results shown that The best method is the bayes estimation ,when the prior distribution for θ is improper distribution with (a=9, b=1) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \& r = 7$),when the true value of $\theta(\theta = 0.5)$.And the non-informative distribution with (c=8) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 1 \& r = 5$), when the true value of $\theta(\theta = 1)$.Also the non-informative distribution with (c=8) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \& r = 9$),when the true value of $\theta(\theta = 1.5)$, according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using Maximum likelihood estimation method (MLE) and Moment estimation method (ME).

Key words: The Exponential, Maximum likelihood estimation, Moment estimation, Bayes method, the prior distributions: Levy distribution, the Gumbel type-II distribution, Inverse Chi-square distribution, Inverted Gamma distribution, improper distribution, non-informative distribution, El-Sayyad's loss function, mean squared errors (MSE).

1. Introduction

The difference between Maximum Likelihood estimation and Bayesian estimation is that in maximum likelihood estimation the parameters are not random variables. In Bayesian analysis the unknown parameter is regarded as being the value of a random variable from a given probability distribution, with the knowledge of some information about its value prior to observing the data $x_1, x_2 \ldots x_n$ (Ross, 2009); we mention some of studies in a brief manner:

In (1967) El-Sayyad [6] introduced some new estimators, which are unbiased with respect to some loss functions, are derived for the parameter in the exponential distribution. The corresponding bayes estimators are also obtained. The comparison between these two kinds of estimation is discussed. In (1998) Rossman, Short, and Parks [9] studied the relationship between Bayesian and classical estimation using the continuous uniform distribution.

In (2001) Elfessi and Reineke [5] show how the classical estimators can be obtained from various choices made within a Bayesian framework .by using some of the relationships for the exponential distribution. In (2005) Ali and Woo and Nadarajah [3] derived bayes estimators under a symmetric squared error loss function as well as an asymmetric loss function, for the parameter of the standard exponential distribution. In (2007) Abu-Taleb and Smadi and Alawneh [1] derive bayes estimates assuming the inverted gamma prior along with the Bayesian credible intervals, for the exponential random censor time. In (2009) Al_Kutubi and Ibrahim [2] used Jeffery prior information to get the modify bayes estimator and then compared it with standard Bayes estimator and maximum likelihood estimator to find the best (less MSE and MPE). Simulation study was used to compare between estimators and Mean Square Error (MSE) and Mean Percentage Error (MPE) of estimators are computed. In (2010) Tahir and Aslam [10] compared Bayesian and classical analysis for parameter of the exponential model for time-to-failure data. Their comparison is based upon the posterior variance, the Bayesian point and interval estimates, the coefficients of skewness of the posterior distribution and the posterior predictive distribution. In (2013) Yang and Zhou and Yuan [14] studied the bayes estimation of parameter of exponential distribution under a bounded loss function, named reflected gamma loss function, which proposed by Towhidi and Behboodian (1999). They used the inverse Gamma prior distribution as the prior distribution of the parameter of exponential distribution. Bayesian estimators are obtained under squared error loss and the reflected gamma loss functions.

So in this paper, we try to find best method to estimate parameter of exponential distribution. According to the smallest value of Mean Square Errors (MSE) were calculated to compare the methods of estimation. We used the maximum likelihood estimator, the moment estimator and the bayes estimator in six types of priors, and then get bayes estimation: Levy distribution, Gumbel type-II distribution, Inverse Chi-square distribution, Inverted Gamma distribution, Improper distribution, Non-informative distribution when the Bayesian estimation based on El-Sayyad's loss function. Several cases from exponential distribution for data generating , for different samples sizes (small, medium, and large) .The results were obtained by using simulation technique, Programs written using MATLAB-R2008a program were used.

2. Exponential Distribution

We consider $x_1, x_2, \ldots, x_n$ is a random sample of n independent observations from an Exponential distribution having the probability density function (pdf) define as [4, 5]:

$$f(x;\theta) = \theta^{-1} \exp(-\frac{x}{\theta}) \quad , \quad x > 0 \qquad \ldots (1)$$

where $\theta > 0$ is mean, standard deviation, and scale parameter of the distribution, θ is a survival parameter in the sense that if a random variable x is the duration of time that a given biological or mechanical system manages to survive and $x \sim Exp(\theta)$ then $E[x] = \theta$. That is to say, the expected duration of survival of the system is θ units of time.

3. Parameter Estimation Methods

In this section, we used several methods to estimation parameter θ.

3.1 Maximum likelihood Estimation

From the Exponential pdf given in (1) the likelihood function will be as follows [4]:

$$L(\underline{x} \setminus \theta) = \prod_{i=1}^{n} f(x;\theta) = \theta^{-n} \exp(-\frac{\sum_{i=1}^{n} x_i}{\theta}) \qquad \ldots (2)$$

By taking the log and differentiating partially with respect to b, we get:

$$\frac{\partial}{\partial\theta} \log L(\underline{x} \setminus \theta) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \qquad \ldots (3)$$

Then the MLE of θ is the solution of equation (2) after equating the first derivative to zero, Hence:

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \qquad \ldots (4)$$

3.2. Moments estimation (ME)

The method of moments is another technique commonly used in the field of estimation of parameters. If $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size (n) represent a set of data, then an unbiased estimator for the rth origin moment is [4]:

$$m_r = \frac{\sum_{i=1}^{n} x_i^r}{n} \qquad \ldots (5)$$

Where m_r stands for the rth sample moment. The first moment of the Exponential distribution as:

$$M_1 = E(x) = \frac{1}{(1/\theta)} = \theta \qquad \ldots (6)$$

Therefore by equating sample and population moments we get

$$m_1 = M_1 = E(x) = \frac{1}{(1/\theta)} = \theta \qquad \ldots (7)$$

From (7) we get $\bar{x} = \theta \quad \Rightarrow \quad \hat{\theta}_{MM} = \bar{x}$ ... (8)

3.3 Bayes Estimation Method

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size n with probability density function given in equation (1) and likelihood function given in equation (2).In this paper the posterior distributions for the unknown parameter θ are derived using the following six types of priors ,and then get bayes estimation [4]:

1. Levy distribution.
2. Gumbel type-II distribution [11].
3. Inverse Chi-square distribution [13].
4. Inverted Gamma distribution [12].
5. Improper distribution.
6. Non-informative distribution.

3.3.1 The posterior distribution using different Priors

It is assumed that θ follows six types of prior distributions with pdf as given in table below:

Table -1: The six types of prior distributions ($P(\theta)$) with pdf for θ.

Prior distribution	$P(\theta)$
$\theta \sim$ Levy(b_3)	$P(\theta)\ \alpha\ \sqrt{\frac{b_3}{2\pi}}\ \theta^{-\frac{3}{2}}\ \exp(-\frac{b_3}{2\theta})$ for $b_3,\ \theta > 0$
$\theta \sim$ Gumbel type-II(b)	$P(\theta)\ \alpha\ b\ \theta^{-2}\ \exp(-\frac{b}{\theta})$ for $b,\ \theta > 0$
$\theta \sim$ Inverse Chi-square(v)	$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp(-\frac{1}{2\theta})$ for $v,\ \theta > 0$
$\theta \sim$ Inverted Gamma(α, β)	$P(\theta)\ \alpha\ \frac{\beta^{\alpha}}{\Gamma\alpha}\ \theta^{-(\alpha+1)}\ \exp(-\frac{\beta}{\theta})$ for $\alpha, \beta,\ \theta > 0$
$\theta \sim$ Improper(a, b)	$P(\theta)\ \alpha\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})$ for $b,\ \theta > 0$ and $-\infty < a < \infty$
$\theta \sim$ Non-informative(c)	$P(\theta)\ \alpha\ \frac{1}{\theta^{c}}$ for $\theta, c > 0$

Then the posterior distribution of given the data $\underline{x} = (x_1, x_2, \ldots, x_n)$ is [4]:

$$P(\theta \backslash x) = \frac{L(\underline{x} \backslash \theta)\ P(\theta)}{\int_{\theta} L(\underline{x} \backslash \theta)\ P(\theta) d\theta} \qquad \ldots(9)$$

Substituting the equation (2) and for each $P(\theta)$ as shown in table -1 in equation (9), we get the posterior distributions for the unknown parameter θ are derived using the following six types of priors (for more details see Appendix-A).

Table -2: The posterior distributions ($P(\theta \backslash x)$) for the unknown parameter (θ) are derived using the following six types of priors.

Prior dist^{n}.	The posterior distribution ($P(\theta \backslash x)$)
Levy	$P_1(\theta \backslash x) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+\frac{1}{2}), \beta_{(new)} = (\sum_{i-1}^{n} x_i + \frac{b_3}{2})$) with pdf $P_1(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})} \theta^{-[(n+\frac{1}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_3}{2}))$ $n,\ b_3,\ \theta > 0$
Gumbel type-II	$P_2(\theta \backslash x) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+1), \beta_{(new)} = (\sum_{i-1}^{n} x_i + b)$) with pdf $P_2(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + b)^{(n+1)} \theta^{-[(n+1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + b))}{\Gamma(n+1)}$ $n,\ b,\ \theta > 0$
Inverse Chi-square	$P_3(\theta \backslash x)) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+\frac{v}{2}), \beta_{(new)} = (\sum_{i-1}^{n} x_i + \frac{1}{2})$) with pdf $P_3(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{1}{2}))$ $n,\ v,\ \theta > 0$
Inverted Gamma	$P_4(\theta \backslash x) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+\alpha), \beta_{(new)} = (\sum_{i-1}^{n} x_i + \beta)$) with pdf $P_4(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \beta))}{\Gamma(n+\alpha)}$ $n, \beta, \alpha, \theta > 0$
Improper	$P_5(\theta \backslash x) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+a), \beta_{(new)} = (\sum_{i-1}^{n} x_i + b)$) with pdf $P_5(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + b)^{(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + b))}{\Gamma(n+a)}$ $n,\ b,\ \theta > 0$ and $-\infty < a < \infty$
Non-informative	$P_6(\theta \backslash x) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+c-1), \beta_{(new)} = (\sum_{i-1}^{n} x_i)$) with pdf $P_6(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)} \theta^{-[(n+c-1)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\Gamma(n+c-1)}$ $n, c,\ \theta > 0$

3.3.2 Bayes' Estimators

Bayes' estimators for the scale parameter θ, was considered with six different priors and under El-Sayyad 's loss function[6,7].The El-Sayyad's loss function is $L(\hat{\theta},\theta)=\theta^{\ell}(\hat{\theta}^{r}-\theta^{r})^{2}$. Where $\hat{\theta}$ is an estimator for is θ, was considered with six different priors, and under El-Sayyad's loss function. Following is the derivation of these estimators:

3.3.2.1 The El-Sayyad's loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L(\hat{\theta},\theta)=\theta^{\ell}(\hat{\theta}^{r}-\theta^{r})^{2} \qquad \ldots (10)$$

After simplified steps, we get Bayes estimator of θ denoted by $\hat{\theta}_{ES}$ for the above prior as follows

$$\hat{\theta}_{ES}=\left[\frac{E(\theta^{r}\backslash x)}{E(\theta^{\ell}\backslash x)}\right]^{\frac{1}{r}}=\left[\frac{\int_0^\infty \theta^{r}P(\theta\backslash x)\,d\theta}{\int_0^\infty \theta^{\ell}P(\theta\backslash x)\,d\theta}\right]^{\frac{1}{r}} \qquad \ldots (11)$$

So, the following results are the derivations of these estimators under the El-Sayyad's loss function with six prior distributions (more details see Appendix-B).

