Some Results on s-prime Compactly Acts over Monoid

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ABSTRACT

In this proper introduce the concept of subact Ṅ of Ṁ iss-prime compactly packed ($sp-\boldsymbol{c.P}$) S-Acts. if for each family $\{\boldsymbol{P_\alpha}\}_{\alpha\in\lambda}$ of s- prime subact of M with Ṅ⊆ $\bigcup_{\alpha\in\lambda}\boldsymbol{P_\alpha}$, $\boldsymbol{N}\subseteq\boldsymbol{P_\beta}$ for some $\boldsymbol{\beta}\in\boldsymbol{\lambda}$. We refer to an S-Act Ṁ as sp- $\boldsymbol{c.P}$. if every subact is s-prime compactly packed. We study various properties of sp-$\boldsymbol{c.P}$ S-Acts.

1.Introduction

Let Ṁ be a unitary module defined on commutative ring with 1.A subset V of R have been said a multiplicatively closed (M.C) subset of R if: 1 ∈ V and for any v1,v2 in V, v1v2 ∈ V. Let V be (m. c. set) of R and M is an R –module, (1) V* a nonempty subset of M have been said V-closed if vm ∈ V^* for every v ∈ V and m ∈ V^*. (2) An V-closed subset V^* have been said saturated if the next provision is hold: where dm ∈ V^* for d ∈ R also m ∈ M, then d ∈ V and m ∈ V^* [1, 2] . A proper subact Ṅ of a S-Act Ṁ is said to be c.P if whenever N is contained in the union of a family of prime subact of Ṁ, then Ṅ is contained in one of the members of the family. And Ṁ is compactly packed S-Act if every proper subact of Ṁ is c.P [3]. Let S ⊆ R be a m.c.set. and P a submodule of M with [P :R M]∩S = ∅ . Then P is said to be an s -prime submodule if there exists s ∈ V and whenever am ∈ P then either sa ∈ [P :R M] or sm ∈ P for each a ∈ R and m ∈ M [4].In [5], we say that a submodule N of an R-module M is S-pure if there exists an s ∈ S such that s(N ∩ IM) ⊆ IN for every ideal I of R. an ideal I of R is called an S -prime ideal if I is an S -prime submodule of R-module R. Note that all prime submodules P whose residual by M is disjoint from S become an S -prime submodule since 1 ∈ S. Also, if we take S ⊆ u(R), where u(R) denotes the set of units in R, the notions of S -prime submodules and prime submodules are equal [4]

2. **s-Prime Compactly Packed S-Acts.**

Definition 2.1 : [6] Let S be a monoid, a subset U of R have been said a multiplicatively closed (m.c) subset of S if : 1 belongs to U and for any u1,u2 ∈ U. Let V be a (M.C) of a monoid S and M be an S-act (1) A nonempty subset U* of M have been said U-closed if um belongs U^* for every v ∈ U and m ∈ U^*. (2) An U-closed subset U^* have been said saturated if the next provisions are hold: where km ∈ U^* for k ∈ S and m ∈ M, then k ∈ U and m ∈ U^* .

Example 2.2: [6] Let Q be act over integer number Z. If U* is saturated U-closed, then U = Z-{0} and U* = Q-{0}.

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Definition 2.3: Let S be a monoid and U (m.c) subset of S with [N:S M] ∩ U = Ø, a subact N of s-act M is said to be s-prime, if there is u belongs to U and whenever am ∈ N then either ua ∈ [N :S M] or um ∈ N for each a ∈ S and m ∈ M.

Definition 2.4 : Let K be a subact of a S-Act Ṁ, if there exist s-prime subact that contain K, then the intersection of all s-prime subact containing K is called the Ṁ- s-radical of K and denoted by rad K. If there is no s-prime subact containing K, then s-rad K =Ṁ. A subact K is called a s-radical subact if s-rad K = K.

Theorem 2.5 : Let Ṁ be an S-Act. The following statements are equivalent:

1- Ṁ is s-c.P.

2- For every a proper subact K of Ṁ, there is a∈K such that's- rad(K) = s-rad(Sa).

3- For every proper subact K of Ṁ, if $\{K_\alpha\}_{(\alpha\in\lambda)}$ is a family of subact of Ṁ and $K \subseteq \bigcup_{(\alpha\in\lambda)} K_\alpha$ then $K \subseteq$s- rad(K_β) for some β∈λ.

4- For every proper subact K of M, if $\{K_\alpha\}_{(\alpha\in\lambda)}$ is a family of radical subact of M and $K\subseteq\bigcup_{(\alpha\in\lambda)}K_\alpha$ then $K\subseteq K_\beta$ for some β∈λ.

