Efficient Numerical Solutions for Fractional Integro-Differential Equations via Conformable Differointegration and the Variational Iteration Method

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ABSTRACT

Multidimensional integro-differential equations (MDIDEs) involve equations where an unknown function, depending on multiple variables, is subjected to both differentiation and integration operations. These equations combine integral and differential operators within a multidimensional framework. The integral equation is referred to as a multidimensional fractional integro-differential equations (MDFIDEs) when the differentiation or integration are of fractional order. Since it is challenging to calculate these problems analytically, the major goal of this study is to use the Variational Iteration Method (VIM) to solve such equations by conformable fractional order integrals and derivatives (CFOID). Beginning, A suggested method of an iterative sequence of approximate solutions was driven, and then verify its convergence to the exact solution in the middle of the kernel of the integro-differential equations (IDE) under specified conditions. Two illustrated examples, linear and nonlinear, are provided.

1. Introduction

Fractional calculus has arisen as a powerful mathematical tool which describing and modeling various complex phenomena in engineering and science [10]. The concept of fractional derivatives has found applications in various fields, inclusive biology, physics, finance, and engineering, because of its ability to long-range interactions, capture memory effects, and irregular diffusion. Fractional (IDE), that involving both fractional derivatives and integrals, have become a prominent mathematical framework for representing complex dynamics and behaviors beyond the scope of classical integer order differential equations [9], [12].

Solving (IDE) by order fractional can be quite challenging, because of their nonlocal and nonlinear characteristics. Numerical methods and traditional analytical may fail to provide effective solutions. Many researchers have been dedicated to creating and developing strategies to tackle these challenges and overcome the limitations of conventional approaches [8], [11]. (VIM) is an approach that has gained attention in solving (IDE). (VIM) initially introduced by Nayfeh and Mook and it has been further Refined by researchers in the field [1], [5]. (VIM) is working by utilizing the concept of iteration, where an auxiliary parameter known as the Lagrange multiplier is introduced to transform the equation into a series of simpler sub problems. This method allows for a solution that gradually improves in accuracy towards approximating the exact solution of the equation under study. Additionally recent advancements in calculus have introduced a concept called conformable differointegration [3], [9]. This novel fractional operator presents an alternative, to the Riemann Liouville fractional derivative offering an advantage of preserving derivative order in resulting equations. [4], [7].

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In this context, this paper explores the application of (VIM) combined with conformable differointegration to solve a variety of challenging fractional (IDE). Through a series of numerical experiments and comparative analyses with other traditional techniques, the effectiveness and efficiency of the proposed method are verified. The results presented herein participate to growing body of knowledge in the field of fractional calculus and showcase the potential of (VIM) with conformable differointegration for solving complex problems in engineering and science [6].

We will introduce (VIM) in this study for solving (IDE) with (CFOID) of the form:

$$\mathcal{D}_p^{\mu} v(p,q) = h(p,q) + \int_m^p \int_n^q \mathcal{K}\big(p,q,r,s,v(p,q)\big)\mathrm{d}s\ \mathrm{d}r \tag{1}$$

$\mathcal{K}$ represent a continuous function, $0 < \mu \le 1, \nu, \xi > 0$, and $p, q \in [m,n] \times [u,v]$. Here, $\mathcal{D}^{\mu}$ denotes the conformable fractional derivative of order μ (CFOD), while I^{ν} and I^{ξ} represent the conformable fractional integrals of orders ν and ξ (CFOI), respectively.

2. Fractional Calculus Fundamentals

The conformable type is one of the most straightforward and significant definitions of fractional order derivatives or integrals that will be utilized in the following sections of this work.

Definition 2.1 ((CFOD) [14])

Assume a function $g: [m, \infty) \to \mathbb{R}$, then the left (CFOD) of order μ is defined by:

$$\mathcal{D}_{\mu}^{m}(g)(p) = \lim_{\epsilon \to 0} \frac{g\big(p + \epsilon (p-m)^{1-\mu}\big) - g(p)}{\epsilon} \tag{2}$$

for all $p > 0$ and $\mu \in (0,1]$. When $m = 0$, we write $\mathcal{D}_{\mu}$. If $\mathcal{D}_{\mu}^{m}(g)(p)$ exists on the interval (m,n), then we define:

$$\mathcal{D}_{\mu}^{m}(g)(m) = \lim_{p \to m^+} g^{(\mu)}(p)$$

The right (CFOD) of order $\mu \in (0,1]$, ending at n, is defined as:

$$ {}_{\mu}^{n}\mathcal{D}(g)(p) = \lim_{\epsilon \to 0} \frac{g\big(p + \epsilon (n-p)^{1-\mu}\big) - g(p)}{\epsilon} \tag{3}$$

If ${}_{\mu}^{n}\mathcal{D}(g)(p)$ exists on the interval (m,n), then we define:

$$ {}_{\mu}^{n}\mathcal{D}(g)(n) = \lim{}_{p \to n^-}^{n} \mathcal{D}_{\mu}(g)(p)$$