Table -3: The estimators ($\hat{\theta}_{ES}$) under the El-Fayyad's loss function with six different priors.

Prior distribution	$\hat{\theta}_{ES}=\left[\frac{E(\theta^{r}\backslash x)}{E(\theta^{\ell}\backslash x)}\right]^{\frac{1}{r}}=\left[\frac{\int_0^\infty \theta^{r}P(\theta\backslash x)\,d\theta}{\int_0^\infty \theta^{\ell}P(\theta\backslash x)\,d\theta}\right]^{\frac{1}{r}}$
Levy	$\hat{\theta}_{SE1}=\left[\frac{\Gamma(n+\frac{1}{2}-r)}{\Gamma(n+\frac{1}{2}-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{1-\frac{\ell}{r}} \quad \ell, r, n\ \&\ b_3>0$
Gumbel type-II	$\hat{\theta}_{SE2}=\left[\frac{\Gamma(n+1-r)}{\Gamma(n+1-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+b)^{1-\frac{\ell}{r}} \quad \ell, r, n\ \&\ b>0$
Inverse Chi-square	$\hat{\theta}_{SE3}=\left[\frac{\Gamma(n+\frac{v}{2}-r)}{\Gamma(n+\frac{v}{2}-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+\frac{1}{2})^{1-\frac{\ell}{r}} \quad \ell, r, n\ \&\ v>0$
Inverted Gamma	$\hat{\theta}_{ES4}=\left[\frac{\Gamma(n+\alpha-r)}{\Gamma(n+\alpha-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+\beta)^{1-\frac{\ell}{r}} \quad \ell, r, n, \beta, \alpha>0$
Improper	$\hat{\theta}_{ES5}=\left[\frac{\Gamma(n+a-r)}{\Gamma(n+a-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+b)^{1-\frac{\ell}{r}} \quad \ell, r, n, a, b>0$
Non-informative	$\hat{\theta}_{ES6}=\left[\frac{\Gamma(n+c-1-r)}{\Gamma(n+c-1-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i)^{1-\frac{\ell}{r}} \quad \ell, r, n, c>0$

4. Simulation Study

In this study, we have generated random samples from Exponential distribution and compared the performance of MLE and MME and Bayes estimator based on them. So we have considered several steps to perform simulation study as follow, for the first stage:

1. We have chosen sample size n = 10, 25, 50 and 100 to represent small, moderate and large sample size.
2. We generated data from Exponential distribution for the scale parameter; we have considered the value for the parameter of Exponential distribution is $\theta = 0.5$.
3. We used three values ($b_3 = 0.2, 0.5, 1$) for the parameters of the Levy distribution as prior distribution for θ.
4. We used three values ($b = 0.04, 0.08, 0.1$) for the parameters of the Gumbel type-II distribution as prior distribution for θ.
5. We used three values ($v = 12, 14, 16$) for the parameters of the Inverse Chi-square distribution as prior distribution for θ.
6. We used the values ($\alpha = 8, 10$ & $\beta = 0.5, 1$) for the parameters of the Inverted Gamma distribution as prior distribution for θ.
7. We used the values ($a = 7, 9$ & $b = 1, 1.5$) for the parameters of the Improper distribution as prior distribution for θ.
8. We used three values $c = 6, 8, 10$ for the function of the non-informative prior distribution.

Then we have considered several steps to perform simulation study as follow, for the second stage:

1. We have chosen sample size n = 10, 25, 50 and 100 to represent small, moderate and large sample size.
2. We generated data from Exponential distribution for the scale parameter; we have considered randomly two values for the parameter of exponential distribution $\theta = 1, 1.5$.
3. We used the value ($b_3 = 0.2$) for the parameters of the Levy distribution as prior distribution for θ.
4. We used the value ($b = 0.04$) for the parameters of the Gumbel type-II distribution as prior distribution for θ.
5. We used the value ($v = 16$) for the parameters of the Inverse Chi-square distribution as prior distribution for θ.
6. We used the value ($\alpha = 10$ & $\beta = 0.5$) for the parameters of the Inverted Gamma distribution as prior distribution for θ.
7. We used the value ($a = 9$ & $b = 1$) for the parameters of the Improper distribution as prior distribution for θ.
8. We used the value ($c = 8$) for the function of the non-informative prior distribution.

We have considered the two steps to perform simulation study for the first and the second stage as follow:

1. We used the values ($\ell = 0.5, 1 \;\&\; r = 5, 7, 9$) for the parameters of these estimators under the El-Sayyad's loss function with six prior distributions, which are listed in table -3.
2. The number of replication used was ($r = 1000$) for each sample size (n).
   We obtained estimators for scale parameter from equations (4), (8) and also the estimators in table -3; it means the estimators $(\hat{\theta}_{ES})$ under the El-Sayyad's loss function with six different priors. The simulation program was written by using MATLAB-R2008a program. After the parameter θ was estimated, Mean Square Errors (MSE) was calculated to compare the methods of estimation, where:

$$MSE = \frac{1}{r}\sum_{r=1}^{1000} \;(\hat{\theta}_{ES}(r) - \theta)^2] \qquad ...(12)$$

See appendix-C, for the programs algorithm. The results of the simulation study are summarized and tabulated in tables (4.1) for the first stage. In each row of table (4.1) ,we have four estimated values for θ $(\hat{\theta})$ with MSE for all samples sizes (n) and values (b_3 , b, v, α , β , a , b, c) respectively as considered in first stage. Also the results of the simulation study are summarized tabulated in tables (4.2-4.3) for the second stage .In each row of tables (4.2-4.3) ,we have four estimated values for θ $(\hat{\theta})$ with MSE for all samples sizes (n) and values (b_3,b,v, α , β , a, b, c,) respectively as considered in second stage.

By using different estimation methods that is maximum likelihood estimator and the moment estimator .And the Bayes estimators in six types of prior distribution .So our criteria is the best method that gives the smallest value of (MSE). We list the results in the following tables (4.1 -4.3).

Table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ $(\hat{\theta})$				MSE			
						Sample Size(n)				Sample Size(n)			
MLE	θ	-	-	-	-	10	25	50	100	10	25	50	100
	0.5					0.4918	0.5030	0.5016	0.5000	0.0224	0.0102	0.0051	0.0026
ME	0.5					0.4918	0.5030	0.5016	0.5000	0.0224	0.0102	0.0051	0.0026
Bayes	θ	b_3	-	ℓ	r	Levy distribution $(P_1(\theta \backslash x))$							
	0.5	0.2	-	0.5	5	0.7254	0.6024	0.5672	0.5503	0.0889	0.0222	0.0098	0.0051
					7	0.8501	0.6203	0.5687	0.5453	0.1782	0.0277	0.0103	0.0047
					9	1.089	0.6462	0.5754	0.5451	0.4413	0.0362	0.0116	0.0048
				1	5	0.7687	0.6419	0.6061	0.5890	0.1062	0.0306	0.0160	0.0102
					7	0.8858	0.6490	0.5962	0.5724	0.2005	0.0345	0.0145	0.0077
					9	1.1244	0.6693	0.597	0.5661	0.4791	0.0428	0.0151	0.0070
		0.5	-	0.5	5	0.7450	0.6089	0.5703	0.5518	0.0979	0.0235	0.0102	0.0052
					7	0.8738	0.6271	0.5718	0.5468	0.1951	0.0293	0.0108	0.0048
					9	1.1199	0.6535	0.5786	0.5467	0.4783	0.0384	0.0121	0.0049
				1	5	0.7874	0.6481	0.6090	0.5904	0.1161	0.0324	0.0166	0.0105
					7	0.9088	0.6556	0.5993	0.5739	0.2183	0.0365	0.0151	0.0079
					9	1.1545	0.6764	0.6001	0.5676	0.5171	0.0452	0.0157	0.0072
		1	-	0.5	5	0.7776	0.6196	0.5753	0.5543	0.1146	0.0259	0.0109	0.0055
					7	0.9132	0.6385	0.5771	0.5494	0.2258	0.0323	0.0115	0.0051
					9	1.1712	0.6655	0.5840	0.5493	0.5441	0.0422	0.0130	0.0052
				1	5	0.8182	0.6582	0.6139	0.5928	0.1342	0.0354	0.0177	0.0109
					7	0.9468	0.6666	0.6044	0.5763	0.2501	0.0400	0.0161	0.0083
					9	1.2045	0.6882	0.6054	0.5701	0.5840	0.0494	0.0168	0.0075

Continue for table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ ($\hat{\theta}$)				MSE			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	b	-	ℓ	r	Gumbel type-II distribution ($P_2(\theta \backslash x)$)							
	0.5	0.04	-	0.5	5	0.6745	0.5879	0.5606	0.5472	0.0642	0.0190	0.0088	0.0047
					7	0.7773	0.6043	0.5617	0.5421	0.1245	0.0235	0.0093	0.0044
					9	0.9615	0.6285	0.5681	0.5418	0.2883	0.0307	0.0105	0.0045
				1	5	0.7193	0.6281	0.5999	0.5860	0.0786	0.0265	0.0146	0.0097
					7	0.8137	0.6333	0.5895	0.5693	0.1430	0.0296	0.0132	0.0073
					9	0.9962	0.6518	0.5898	0.5629	0.3181	0.0366	0.0136	0.0065
		0.08	-	0.5	5	0.6794	0.5896	0.5615	0.5476	0.0659	0.0193	0.0089	0.0048
					7	0.7832	0.6060	0.5626	0.5425	0.0659	0.0239	0.0094	0.0044
					9	0.9688	0.6304	0.5690	0.5423	0.2951	0.0312	0.0106	0.0045
				1	5	0.7241	0.6297	0.6006	0.5864	0.0806	0.0269	0.0148	0.0097
					7	0.8194	0.6351	0.5903	0.5697	0.1465	0.0301	0.0133	0.0073
					9	1.0034	0.6537	0.5907	0.5633	0.3252	0.0371	0.0138	0.0066
		0.1	-	0.5	5	0.6819	0.5905	0.5619	0.5478	0.0667	0.0194	0.0090	0.0048
					7	0.7861	0.6069	0.5630	0.5427	0.1294	0.0241	0.0095	0.0044
					9	0.9725	0.6313	0.5694	0.5425	0.2985	0.0314	0.0107	0.0045
				1	5	0.7264	0.6305	0.6010	0.5866	0.0816	0.0271	0.0149	0.0098
					7	0.8222	0.6359	0.5907	0.5699	0.1483	0.0303	0.0134	0.0073
					9	1.007	0.6546	0.5911	0.5635	0.3287	0.0374	0.0139	0.0066