Proof: (1→2) Let K be a proper subact of Ṁ. Suppose s-radṄ⊄s- rad(Sa) for each a∈K, there is a s-prime subact N_a which contains Sa and K $\not\subset N_a$. But $K = \bigcup_{(a\in K)} S_a \subseteq \bigcup_{(a\in K)} N_a$, that is Ṁ is not s- c-P which contradicts (1).

(2→3) Let K be a proper subact of Ṁ and let $\{K_\alpha\}_{(\alpha\in\lambda)}$ be a family of subact of Ṁ such that $K \subseteq \bigcup_{(\alpha\in\lambda)} K_\alpha$. By (b) there is a ∈ K such that s-radK = s-rad(Sa). Then $a\in\bigcup_{(\alpha\in\lambda)} K_\alpha$ and hence $a\in K_\beta$ for some β∈λ, so that Sa $\subseteq K_\beta$ and K ⊆ s-rad(K) = s-rad(Sa) ⊆ s-rad(K_β).

(3→4) & (4→1) are clear

Recall that an S-Act M is called a multiplication S-Act if each subact N of M has the form N=IM for an ideal I of R. In fact N=[N:M]M. [7].

Remark 2.6:

1- If Ṁ is a multiplication S-Act and K is a subact of Ṁ with $K \subseteq\bigcup_{(\alpha\in\lambda)}K_\alpha$, where K is s- prime subact of Ṁ and λ is a finite set, then $K \subseteq K_\beta$ for some β∈λ .

2- If Ṁ is a multiplication S-Act containing finite number of s-prime subact then Ṁ is s-c.p.

Definition 2.7 : A subact N of a S-act M is called S-pure if there exists an s ∈ U such that s(N ∩ IM) ⊆ IN for every ideal I of S.

Proposition 2.8 : Let Ṁ be S-Act and every subact is s-pure , then Ṁ is $c.P$ if and only if, each proper subact K of M is cyclic.

Proof : The sufficiency is clear. To prove the necessity, let K be a proper subact of M. Since M is s-c.p then by theorem 2.5, there exists a∈ K such that $-rad\ K = s - radSa$. But every subact is s-pure,

Theorem 2.8 : If M is $s - c.P$ S-Act which has at least one maximal subact then M satisfies the ACC on s-radical subact.

Proof: let $K_1 \subseteq K_2 \subseteq \cdots$ be an ascending chain of s-radical subact of M and let $K = \bigcup_i K_i$. If$K = \dot{M}$ and B is a maximal subact of Ṁ, then $B \subsetneq \bigcup_i K_i$. Since Ṁ is s-c.P then $B \subseteq K_j$ for some j. Therefore $B \subseteq K_j$ and therefore $\bigcup_i K_i \subseteq K_j$, that is $\dot{M} \subseteq K_j$ which is impossible. Thus L is a proper subact of Ṁ. Thus $L \subseteq K_j$ for some j and therefore $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_j = K_{j+1} = K_{j+2} = \cdots$, thus the ACC is satisfied for s-radical subact.

Because every finitely generated S-Act and every multiplication S-Act has a proper maximal subact, [8] then we have:-then K = Sa

Corollary 2.9 : If Ṁ is finitely generated or multiplication $c.P$ S-Act, then M satisfies the ACC on radical subact.

Definition 2. 10 : A s-prime subact L of an S-Act Ṁ is called a minimal s-prime subact of a subact K if L ⊆ K and there exist no smaller s-prime subact with this property. Remember that if Ṁ is an S-Act that satisfies the ACC on s-radical subact then the s-radical of any proper subact K of Ṁ is the intersection of a finite number of minimal s-prime subact of K.

Lemma 2.11: If Ṁ be a proper multiplication S-Act that satisfies the ACC on radical subact, then for every proper subact Ṅ of Ṁ there exists a finite number of minimal prime subact of Ṅ.

Proof: let K be a proper subact of Ṁ, then $s - radK$ is the intersection of a finite number of minimal s-prime subact of Ṁ say $L_1, L_2, \dots, L_n$. We shall prove that these L_i's are the only minimal s-prime subact of Ṅ. Suppose H is a minimal prime subact. It is clear that $s - rad\ K \subseteq H$ that is $\bigcap_{i=1}^{n} L_i \subseteq H$ and hence $\bigcap_{i=1}^{n} [L_i : \dot{M}] = [\bigcap_{i=1}^{n} l_i : \dot{M}] \subseteq [\grave{U}_i : \dot{M}]$. And $[\grave{U} : \dot{M}]$ is s-prime ideal [7] then there exists $j \in \{1,2, \dots, n\}$ such that $[L_j : \dot{M}] \subseteq [H : \dot{M}]$, but Ṁ is a multiplication S-Act thus $L_j \subseteq H$ because H is minimal s-prime subact.