Continue for table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ ($\hat{\theta}$)				MSE			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	v	-	ℓ	r	Inverse Chi-square distribution ($P_3(\theta \backslash x)$)							
	0.5	12	-	0.5	5	0.4634	0.5076	0.5210	0.5275	0.0146	0.0078	0.0047	0.0030
					7	0.4919	0.5149	0.5198	0.5217	0.0160	0.0088	0.0049	0.0028
					9	0.536	0.5290	0.5239	0.5209	0.0208	0.0102	0.0053	0.0029
				1	5	0.5108	0.5504	0.5618	0.5672	0.0129	0.0098	0.0078	0.0066
					7	0.5272	0.5455	0.5485	0.5495	0.0164	0.0102	0.0067	0.0047
					9	0.5656	0.5532	0.5463	0.5423	0.0236	0.0119	0.0067	0.0041
		14	-	0.5	5	0.4327	0.4917	0.5123	0.5229	0.0161	0.0074	0.0043	0.0041
					7	0.4551	0.4976	0.5106	0.5170	0.0156	0.0080	0.0045	0.0026
					9	0.4906	0.5101	0.5143	0.5160	0.0165	0.0088	0.0048	0.0027
				1	5	0.4801	0.5349	0.5534	0.5628	0.0117	0.0080	0.0067	0.0060
					7	0.4901	0.5284	0.5395	0.5449	0.0136	0.0085	0.0057	0.0042
					9	0.5197	0.5345	0.5368	0.5375	0.0167	0.0096	0.0058	0.0037
		16	-	0.5	5	0.406	0.4767	0.5038	0.5184	0.0190	0.0074	0.0040	0.0025
					7	0.4237	0.4814	0.5018	0.5124	0.0176	0.0078	0.0042	0.0024
					9	0.4527	0.4926	0.5051	0.5113	0.0162	0.0081	0.0045	0.0025
				1	5	0.4532	0.5203	0.5453	0.5585	0.0122	0.0069	0.0057	0.0054
					7	0.4583	0.5124	0.5309	0.5403	0.0135	0.0073	0.0050	0.0038
					9	0.4812	0.5170	0.5277	0.5329	0.0143	0.0082	0.0051	0.0034

Continue for table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ ($\hat{\theta}$)				MSE			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	α	β	ℓ	r	Inverted Gamma distribution $(P_4(\theta \backslash x))$							
	0.5	8	0.5	0.5	5	0.406	0.4767	0.5038	0.5184	0.0190	0.0074	0.0040	0.0025
					7	0.4237	0.4814	0.5018	0.5124	0.0176	0.0078	0.0042	0.0024
					9	0.4527	0.4926	0.5051	0.5113	0.0162	0.0081	0.0045	0.0025
				1	5	0.4532	0.5203	0.5453	0.5585	0.0122	0.0069	0.0057	0.0054
					7	0.4583	0.5124	0.5309	0.5403	0.0135	0.0073	0.0050	0.0038
					9	0.4812	0.5170	0.5277	0.5329	0.0143	0.0082	0.0051	0.0034
		8	1	0.5	5	0.4398	0.4932	0.5127	0.5230	0.0136	0.0068	0.0041	0.0027
					7	0.4601	0.4986	0.5109	0.5171	0.0132	0.0074	0.0043	0.0026
					9	0.4922	0.5104	0.5144	0.5161	0.0139	0.0081	0.0046	0.0026
				1	5	0.4869	0.5363	0.5538	0.5629	0.0099	0.0077	0.0066	0.0060
					7	0.4947	0.5292	0.5398	0.5449	0.0115	0.0080	0.0056	0.0042
					9	0.5208	0.5347	0.5369	0.5376	0.0141	0.0090	0.0056	0.0037
		10	0.5	0.5	5	0.3618	0.4496	0.4878	0.5097	0.0272	0.0086	0.0039	0.0022
					7	0.3727	0.4522	0.4850	0.5034	0.0253	0.0088	0.0042	0.0022
					9	0.3927	0.4610	0.4876	0.5021	0.0220	0.0086	0.0043	0.0023
				1	5	0.4085	0.4936	0.5297	0.5501	0.0165	0.0058	0.0044	0.0045
					7	0.4064	0.4834	0.5144	0.5315	0.0180	0.0067	0.0040	0.0031
					9	0.4200	0.4856	0.5104	0.5238	0.0170	0.0071	0.0041	0.0028
		10	1	0.5	5	0.3919	0.4651	0.4964	0.5142	0.0196	0.0073	0.0037	0.0023
					7	0.4047	0.4683	0.4938	0.5080	0.0181	0.0075	0.0040	0.0023
					9	0.4270	0.4777	0.4966	0.5068	0.0157	0.0075	0.0041	0.0023
				1	5	0.4388	0.5088	0.5380	0.5544	0.0116	0.0058	0.0049	0.0049
					7	0.4387	0.4993	0.5231	0.5361	0.0128	0.0063	0.0043	0.0034
					9	0.4546	0.5021	0.5193	0.5284	0.0125	0.0069	0.0044	0.0030

Continue for table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ ($\hat{\theta}$)				MSE			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	a	b	ℓ	r	Improper distribution $(P_5(\theta \backslash x))$							
	0.5	7	1	0.5	5	0.4687	0.5086	0.5213	0.5276	0.0123	0.0073	0.0046	0.0030
					7	0.4942	0.5153	0.5199	0.5217	0.0135	0.0082	0.0048	0.0028
					9	0.5335	0.5286	0.5238	0.5209	0.0173	0.0094	0.0052	0.0028
				1	5	0.5157	0.5513	0.5621	0.5673	0.0111	0.0093	0.0076	0.0066
					7	0.5290	0.5458	0.5486	0.5495	0.0140	0.0097	0.0065	0.0046
					9	0.5625	0.5527	0.5462	0.5423	0.0199	0.0111	0.0065	0.0041
		7	1.5	0.5	5	0.5044	0.5255	0.5303	0.5322	0.0112	0.0078	0.0050	0.0033
					7	0.5330	0.5329	0.5292	0.5265	0.0144	0.0090	0.0052	0.0030
					9	0.5761	0.5470	0.5333	0.5257	0.0219	0.0108	0.0057	0.0031
				1	5	0.5507	0.5676	0.5707	0.5717	0.0131	0.0112	0.0088	0.0072
					7	0.5674	0.5630	0.5576	0.5541	0.0174	0.0115	0.0074	0.0051
					9	0.6048	0.5708	0.5555	0.5470	0.0267	0.0133	0.0075	0.0045
		9	1	0.5	5	0.4144	0.4787	0.5044	0.5186	0.0162	0.0069	0.0039	0.0025
					7	0.4305	0.4829	0.5022	0.5125	0.0150	0.0072	0.0041	0.0024
					9	0.4572	0.4935	0.5053	0.5114	0.0137	0.0076	0.0043	0.0024
				1	5	0.4615	0.5221	0.5458	0.5586	0.0102	0.0065	0.0057	0.0054
					7	0.4648	0.5138	0.5313	0.5405	0.0114	0.0069	0.0049	0.0038
					9	0.4852	0.5178	0.5279	0.5329	0.0121	0.0076	0.0049	0.0033
		9	1.5	0.5	5	0.446	0.4946	0.5131	0.5232	0.0116	0.0064	0.0040	0.0027
					7	0.4644	0.4995	0.5112	0.5172	0.0113	0.0069	0.0042	0.0025
					9	0.4937	0.5107	0.5145	0.5161	0.0118	0.0076	0.0045	0.0026
				1	5	0.4928	0.5376	0.5542	0.5630	0.0085	0.0073	0.0065	0.0059
					7	0.4986	0.5300	0.5400	0.5450	0.0099	0.0075	0.0055	0.0041
					9	0.5218	0.5348	0.5369	0.5376	0.0121	0.0085	0.0055	0.0036

Continue for table 4.1: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ $(\hat{\theta})$				MSE			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	c	-	ℓ	r	Non-informative distribution $(P_6(\theta \setminus x))$							
	0.5	6	-	0.5	5	0.4571	0.5065	0.5207	0.5274	0.0175	0.0084	0.0049	0.0031
					7	0.4892	0.5146	0.5197	0.5217	0.0193	0.0094	0.0051	0.0029
					9	0.5393	0.5294	0.5240	0.5209	0.0256	0.0110	0.0055	0.0029
				1	5	0.5050	0.5494	0.5616	0.5671	0.0153	0.0102	0.0079	0.0066
					7	0.5251	0.5452	0.5484	0.5494	0.0195	0.0109	0.0068	0.0047
					9	0.5697	0.5538	0.5464	0.5423	0.0287	0.0127	0.0069	0.0042
		8	-	0.5	5	0.3962	0.4746	0.5032	0.5182	0.0226	0.0080	0.0042	0.0026
					7	0.4157	0.4799	0.5013	0.5122	0.0209	0.0084	0.0044	0.0025
					9	0.4476	0.4916	0.5048	0.5112	0.0193	0.0088	0.0046	0.0025
				1	5	0.4437	0.5183	0.5447	0.5583	0.0149	0.0073	0.0058	0.0055
					7	0.4506	0.5109	0.5305	0.5402	0.0163	0.0078	0.0051	0.0038
					9	0.4764	0.5162	0.5275	0.5328	0.0172	0.0088	0.0052	0.0034
		10	-	0.5	5	0.3503	0.4467	0.4869	0.5094	0.0316	0.0093	0.0041	0.0023
					7	0.3622	0.4497	0.4843	0.5031	0.0295	0.0095	0.0043	0.0023
					9	0.3836	0.4590	0.4870	0.5019	0.0257	0.0093	0.0045	0.0023
				1	5	0.3970	0.4909	0.5289	0.5499	0.0200	0.0063	0.0045	0.0045
					7	0.3959	0.4810	0.5137	0.5313	0.0215	0.0072	0.0041	0.0031
					9	0.4110	0.4836	0.5099	0.5236	0.0203	0.0077	0.0043	0.0028

Table 4.2: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ $(\hat{\theta})$				MSE			
						Sample Size(n)				Sample Size(n)			
MLE	θ	-	-	-	-	10	25	50	100	10	25	50	100
	1					1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
ME	1					1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
Bayes	θ	b_3	-	ℓ	r	Levy distribution $(P_1(\theta \setminus x))$							
	1	0.2	-	0.5	5	1.3698	1.1216	1.0488	1.0245	0.2857	0.0547	0.0209	0.0091
					7	1.6383	1.1779	1.0722	1.0354	0.6344	0.0784	0.0258	0.0105
					9	2.1224	1.2406	1.0967	1.0465	1.6542	0.1116	0.0317	0.0120
				1	5	1.3524	1.1154	1.0468	1.0235	0.2384	0.0446	0.0167	0.0072
					7	1.6228	1.1731	1.0706	1.0346	0.5772	0.0696	0.0225	0.0091
					9	2.1066	1.2366	1.0954	1.0459	1.5679	0.1033	0.0288	0.0108
Bayes	θ	b	-	ℓ	r	Gumbel type-II distribution $(P_2(\theta \setminus x))$							
	1	0.04	-	0.5	5	1.2808	1.097	1.0378	1.0193	0.2106	0.0478	0.0196	0.0088
					7	1.5065	1.15	1.0603	1.0299	0.4509	0.0673	0.0238	0.0101
					9	1.8848	1.2092	1.0841	1.0408	1.0977	0.0951	0.0289	0.0114
				1	5	1.2718	1.0934	1.0369	1.0188	0.1762	0.0389	0.0157	0.0070
					7	1.4986	1.1471	1.0597	1.0295	0.4119	0.0597	0.0207	0.0087
					9	1.8768	1.2069	1.0836	1.0405	1.0446	0.0881	0.0263	0.0102
Bayes	θ	v	-	ℓ	r	Inverse Chi-square distribution $(P_3(\theta \setminus x))$							
	1	16	-	0.5	5	0.7406	0.8753	0.9251	0.9617	0.1075	0.0391	0.0198	0.0089
					7	0.7879	0.9012	0.9393	0.9693	0.0935	0.0363	0.0192	0.0090
					9	0.8510	0.9320	0.9556	0.9778	0.0808	0.0340	0.0186	0.0090
				1	5	0.7731	0.8929	0.9357	0.9673	0.0860	0.0308	0.0156	0.0070
					7	0.8122	0.914	0.9469	0.9733	0.0791	0.0307	0.0163	0.0076
					9	0.8713	0.9422	0.9617	0.9810	0.0709	0.0299	0.0164	0.0079

Continue for table 4.2: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ $(\hat{\theta})$				MES			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	α	β	ℓ	r	Inverted Gamma distribution $(P_4(\theta \setminus x))$							
	1	10	0.5	0.5	5	0.6600	0.8254	0.8957	0.9454	0.1475	0.0514	0.0242	0.0102
					7	0.6931	0.8464	0.9079	0.9522	0.1317	0.0470	0.0230	0.0100
					9	0.7382	0.8723	0.9226	0.9602	0.1127	0.0420	0.0215	0.0097
				1	5	0.6968	0.8472	0.9091	0.9527	0.1199	0.0408	0.0190	0.0080
					7	0.7203	0.8622	0.9176	0.9575	0.1127	0.0397	0.0194	0.0085
					9	0.7605	0.8849	0.9302	0.9643	0.0987	0.0367	0.0188	0.0086
Bayes	θ	a	b	ℓ	r	Improper distribution $(P_5(\theta \setminus x))$							
	1	9	1	0.5	5	0.7277	0.8645	0.9183	0.9578	0.1095	0.0404	0.0204	0.0091
					7	0.7699	0.8887	0.9319	0.9651	0.0952	0.0372	0.0196	0.0091
					9	0.8258	0.9177	0.9476	0.9735	0.0807	0.0341	0.0188	0.0090
				1	5	0.7607	0.8830	0.9296	0.9638	0.0876	0.0319	0.0160	0.0071
					7	0.7945	0.9021	0.9400	0.9694	0.0804	0.0314	0.0166	0.0077
					9	0.8462	0.9284	0.9302	0.9769	0.0703	0.0300	0.0165	0.0079
Bayes	θ	c	-	ℓ	r	Non-informative distribution $(P_6(\theta \setminus x))$							
	1	8	-	0.5	5	0.7553	0.8868	0.9321	0.9656	0.1060	0.0379	0.0193	0.0087
					7	0.8089	0.9145	0.9470	0.9735	0.0929	0.0357	0.0189	0.0089
					9	0.8808	0.9474	0.964	0.9823	0.0835	0.0343	0.0186	0.0089
				1	5	0.7871	0.9035	0.9421	0.9709	0.0848	0.0300	0.0152	0.0069
					7	0.8328	0.9266	0.9541	0.9772	0.0788	0.0303	0.0160	0.0075
					9	0.9009	0.9571	0.9696	0.9852	0.0739	0.0304	0.0164	0.0079

Table 4.3: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ $(\hat{\theta})$				MSE			
						Sample Size(n)				Sample Size(n)			
MLE	θ	-	-	-	-	10	25	50	100	10	25	50	100
	1.5	-	-	-	-	1.5027	1.5091	1.5028	1.4951	0.2229	0.0925	0.0448	0.0227
ME	1.5	-	-	-	-	1.5027	1.5091	1.5028	1.4951	0.2229	0.0925	0.0448	0.0227
Bayes	θ	b_3	-	ℓ	r	Levy distribution $(P_1(\theta \setminus x))$							
	1.5	0.2	-	0.5	5	1.9577	1.6117	1.5191	1.473	0.5111	0.0973	0.0373	0.0185
					7	2.368	1.7121	1.571	1.5059	1.2236	0.1469	0.0472	0.0198
					9	3.0871	1.8149	1.6178	1.5318	3.3459	0.2177	0.0600	0.0222
				1	5	1.8575	1.5395	1.4551	1.4132	0.3421	0.0627	0.0288	0.0204
					7	2.2801	1.6567	1.5237	1.462	0.9796	0.1059	0.0343	0.0173
					9	2.9973	1.769	1.5794	1.497	2.9317	0.1721	0.0453	0.0179
Bayes	θ	b	-	ℓ	r	Gumbel type-II distribution $(P_2(\theta \setminus x))$							
	1.5	0.04	-	0.5	5	1.8337	1.5774	1.5037	1.4657	0.3781	0.0875	0.0363	0.0188
					7	2.1815	1.6728	1.5546	1.4981	0.8667	0.1275	0.0442	0.0196
					9	2.7465	1.7704	1.5998	1.5237	2.2138	0.1862	0.0552	0.0215
				1	5	1.7497	1.51	1.4419	1.407	0.2540	0.0591	0.0297	0.0214
					7	2.1091	1.6212	1.5085	1.455	0.6909	0.0928	0.0332	0.0177
					9	2.675	1.7277	1.5628	1.4895	1.9345	0.1472	0.04221	0.0178
Bayes	θ	v	-	ℓ	r	Inverse Chi-square distribution $(P_3(\theta \setminus x))$							
	1.5	16	-	0.5	5	1.0465	1.252	1.3367	1.381	0.2874	0.1116	0.0550	0.0297
					7	1.1257	1.3037	1.3732	1.408	0.2409	0.0964	0.0479	0.0256
					9	1.2233	1.3569	1.4062	1.4296	0.1997	0.0853	0.0433	0.0233
				1	5	1.0512	1.2273	1.298	1.3343	0.2665	0.1124	0.0619	0.0389
					7	1.129	1.2851	1.3445	1.3738	0.2239	0.0940	0.0501	0.0298
					9	1.2259	1.3417	1.3833	1.4025	0.1846	0.0812	0.0432	0.0251

Continue for table 4.3: Shows the values for $\hat{\theta}$ under El-Sayyad's loss function with MSE.

Method	parameters					Estimate for θ ($\hat{\theta}$)				MES			
						Sample Size(n)				Sample Size(n)			
						10	25	50	100	10	25	50	100
Bayes	θ	α	β	ℓ	r	Inverted Gamma distribution ($P_4(\theta \setminus x)$)							
	1.5	10	0.5	0.5	5	0.9326	1.1806	1.2942	1.3577	0.3868	0.1465	0.0689	0.0352
					7	0.9903	1.2245	1.3273	1.3833	0.3378	0.1269	0.0595	0.0302
					9	1.0611	1.2699	1.3575	1.4038	0.2853	0.1097	0.0524	0.0269
				1	5	0.9474	1.1644	1.2611	1.3143	0.3582	0.1468	0.0770	0.0456
					7	1.0012	1.2123	1.3029	1.3515	0.3166	0.1254	0.0632	0.0355
					9	1.0701	1.26	1.338	1.3786	0.2682	0.1071	0.0539	0.0298
Bayes	θ	a	b	ℓ	r	Improper distribution ($P_5(\theta \setminus x)$)							
	1.5	9	1	0.5	5	1.0149	1.2296	1.323	1.3734	0.3074	0.1202	0.0587	0.0313
					7	1.0852	1.2782	1.3581	1.3999	0.2600	0.1034	0.0508	0.0269
					9	1.1709	1.3282	1.39	1.421	0.2142	0.0900	0.0453	0.0242
				1	5	1.0223	1.2075	1.286	1.3278	0.2859	0.1214	0.0662	0.0409
					7	1.0905	1.2616	1.3309	1.3664	0.2431	0.1018	0.0537	0.0315
					9	1.1752	1.3147	1.3682	1.3945	0.1998	0.0868	0.0459	0.0265
Bayes	θ	c	-	ℓ	r	Non-informative distribution ($P_6(\theta \setminus x)$)							
	1.5	8	-	0.5	5	1.0827	1.2759	1.351	1.3888	0.2677	0.1037	0.0515	0.0282
					7	1.1728	1.3311	1.3888	1.4163	0.2239	0.0905	0.0453	0.0245
					9	1.2852	1.3877	1.4229	1.4383	0.1914	0.0822	0.0417	0.0225
				1	5	1.0841	1.2483	1.3104	1.3411	0.2469	0.1037	0.0577	0.0369
					7	1.1736	1.3103	1.3587	1.3813	0.2061	0.0871	1.3587	0.0282
					9	1.2857	1.3708	1.3989	1.4106	0.1745	0.0769	0.0409	0.0239

5. Discussion

In general, as we see in the tables (4.1-4.3) by using different estimation methods, we find the Mean Square Errors (MSE) decreased when sample size increased in all cases. And we obtained the same results for θ & MSE by using maximum likelihood estimation (MLE) and the moment estimation(ME) for all sample sizes (n), because they have the same formula see formula from equations (4), (8).

For the first stage in table (4.1), when the true value of θ ($\theta = 0.5$):

When the prior distribution for θ is Levy distribution with b_3.

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Levy distribution with b_3=0.2 and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \,\&\, r = 5$).

When the prior distribution for θ is the Gumbel type-II distribution with b.

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Gumbel type-II distribution with b=0.04 and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \,\&\, r = 7$).

When the prior distribution for θ is the Inverse Chi-square distribution with v.