Corollary 2.12: If Ṁ is a multiplication $s - c.P$ S-Act, then for every proper subact K of Ṁ there exist a finite number of minimal prime subact of K.

Let L be s-prime subact of an S-Act Ṁ. The height of L equals n (denoted by $ht(L) = n$) if there exists a chain of distinct s-prime subact of L_i of Ṁ of the form $L = L_0 \supset L \supset \cdots \supset L_n$ and it is the longest chain such that $L = L_0$.

The Krull dimension of Ṁ, denoted by dimṀ, is defined as:$dim\dot{M} = \{ht(L): L\ is\ s - prime\ subact\ of\ \dot{M}\}$.

Following [8, 9], the Prime Avoidance Theorem for modules states as follows: Put U is module, $K_1, K_2, \ldots, K_n$ a finite number of submodules of U and K is a submodule of U such that $K \subseteq K_1 \cup K_2 \cup \ldots \cup K_n$. Assume that at most two of the K_i;s are not prime, and that

$(K_j : M) \nsubseteq \sqrt{K_l : M}$ (whenever $j \neq l$; Then $K \subseteq K_l$ for some $l \in \{1,2,\ldots, n\}$. We examine how this theory can be extended to the Primal Avoidance theory for acts over monoid .

Theorem 2.13: Let Ṁ be a multiplication S-Act. If dimṀ=0 then Ṁ is $c.P$ iff Ṁ has finite number of prime subact.

Proof: Suppose Ṁ is $c.P$. If {0} is prime subact then the necessity is trivial. If {0} is not prime, then every prime subact is minimal in Ṁ and by (corollary 2.12) the number of prime subact of Ṁ is finite. The sufficiency follows from the Prime Avoidance Theorem

A partial converse of theorem 1.5 can be found in the subsequent theorem.

Theorem 2.14: Put Ṁ is an S-act and every finitely generated subact is cyclic. If Ṁ satisfies the ACC on radical subact, then Ṁ is $c.P$.

Proof: Put Ṅ is a proper subact of Ṁ. By [8], there exists a finitely generated subact Ù of Ṁ such that s-radṄ = s-radÙ and hence Ù is cyclic subact, and by theorem 2.5 Ṁ is $s - c.P$.

Definition 2.15 : An S-act Ṁ is called s-semilocal, if Ṁ/s-rad(Ṁ) is semisimple.

Proposition 2.16: Let Ṁ be a multiplication s-semilocal S-Act with dim Ṁ ≤ 1. If M satisfies the ACC on s-radical subact, then Ṁ is $s - c.P$.

Proof: We have two cases, first if {0} is s-prime subact then every non-zero s-prime subact of Ṁ is maximal and hence the number of s-prime subact of Ṁ is finite. On the other hand if {0} is not s-prime, let Á be the set of all s-prime subact of Ṁ and let Ĕ = {Ù∈ Á: Ù is maximal subact of Ṁ}, Ķ={L∈ Á: $L \notin$ Ĕ}. By lemma 2.11. we have Ķ is finite set and hence Á is finite. In any case we get Ṁ is $s - c.P$ from the s-Prime Avoidance Theorem

Definition 2.17: An S-Act Ṁ is said to be satisfy the Cyclic Subact Condition (s-CSC) if for each x∈ Ṁ and each s-prime subact Ķ of Ṁ minimal over Sx, therefore ht(K) ≤ 1.

Proposition 2.18: Suppose that Ṁ is a $c.P$ S-Act. If the s-CSC is satisfied for Ṁ, then dim Ṁ ≤ 1.

Proof: Put Ķ be a maximal subact of Ṁ, then by theorem (2.5) there exists a∈Ṁ such that Ķ = s-radSa. This implies that Ķ is minimal s-prime subact over Sa. By s-CSC, ht(K) ≤ 1, therefore $dim\ M \leq 1$.

Proposition 2.19: Let f: $M \to M'$ be an epimorphism. If M is $s - c.P$ then so is M'. The converse is true when M is finitely generated or (multiplication) S-Act and $ker\ f \subseteq s - rad\{0\}$.

Proof: It is clear.

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