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Inverse Chi-square distribution with v=16 and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \,\&\, r = 7$), which is best estimation ,the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Inverted Gamma distribution with (α, β).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Inverted Gamma distribution with $(\alpha = 10, \beta = 0.5)$ and for the values for the parameters of the El-Sayyad 's loss function is $(\ell = 0.5 \,\&\, r = 5)$, which is best estimation ,the according to the smallest values of MSE for the samples sizes $n \geq 25$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Improper distribution with (a, b).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Improper distribution with (a=9, b=1) and for the values for the parameters of the El-Sayyad's loss function is $(\ell = 0.5 \,\&\, r = 7)$, which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Non-informative distribution with c.

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values by using the prior distribution for θ is Non-informative distribution with c=8 and for the values for the parameters of the El-Sayyad 's loss function is $(\ell = 0.5 \,\&\, r = 5)$, which is best estimation ,the according to the smallest values of MSE for the samples sizes $n \leq 50$ comparative to the estimated values by using MLE and ME.

For the second stage in table (4.2), when the true value of $\theta\,(\theta = 1)$:

When the prior distribution for θ is Levy distribution with (b_3=0.2).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is $(\ell = 1 \,\&\, r = 5)$, which is best estimation, the according to the smallest values of MSE for the samples sizes $n \geq 50$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Gumbel type-II distribution with (b=0.04).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with and the values for the parameters of the El-Sayyad 's loss function is $(\ell = 1 \,\&\, r = 5)$,which is best estimation ,the according to the smallest values of MSE for the samples sizes $n \geq 25$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Inverse Chi-square distribution with (v=16).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is $(\ell = 1 \,\&\, r = 5)$, which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Inverted Gamma distribution with $(\alpha = 10, \beta = 0.5)$.

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 1 \& r = 5$), which is best estimation, the according to the smallest values of MSE for the samples sizes $n \geq 50$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Improper distribution with (a=9, b=1).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 1 \& r = 5$), which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Non-informative distribution with (c=8).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 1 \& r = 5$), which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

For the second stage in table (4.3), when the true value of $\theta\,(\theta = 1.5)$:

When the prior distribution for θ is Levy distribution with (b_3=0.2).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 1 \& r = 7$), which is best estimation, the according to the smallest values of MSE for the samples sizes $n \geq 50$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Gumbel type-II distribution with (b=0.04).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with and the values for the parameters of the El-Sayyad 's loss function is ($\ell = 1 \& r = 7$),which is best estimation ,the according to the smallest values of MSE for the samples sizes $n \geq 50$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Inverse Chi-square distribution with (v=16).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 0.5 \& r = 9$), which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Inverted Gamma distribution with ($\alpha = 10\,, \beta = 0.5$).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values ,with the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \& r = 9$).

When the prior distribution for θ is the Improper distribution with (a=9, b=1).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 0.5 \& r = 9$), which is best estimation, the according to the smallest values of MSE for the samples sizes $n \leq 25$ comparative to the estimated values by using MLE and ME.

When the prior distribution for θ is the Non-informative distribution with (c=8).

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values, with the values for the parameters of the El-Sayyad's loss function is ($\ell = 0.5 \,\&\, r = 9$), which is best estimation, the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME.

6. Conclusion

When we compared the estimated values for θ ($\hat{\theta}$) for the scale parameter of the Exponential distribution by using the methods in this study .We find that Mean Square Errors (MSE) was decreased when sample size increased in all cases. And the MSE increased in all samples sizes (n) when the true value of θ increased. The best method is the bayes estimation according to the smallest values of MSE for all sample sizes (n) when the prior distribution is

- Improper distribution with (a=9, b=1) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \,\&\, r = 7$), which is best estimation ,the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME, when the true value of θ ($\theta = 0.5$) see table (4.1).
- Non-informative distribution with (c=8) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 1 \,\&\, r = 5$), which is best estimation ,the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME, when the true value of θ ($\theta = 1$) see table (4.2).
- Non-informative distribution with (c=8) and for the values for the parameters of the El-Sayyad 's loss function is ($\ell = 0.5 \,\&\, r = 9$), which is best estimation ,the according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME, when the true value of θ ($\theta = 1.5$) see table (4.3).

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Appendix-A: The posterior distribution using different Priors

1. The posterior distribution using Levy distribution as prior:

It is assumed that θ follows the Levy distribution with pdf as given below:

$$P(\theta)\ \alpha\ \sqrt{\frac{b_3}{2\pi}}\,\theta^{\frac{3}{2}}\exp(-\frac{b_3}{2\theta})\quad \text{for}\quad b_3,\theta>0 \qquad ...(A.1)$$

Where b_3 = hyperparameter

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ is:

$$P(\theta\backslash x)=\frac{L(\underline{x}\backslash\theta)\ P(\theta)}{\int_{\theta} L(\underline{x}\backslash\theta)\ P(\theta)d\theta} \qquad ...(A.2)$$

Substituting the equation (2) and the equation (A.1) in equation (A.2), we get:

$$P_1(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})\,[\sqrt{\frac{b_3}{2\pi}}\,\theta^{-\frac{3}{2}}\exp(-\frac{b_3}{2\theta})\,]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})[\sqrt{\frac{b_3}{2\pi}}\,\theta^{-\frac{3}{2}}\exp(-\frac{b_3}{2\theta})\,]d\theta} \qquad ...(A.3)$$

$$P_1(\theta\backslash x)=\frac{\theta^{-n-\frac{3}{2}}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\int_0^\infty \theta^{-n-\frac{3}{2}}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))d\theta} \qquad ...(A.4)$$

We can write $\theta^{-n+\frac{3}{2}}$ as $\theta^{-[(n+\frac{1}{2})+1]}$, and by multiplying the integral in equation (A.4) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\right)\left(\frac{\Gamma(n+\frac{1}{2})}{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}\right)$$, where $\Gamma(.)$ is a gamma function

. Then we get,

$$P(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}\,\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\Gamma(n+\frac{1}{2})\ A(x;\theta)} \qquad ...(A.5)$$

Where $A(x;\theta)$ equals to

$$A(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}\,\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\Gamma(n+\frac{1}{2})}d\theta=1$$

. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_1(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2})) \qquad ...(A.6)$$

It means that $P_1(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+\frac{1}{2}),\beta_{(new)}=(\sum_{i=1}^{n}x_i+\frac{b_3}{2})$).

2. The posterior distribution using Gumbel type-II distribution as prior:

It is assumed that θ follows the Gumbel type-II distribution with pdf as given below: $P(\theta)\ \alpha\ a\,b\,\theta^{-(a+1)}\exp(-\frac{b}{\theta^{a}})$ for $a,b,\theta>0$ ...(A.7)

If a=1 then we get

$$P(\theta)\ \alpha\ b\,\theta^{-2}\exp(-\frac{b}{\theta})\quad \text{for}\quad b,\theta>0 \qquad ...(A.8)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.8) in equation (A.2), so we have

$$P_2(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})[b\,\theta^{-2}\exp(-\frac{b}{\theta})]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})[b\,\theta^{-2}\exp(-\frac{b}{\theta})]d\theta} \qquad ...(A.9)$$

$$P_2(\theta\backslash x)=\frac{\theta^{-n-2}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))}{\int_0^\infty \theta^{-n-2}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))d\theta} \qquad ...(A.10)$$

We can write θ^{-n-2} as $\theta^{-[(n+1)+1]}$, and by multiplying the integral in equation (A.10) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}x_i+b)^{(n+1)}}{\Gamma(n+1)}\right)\left(\frac{\Gamma(n+1)}{(\sum_{i=1}^{n}x_i+b)^{(n+1)}}\right)$$, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_2(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))}{\Gamma(n+1)\ \ B(x;\theta)} \qquad ...(A.11)$$

Where $B(x;\theta)$ equals to

$$B(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\Gamma(n+1)}d\theta=1$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_2(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\Gamma(n+1)} \qquad ...(A.12)$$

It means that $P_2(\theta\backslash x)\sim$Inverted Gamma distribution with new parameters $(\alpha_{(new)}=(n+1),\beta_{(new)}=(\sum_{i=1}^n x_i+b))$.

3. The posterior distribution using Inverse Chi-square distribution as prior:

It is assumed that θ follows the Inverse Chi-square distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta}) \qquad \text{for } v,\theta>0 \qquad ...(A.13)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.13) in equation (A.2), so we have

$$P_3(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta})]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta})]d\theta} \qquad ...(A.14)$$

$$P_3(\theta\backslash x)=\frac{\theta^{-[(n+\frac{v}{2})+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2})}}{\int_0^\infty \theta^{-[(n+\frac{v}{2})+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2})}d\theta} \qquad ...(A.15)$$

By multiplying the integral in equation (A.15) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\right)\left(\frac{\Gamma(n+\frac{v}{2})}{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}}\right)$$

, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_3(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}\theta^{-[(n-\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2}))}{\Gamma(n+\frac{v}{2})\ \ C(x;\theta)} \qquad ...(A.16)$$

Where $C(x;\theta)$ equals to

$$C(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}\ \theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2}))}{\Gamma(n+\frac{v}{2})}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_3(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2})) \qquad ...(A.17)$$

It means that $P_3(\theta\backslash x))\sim$Inverted Gamma distribution with new parameters $(\alpha_{(new)}=(n+\frac{v}{2}),\beta_{(new)}=(\sum_{i=1}^n x_i+\frac{1}{2}))$.

4. The posterior distribution using Inverted Gamma distribution as prior:

It is assumed that θ follows the Inverted Gamma distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{\beta^\alpha}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta}) \qquad \text{for } \alpha,\beta,\theta>0 \qquad ...(A.18)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.18) in equation (A.2), so we have

$$P_4(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\frac{\beta^\alpha}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta})]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\frac{\beta^\alpha}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta})]d\theta} \qquad ...(A.19)$$

$$P_4(\theta\backslash x)=\frac{\theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\beta))}{\int_0^\infty \theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\beta))d\theta} \qquad ...(A.20)$$

By multiplying the integral in equation (A.20) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^n x_i+\beta)^{(n+\alpha)}}{\Gamma(n+\alpha)}\right)\left(\frac{\Gamma(n+\alpha)}{(\sum_{i=1}^n x_i+\beta)^{(n+\alpha)}}\right)$$

, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_4(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\beta))}{\Gamma(n+\alpha)\ \ D(x;\theta)} \qquad ...(A.21)$$

Where $D(x;\theta)$ equals to

$$D(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\dots,x_n)$ is

$$P_4(\theta\setminus x)=\frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)} \quad \dots(A.22)$$

It means that $P_4(\theta\setminus x)$~Inverted Gamma distribution with new parameters($\alpha_{(new)}=(n+\alpha),\ \beta_{(new)}=(\sum_{i=1}^{n} x_i+\beta)$).

5. The posterior distribution using improper distribution as prior:

It is assumed that θ follows the improper distribution with pdf as given below:

$$P(\theta)\ \alpha\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta}) \quad \text{for } b,\theta>0 \text{ and } -\infty<a<\infty \quad \dots(A.23)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,\dots,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.22) in equation (A.2), so we have

$$P_5(\theta\setminus x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})\ ]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})\ ]d\theta} \quad \dots(A.24)$$

$$P_5(\theta\setminus x)=\frac{\theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\int_0^\infty \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))d\theta} \quad \dots(A.25)$$

By multiplying the integral in equation (A.25) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}}{\Gamma(n+a)}\right)\left(\frac{\Gamma(n+a)}{(\sum_{i=1}^{n} x_i+b)^{(n+a)}}\right)$$, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_5(\theta\setminus x)=\frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)\quad E(x;\theta)} \quad \dots(A.26)$$

Where $E(x,\theta)$ equals to

$$E(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\dots,x_n)$ is

$$P_5(\theta\setminus x)=\frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)} \quad \dots(A.27)$$

It means that $P_5(\theta\setminus x)$~ Inverted Gamma distribution with new parameters($\alpha_{(new)}=(n+a),\ \beta_{(new)}=(\sum_{i=1}^{n} x_i+b)$).

6. The posterior distribution using Non-informative distribution as prior:

It is assumed that θ follows the non-informative distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{\theta^c} \quad \text{for } \theta,c>0 \quad \dots(A.28)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,\dots,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.28) in equation (A.2), so we have

$$P_6(\theta\setminus x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\ \theta^{-c}\ ]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\ \theta^{-c}\ ]d\theta} \quad \dots(A.29)$$

$$P_6(\theta\setminus x)=\frac{\theta^{-(n+c)}\ \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\int_0^\infty \theta^{-(n+c)}\ \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)d\theta} \quad \dots(A.30)$$

We can write $\theta^{-(n+c)}$ as $\theta^{-[(n+c-1)+1]}$, and by multiplying the integral in equation (A.30), by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)}}{\Gamma(n+c-1)}\right)\left(\frac{\Gamma(n+c-1)}{(\sum_{i=1}^{n} x_i)^{(n+c-1)}}\right)$$, where $\Gamma(.)$ is a gamma function. Then we get

$$P_6(\theta\setminus x)=\frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\Gamma(n+c-1)\quad F(x;\theta)} \quad \dots(A.31)$$

Where $F(x,\theta)$ equals to

$$F(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\Gamma(n+c-1)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\dots,x_n)$ is

$$P_6(\theta\setminus x)=\frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\ \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\Gamma(n+c-1)} \quad \dots(A.32)$$

It means that $P_6(\theta\setminus x)$~Inverted Gamma distribution with new parameters($\alpha_{(new)}=(n+c-1),\ \beta_{(new)}=(\sum_{i=1}^{n} x_i)$).

Appendix-B: The following is the derivation of these estimators under the El-Sayyad's loss function.

The El-Sayyad's loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L(\hat{\theta}, \theta) = \theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}$, the risk function is:

$$R(\hat{\theta} - \theta) = E[L(\hat{\theta}, \theta)] \quad \text{...(B.1)}$$

$$R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = \int_{\theta} L(\hat{\theta}, \theta)\, P(\theta \backslash x)d\theta$$

$$R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = \int_{\theta} \theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}\, P(\theta \backslash x)d\theta$$

$$R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = \int_{\theta} \theta^{\ell}(\hat{\theta}^{2r} - 2\hat{\theta}^{r}\theta^{r} + \theta^{2r})\, P(\theta \backslash x)d\theta$$

$$R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = \hat{\theta}^{2r}\int_0^{\infty} \theta^{\ell} P(\theta \backslash x)d\theta - 2\hat{\theta}^{r}\int_0^{\infty} \theta^{r} P(\theta \backslash x)d\theta + \int_0^{\infty} \theta^{2r} P(\theta \backslash x)d\theta$$

$$R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = \hat{\theta}^{2r}E(\theta^{\ell} \backslash x) - 2\hat{\theta}^{r}E(\theta^{r} \backslash x) + E(\theta^{2r} \backslash x) \quad \text{... (B.2)}$$

Let

$\frac{\partial}{\partial \hat{\theta}^{r}} R(\theta^{\ell}(\hat{\theta}^{r} - \theta^{r})^{2}) = 0 \Rightarrow 2\hat{\theta}^{r}E(\theta^{\ell} \backslash x) - 2E(\theta^{r} \backslash x) = 0 \Rightarrow \hat{\theta}^{r} = \frac{E(\theta^{r} \backslash x)}{E(\theta^{\ell} \backslash x)}$, we

get Bayes estimator of θ denoted by $\hat{\theta}_{ES}$ for the above prior as follows

$$\hat{\theta}_{ES} = \left[\frac{E(\theta^{r} \backslash x)}{E(\theta^{\ell} \backslash x)}\right]^{\frac{1}{r}} = \left[\frac{\int_0^{\infty} \theta^{r} P(\theta \backslash x)d\theta}{\int_0^{\infty} \theta^{\ell} P(\theta \backslash x)d\theta}\right]^{\frac{1}{r}} \quad \text{... (B.3)}$$

1. **Bayes estimation using Levy distribution as prior**

To obtain the Bayes' estimator under Levy distribution as prior. Substituting the equation (A.6) in the integral in equation (B.3), we get:

$$E(\theta^{r} \backslash x) = \int_0^{\infty} \theta^{r} P_1(\theta \backslash x)d\theta \quad \text{...(B.4)}$$

$$E(\theta^{r} \backslash x) = \int_0^{\infty} \theta^{r} \frac{(\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})} \theta^{-[(n+\frac{1}{2})+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_2}{2})} d\theta \quad \text{... (B.5)}$$

$$E(\theta^{r} \backslash x) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})} \theta^{-[(n+\frac{1}{2})+1]+r} e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_2}{2})} d\theta \quad \text{... (B.6)}$$

For the equation (B.6), we can write

$-[(n+\frac{1}{2})+1]+r = -[n+\frac{1}{2}+1-r] = -[(n+\frac{1}{2}-r)+1]$. By multiplying the integral in equation (B.6) by the quantity which equals to $A_1 = (\frac{\Gamma(n+\frac{1}{2}-r)}{\Gamma(n+\frac{1}{2}-r)})$, where $\Gamma(.)$ is a gamma function. Then, we have

$$E(\theta^{r} \backslash x) = A_1 \int_0^{\infty} \frac{(\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{(n+\frac{1}{2})+r-r}}{\Gamma(n+\frac{1}{2})} \theta^{-[(n+\frac{1}{2}-r)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_2}{2})} d\theta \quad \text{...(B.7)}$$

Then, we have

$$E(\theta^{r} \backslash x) = \frac{\Gamma(n+\frac{1}{2}-r)}{\Gamma(n+\frac{1}{2})} (\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{r}\ (A_2(x;\theta) \quad \text{... (B.8)}$$

Where $A_2(x;\theta)$ equals to

$A_2(x;\theta) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{(n+\frac{1}{2}-r)}}{\Gamma(n+\frac{1}{2}-r)} \theta^{-[(n+\frac{1}{2}-r)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_2}{2})} d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^{r} \backslash x) = \frac{\Gamma(n+\frac{1}{2}-r)}{\Gamma(n+\frac{1}{2})} (\sum_{i=1}^{n} x_i + \frac{b_2}{2})^{r} \qquad r, n \,\&\, b_2 > 0 \quad \text{... (B.9)}$$

Also, we have

$$E(\theta^{\ell} \backslash x) = \int_0^{\infty} \theta^{\ell} P_1(\theta \backslash x)d\theta \quad \text{... (B.10)}$$

$$E(\theta^\ell \mid x)=\int_0^\infty \theta^\ell \frac{(\sum_{i=1}^n x_i+\frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{b_2}{2})} d\theta \quad \text{... (B.11)}$$

By the same way we can get

$$E(\theta^\ell \mid x)=\frac{\Gamma(n+\frac{1}{2}-\ell)}{\Gamma(n+\frac{1}{2})}(\sum_{i=1}^n x_i+\frac{b_2}{2})^\ell \qquad \ell, n \,\&\, b_2>0 \quad \text{... (B.12)}$$

Substituting the equations (B.9) and (B.12) in equation (B.3), we get:

$$\hat{\theta}_{ES1}=[\frac{\Gamma(n+\frac{1}{2}-r)(\sum_{i=1}^n x_i+\frac{b_2}{2})^r}{\Gamma(n+\frac{1}{2}-\ell)(\sum_{i=1}^n x_i+\frac{b_2}{2})^\ell}]^{\frac{1}{r}} \qquad \ell, r, n \,\&\, b_2>0 \quad \text{... (B.13)}$$

$$\hat{\theta}_{ES1}=[\frac{\Gamma(n+\frac{1}{2}-r)}{\Gamma(n+\frac{1}{2}-\ell)}]^{\frac{1}{r}}(\sum_{i=1}^n x_i+\frac{b_2}{2})^{1-\frac{\ell}{r}} \qquad \ell, r, n \,\&\, b_2>0 \quad \text{... (B.14)}$$

2. <u>Bayes estimation using Gumbel type-II distribution as prior:</u>

To obtain the Bayes' estimator under the Gumbel type-II distribution as prior. Substituting the equation (A.12) in the integral in equation (B.3), we get:

$$E(\theta^r \mid x)=\int_0^\infty \theta^r P_2(\theta \mid x)d\theta \quad \text{...(B.15)}$$

$$E(\theta^r \mid x)=\int_0^\infty \theta^r \frac{(\sum_{i=1}^n x_i+b)^{(n+1)}\, \theta^{-[(n+1)+1]}\, e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+b)}}{\Gamma(n+1)} d\theta \quad \text{... (B.16)}$$

$$E(\theta^r \mid x)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+b)^{(n+1)+r-r}\, \theta^{-[(n+1+1)-r]}\, e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+b)}}{\Gamma(n+1)} d\theta \quad \text{... (B.17)}$$

For the equation (B.17), we can write $-[(n+1-r+1]=-[(n+1-r)+1]$. And by multiplying the integral in equation (B.17) by the quantity which equals to $B_1=(\frac{\Gamma(n+1-r)}{\Gamma(n+1-r)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\theta^r \mid x)=B_1\int_0^\infty \frac{(\sum_{i=1}^n x_i+b)^{(n+1-r)}\, \theta^{-[(n+1-r)+1]}\, e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+b)}}{\Gamma(n+1)} d\theta \quad \text{... (B.18)}$$

Then we have

$$E(\theta^r \mid x)=\frac{\Gamma(n+1-r)}{\Gamma(n+1)}(\sum_{i=1}^n x_i+b)^r \quad (B_2(x;\theta)) \quad \text{... (B.19)}$$

Where $B_2(x;\theta)$ equals to

$$B_2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+b)^{(n+1-r)}\, \theta^{-[(n+1-r)+1]}\, e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+b)}}{\Gamma(n+1-r)} d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^r \mid x)=\frac{\Gamma(n+1-r)}{\Gamma(n+1)}(\sum_{i=1}^n x_i+b)^r \quad ,r, n \,\&\, b>0 \quad \text{... (B.20)}$$

Also, we have

$$E(\theta^\ell \mid x)=\int_0^\infty \theta^\ell P_2(\theta \mid x)d\theta \quad \text{... (B.21)}$$

$$E(\theta^\ell \mid x)=\int_0^\infty \theta^\ell \frac{(\sum_{i=1}^n x_i+b)^{(n+1)}\, \theta^{-[(n+1)+1]}\, e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+b)}}{\Gamma(n+1)} d\theta \quad \text{... (B.22)}$$

By the same way we can get

$$E(\theta^\ell \mid x)=\frac{\Gamma(n+1-\ell)}{\Gamma(n+1)}(\sum_{i=1}^n x_i+b)^\ell \quad ,\ell, n \,\&\, b>0 \quad \text{... (B.23)}$$

Substituting the equations (B.20) and (B.23) in equation (B.3), we get:

$$\hat{\theta}_{ES2}=[\frac{\Gamma(n+1-r)(\sum_{i=1}^n x_i+b)^r}{\Gamma(n+1-\ell)(\sum_{i=1}^n x_i+b)^\ell}]^{\frac{1}{r}} \qquad \ell, r, n \,\&\, b>0 \quad \text{... (B.24)}$$

$$\hat{\theta}_{ES2}=[\frac{\Gamma(n+1-r)}{\Gamma(n+1-\ell)}]^{\frac{1}{r}}(\sum_{i=1}^n x_i+b)^{1-\frac{\ell}{r}} \qquad \ell, r, n \,\&\, b>0 \quad \text{... (B.25)}$$

3. <u>Bayes estimation using Inverse chi-squared distribution as prior:</u>

To obtain the Bayes' estimator under inverse chi-squared distribution as prior . Substituting the equation (A.16) in the integral in equation (B.3), we get:

$$E(\theta^r \mid x)=\int_0^\infty \theta^r P_3(\theta \mid x)d\theta \quad \text{...(B.26)}$$

$$E(\theta^r \mid x)=\int_0^\infty \theta^r \frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2})+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2})} d\theta \quad \text{... (B.27)}$$

$$E(\theta^r \mid x)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2})+1]+r} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i+\frac{1}{2})} d\theta \quad \text{... (B.28)}$$

For the equation (B.28), we can write $-[((n+\frac{v}{2})+1)-r]=-[(n+\frac{v}{2}-r)+1]$ .And by multiplying the integral in equation (B.28) by the quantity which equals to $C_1 = \left(\frac{\Gamma(n+\frac{v}{2}-r)}{\Gamma(n+\frac{v}{2}-r)}\right)$, where $\Gamma(.)$ is a gamma function. Then, we have

$$E(\theta^r \backslash x) = C_1 \int_0^\infty \frac{(\sum_{i=1}^n x_i + \frac{1}{2})^{(n+\frac{v}{2})+r-r}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2}-r])+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \frac{1}{2})} d\theta \quad \dots (B.29)$$

Then we have

$$E(\theta^r \backslash x) == \frac{\Gamma(n+\frac{v}{2}-r)}{\Gamma(n+\frac{v}{2})} (\sum_{i=1}^n x_i + \frac{1}{2})^r \; (C_2(x;\theta)) \quad \dots (B.30)$$

Where $C_2(x;\theta)$ equals to

$$C_2(x;\theta) = \int_0^\infty \frac{(\sum_{i=1}^n x_i + \frac{1}{2})^{(n+\frac{v}{2}-r)}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2}-r)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \frac{1}{2})} d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^r \backslash x) = \frac{\Gamma(n+\frac{v}{2}-r)}{\Gamma(n+\frac{v}{2})} (\sum_{i=1}^n x_i + \frac{1}{2})^r \quad , n \,\&\, v > 0 \quad \dots (B.31)$$

Also, we have

$$E(\theta^\ell \backslash x) = \int_0^\infty \theta^\ell \; P_3(\theta \backslash x) d\theta \quad \dots (B.32)$$

$$E(\theta^\ell \backslash x) = \int_0^\infty \theta^\ell \frac{(\sum_{i=1}^n x_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \frac{1}{2})} d\theta \quad \dots (B.33)$$

By the same way we can get

$$E(\theta^\ell \backslash x) = \frac{\Gamma(n+\frac{v}{2}-\ell)}{\Gamma(n+\frac{v}{2})} (\sum_{i=1}^n x_i + \frac{1}{2})^\ell \quad , n \,\&\, v > 0 \quad \dots (B.34)$$

Substituting the equations (B.31) and (B.34) in equation (B.3), we get:

$$\hat{\theta}_{ESL} = \left[\frac{\Gamma(n+\frac{v}{2}-r)(\sum_{i=1}^n x_i + \frac{1}{2})^r}{\Gamma(n+\frac{v}{2}-\ell)(\sum_{i=1}^n x_i + \frac{1}{2})^\ell}\right]^{\frac{1}{r}} \quad \ell, r, n \,\&\, v > 0 \quad \dots (B.35)$$

$$\hat{\theta}_{ESL} = \left[\frac{\Gamma(n+\frac{v}{2}-r)}{\Gamma(n+\frac{v}{2}-\ell)}\right]^{\frac{1}{r}} (\sum_{i=1}^n x_i + \frac{1}{2})^{1-\frac{\ell}{r}} \quad \ell, r, n \,\&\, v > 0 \quad \dots (B.36)$$

4. <u>Bayes estimation using Inverted gamma distribution as prior:</u>

To obtain the Bayes' estimator under the inverted gamma distribution as prior . Substituting the equation (A.21) in the integral in equation (B.3), we get:

$$E(\theta^r \backslash x) = \int_0^\infty \theta^r \; P_4(\theta \backslash x) d\theta \quad \dots (B.37)$$

$$E(\theta^r \backslash x) = \int_0^\infty \theta^r \frac{(\sum_{i=1}^n x_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \beta)}}{\Gamma(n+\alpha)} d\theta \quad \dots (B.38)$$

$$E(\theta^r \backslash x) = \int_0^\infty \frac{(\sum_{i=1}^n x_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]+r} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \beta)}}{\Gamma(n+\alpha)} \quad \dots (B.39)$$

For the equation (B.39), we can write $-[(n+\alpha)+1]+r = -[(n+\alpha-r)+1]$ .And by multiplying the integral in equation (B.39) by the quantity which equals to $D_1 = \left(\frac{\Gamma(n+\alpha-r)}{\Gamma(n+\alpha-r)}\right)$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\theta^r \backslash x) = D_1 \int_0^\infty \frac{(\sum_{i=1}^n x_i + \beta)^{(n+\alpha)-r+r} \theta^{-[(n+\alpha-r)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \beta)}}{\Gamma(n+\alpha)} d\theta \quad \dots (B.40)$$

Then we have

$$E(\theta^r \backslash x) = \frac{\Gamma(n+\alpha-r)}{\Gamma(n+\alpha)} (\sum_{i=1}^n x_i + \beta)^r \, (D_2(x;\theta)) \quad \dots (B.41)$$

Where $D_2(x;\theta)$ equals to

$$D_2(x;\theta) = \int_0^\infty \frac{(\sum_{i=1}^n x_i + \beta)^{(n+\alpha-r)} \theta^{-[(n+\alpha-r)+1]} e^{-\frac{1}{\theta}(\sum_{i=1}^n x_i + \beta)}}{\Gamma(n+\alpha-r)} d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^r \backslash x) = \frac{\Gamma(n+\alpha-r)}{\Gamma(n+\alpha)} (\sum_{i=1}^n x_i + \beta)^r \quad , r, n, \beta, \alpha > 0 \quad \dots (B.42)$$

Also, we have

$$E(\theta^\ell \backslash x) = \int_0^\infty \theta^\ell \; P_4(\theta \backslash x) d\theta \quad \dots (B.43)$$

$$E(\theta^{\ell}\setminus x)=\int_0^{\infty}\theta^{\ell}\frac{(\sum_{i=1}^{n}x_i+\beta)^{(n+\alpha)}\theta^{-[(n+\alpha)+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\beta)}}{\Gamma(n+\alpha)}d\theta \quad \text{... (B.44)}$$

By the same way we can get

$$E(\theta^{\ell}\setminus x)=\frac{\Gamma(n+\alpha-\ell)}{\Gamma(n+\alpha)}(\sum_{i=1}^{n}x_i+\beta)^{\ell} \qquad ,\ell,n,\beta,\alpha>0 \quad \text{... (B.45)}$$

Substituting the equations (B.41) and (B.45) in equation (B.3), we get:

$$\hat{\theta}_{BSE}=[\frac{\Gamma(n+\alpha-r)(\sum_{i=1}^{n}x_i+\beta)^{r}}{\Gamma(n+\alpha-\ell)(\sum_{i=1}^{n}x_i+\beta)^{\ell}}]^{\frac{1}{r}} \qquad \ell,r,n,\beta,\alpha>0 \quad \text{... (B.46)}$$

$$\hat{\theta}_{BSE}=[\frac{\Gamma(n+\alpha-r)}{\Gamma(n+\alpha-\ell)}]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+\beta)^{1-\frac{\ell}{r}} \qquad \ell,r,n,\beta,\alpha>0 \quad \text{... (B.47)}$$

5. Bayes estimation using improper distribution as prior:

To obtain the Bayes' estimator under improper distribution as prior. Substituting the equation (A.28) in the integral in equation (B.3), we get:

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\theta^{r}P_r(\theta\setminus x)d\theta \quad \text{...(B.48)}$$

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\theta^{r}\frac{(\sum_{i=1}^{n}x_i+b)^{(n+a)}\theta^{-[(n+a)+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b)}}{\Gamma(n+a)}d\theta \quad \text{... (B.49)}$$

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\frac{(\sum_{i=1}^{n}x_i+b)^{(n+a)}\theta^{-[(n+a)+1]+r}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b)}}{\Gamma(n+a)}d\theta \quad \text{... (B.50)}$$

For the equation (B.50), we can write $-[(n+a)+1]+r=-[(n+a-r)+1]$. And by multiplying the integral in equation (B.50) by the quantity which equals to $E_1=(\frac{\Gamma(n+a-r)}{\Gamma(n+a-r)})$, where $\Gamma(.)$ is a gamma function. Then, we have

$$E(\theta^{r}\setminus x)=E_1\int_0^{\infty}\frac{(\sum_{i=1}^{n}x_i+b)^{(n+a)+r-r}\theta^{-[(n+a-r)+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b)}}{\Gamma(n+a)}d\theta \quad \text{... (B.51)}$$

Then we have

$$E(\theta^{r}\setminus x)=\frac{\Gamma(n+a-r)}{\Gamma(n+a)}(\sum_{i=1}^{n}x_i+b)^{r}(E_2(x;\theta)) \quad \text{... (B.52)}$$

Where $E_2(x;\theta)$ equals to

$$E_2(x;\theta)=\int_0^{\infty}\frac{(\sum_{i=1}^{n}x_i+b)^{(n+a-r)}\theta^{-[(n+a-r)+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b)}}{\Gamma(n+a-r)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^{r}\setminus x)=\frac{\Gamma(n+a-r)}{\Gamma(n+a)}(\sum_{i=1}^{n}x_i+b)^{r} \qquad ,r,n,b,a>0 \quad \text{... (B.53)}$$

Also, we have

$$E(\theta^{\ell}\setminus x)=\int_0^{\infty}\theta^{\ell}P_r(\theta\setminus x)d\theta \quad \text{... (B.54)}$$

$$E(\theta^{\ell}\setminus x)=\int_0^{\infty}\theta^{\ell}\frac{(\sum_{i=1}^{n}x_i+b)^{(n+a)}\theta^{-[(n+a)+1]}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b)}}{\Gamma(n+a)}d\theta \quad \text{...(B.55)}$$

By the same way we can get

$$E(\theta^{\ell}\setminus x)=\frac{\Gamma(n+a-\ell)}{\Gamma(n+a)}(\sum_{i=1}^{n}x_i+b)^{\ell} \qquad ,\ell,n,b,a>0 \quad \text{... (B.56)}$$

Substituting the equations (B.53) and (B.56) in equation (B.3), we get:

$$\hat{\theta}_{BSE}=[\frac{\Gamma(n+a-r)(\sum_{i=1}^{n}x_i+b)^{r}}{\Gamma(n+a-\ell)(\sum_{i=1}^{n}x_i+b)^{\ell}}]^{\frac{1}{r}} \qquad \ell,r,n,a,b>0 \quad \text{... (B.57)}$$

$$\hat{\theta}_{BSE}=[\frac{\Gamma(n+a-r)}{\Gamma(n+a-\ell)}]^{\frac{1}{r}}(\sum_{i=1}^{n}x_i+b)^{1-\frac{\ell}{r}} \qquad \ell,r,n,a,b>0 \quad \text{... (B.58)}$$

6. Bayes estimation using non-informative distribution as prior:

To obtain the Bayes' estimator under non informative distribution as prior. Substituting the equation (A.32) in the integral in equation (B.3), we get:

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\theta^{r}P_r(\theta\setminus x)d\theta \quad \text{... (B.59)}$$

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\theta^{r}\frac{(\sum_{i=1}^{n}x_i)^{(n+c-1)}\theta^{-[(n+c-1)+1]}e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i}}{\Gamma(n+c-1)}d\theta \quad \text{...(B.60)}$$

$$E(\theta^{r}\setminus x)=\int_0^{\infty}\frac{(\sum_{i=1}^{n}x_i)^{(n+c-1)}\theta^{-[(n+c-1)+1]+r}e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i}}{\Gamma(n+c-1)}d\theta \quad \text{... (B.61)}$$

For the equation (B.61), we can write $-[(n+c-1)+1]+r=-[(n+c-1-r)+1]$. And by multiplying the integral in equation

(B.61) by the quantity which equals $F_1=\left(\frac{\Gamma(n+c-1-r)}{\Gamma(n+c-1-r)}\right)$, where $\Gamma(.)$ a gamma function is. Then, we have

$$E(\theta^r \backslash x)=F_1\int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c-1)+r-r}\,\theta^{-[(n+c-1-r)+1]}\,e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}}{\Gamma(n+c-1)}d\theta \quad \ldots (B.62)$$

Then we have

$$E(\theta^r \backslash x)=\frac{\Gamma(n+c-1-r)}{\Gamma(n+c-1)}(\sum_{i=1}^n x_i)^r(F_2(x;\theta)) \quad \ldots (B.63)$$

Where $F_2(x;\theta)$ equals to

$$F_2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c)}\,\theta^{-[(n+c)+1]}\,e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}}{\Gamma(n+c)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\theta^r \backslash x)=\frac{\Gamma(n+c-1-r)}{\Gamma(n+c-1)}(\sum_{i=1}^n x_i)^r \qquad r,n,c>0 \quad \ldots (B.64)$$

Also, we have

$$E(\theta^\ell \backslash x)=\int_0^\infty \theta^\ell\, P_\ell(\theta \backslash x)d\theta \quad \ldots (B.65)$$

$$E(\theta^\ell \backslash x)=\int_0^\infty \theta^\ell \frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\,\theta^{-[(n+c-1)+1]}\,e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}}{\Gamma(n+c-1)}d\theta \quad \ldots (B.66)$$

By the same way we can get

$$E(\theta^\ell \backslash x)=\frac{\Gamma(n+c-1-\ell)}{\Gamma(n+c-1)}(\sum_{i=1}^n x_i)^\ell \qquad r,n,c>0 \quad \ldots (B.67)$$

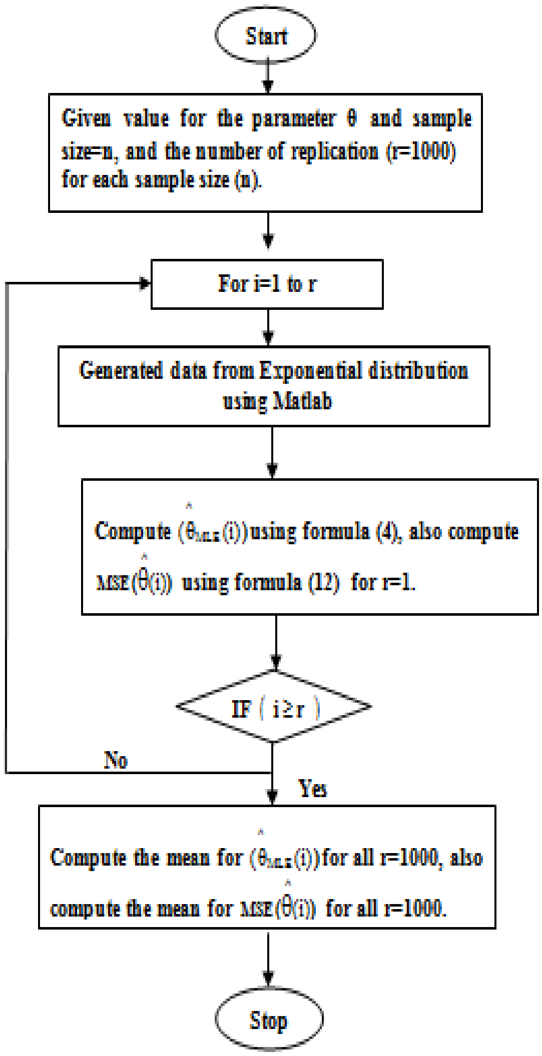
Substituting the equations (B.65) and (B.67) in equation (B.3), we get:

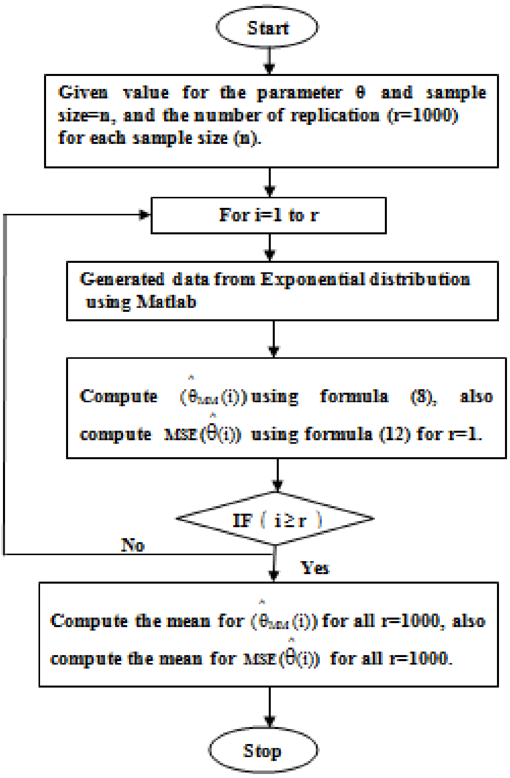
$$\hat{\theta}_{ESS}=\left[\frac{\Gamma(n+c-1-r)(\sum_{i=1}^n x_i)^r}{\Gamma(n+c-1-\ell)(\sum_{i=1}^n x_i)^\ell}\right]^{\frac{1}{r}} \qquad \ell,r,n,c>0 \quad \ldots (B.68)$$

$$\hat{\theta}_{ESS}=\left[\frac{\Gamma(n+c-1-r)}{\Gamma(n+c-1-\ell)}\right]^{\frac{1}{r}}(\sum_{i=1}^n x_i)^{1-\frac{\ell}{r}} \qquad \ell,r,n,c>0 \quad \ldots (B.69)$$

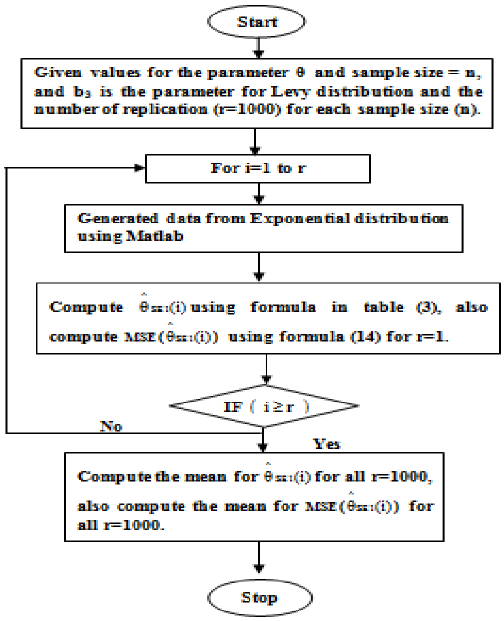
<u>Appendix-C:</u> The following is the programs algorithm.

Algorithm (1): To compute MLE for scale parameter ($\hat{\theta}$) with MSE.





Algorithm (3): To compute Bayes estimators ($\hat{\theta}_{ES1}$) using Levy distribution as prior distribution for θ with MSE.



Note (1): we can reformulate the Algorithm (3) to compute Bayes estimators $\hat{\theta}_{ESk}, k = 2,3,4,5,6$ under using other distributions as prior distribution for θ with MSE.

Algorithm (2): To compute MM for scale parameter ($\hat{\theta}$) with MSE.