Definition 2.2 ((CFOI) [15])

Let $g: [m, \infty) \to \mathbb{R}$ be a continuous function. The left (CFOI) of order μ of g is:

$$I_{\mu}^{m}(g)(p) = \int_m^p g(r) d\mu(r,m) = \int_m^p \frac{g(r)}{(r-m)^{1-\mu}} dr \tag{4}$$

where the given integration is understood as a standard Riemann improper integral and $m \ge 0$.

and the right (CFOI) of order μ of g, is given by:

$$ {}_{\mu}^{n}I_p(g)(p) = \int_p^n g(r) d\mu(n,r) = \int_p^n \frac{g(r)}{(n-r)^{1-\mu}} dr, \tag{5}$$

The following are some important properties that related fractional derivative and integrals which are extremely beneficial in applied examples, where $\mathcal{D}^{\mu}$ and I^{μ} refers to conformable fractional order and integral correspondingly:

- $\mathcal{D}^{\mu}(c_1 g_1 + c_2 g_2) = c_1 \mathcal{D}^{\mu}(g_1) + c_2 \mathcal{D}^{\mu}(g_2)$, for all $c_1, c_2 \in \mathbb{R}$.
- $\mathcal{D}^{\mu}(p^k) = kp^{k-\mu}$, for all $k \in \mathbb{R}$.
- $\mathcal{D}^{\mu}(\lambda) = 0$, for any constant function $g(p) = \lambda$.
- $\mathcal{D}^{\mu}(g_1 g_2) = g_1 \mathcal{D}^{\mu}(g_2) + g_2 \mathcal{D}^{\mu}(g_1)$.
- $\mathcal{D}^{\mu}\left(\frac{g_1}{g_2}\right) = \frac{g_2 \mathcal{D}^{\mu}(g_1) - g_1 \mathcal{D}^{\mu}(g_2)}{g_2^2}$, $g_2 \neq 0$.
- If g is differentiable, then $\mathcal{D}^{\mu}(g)(p) = p^{1-\mu} \frac{dg}{dp}(p)$.
- $\mathcal{D}_{\mu}^{m} I_{\mu}^{m}(g)(p) = g(p)$, for $p > m$, where $g(p)$ is continuous for all $p \in$ domain(I_{μ}^{m}).
- ${}_{\mu}^{n}\mathcal{D}_{\mu}^{n} I_{p}(g)(p) = g(p)$, for $p < n$, where $g(p)$ is continuous for all $p \in$ domain(I_{μ}^{m}).

3. Variational Iteration Method

As mentioned earlier in the literature, the important component of the (VIM) is finding the exact solution to a mathematical problem, based on the hypothesis of the linearization, serves as a trial function or an initial approximation to determine the following subsequent approximate solution to the issue along with specific conditions [16]. To exemplify the core concept of the (VIM), consider the following general nonlinear equation in operator form [17]:

$$\mathcal{A}v(p) = h(p) \tag{6}$$

and assume that equation (1.1) can be decomposed as:

$$\mathcal{L}(v(p)) + \mathcal{N}(v(p)) = h(p),\ p \in [m, n] \tag{7}$$

where $\mathcal{A}$ is a general operative that can be divided into linear $(\mathcal{L})$ and non-linear $(\mathcal{N})$ components, and $h(p)$ is a function representing the non-homogeneous term.
Equation (7) can be solved iteratively by (VIM) along with the following associated correction functional:

$$v_{k+1}(p) = v_k(p) + \int_m^p \lambda(p, r)\{\mathcal{L}(v_k(r)) + \mathcal{N}(\tilde{v}_k(r)) - h(r)\}dr,\ k = 0,1, \ldots, \tag{8}$$

where λ represent the Lagrange multiplier, determined using variational theory. The subscript k represents the iteration number, v_k is the k-th approximated solution, and $\tilde{v}_k$ represented a restricted variation, i.e., $\delta \tilde{v}_k = 0$, where δ denotes the first variation [17].

4. The General Form of the (MDIDEs) with Fractional Derivative

The formulation of (VIM) will be presented to used later in evaluation of the approximate solution of (MDIDE) with conformable integrals and the Lagrange multiplier [18]. The state of the derivation and prove are present in the following theorem:

Theorem1

Assuming v, v_k the exact and approximate solutions respectively of the equation:

$$\mathcal{D}_p^{\mu} v(p,q) = h(p,q) + \int_a^p \int_b^q \mathcal{K}(p,q,r,s,v(r,s))dsdr,\ p \in [a,b], t \in [0,T] \quad (9)$$

Where $v, v_k \in C_p^d([a,b] \times [c,d])$ and $\mathcal{D}_p^{\mu}$ denotes the conformable fractional derivative.
If $E_k(p,q) = v_k(p,q) - v(p,q)$ and the kernel $\mathcal{K}$ satisfies the Lipschitz condition with a constant L such that:

$$L < \left[\frac{(\mu+1)(\mu+2)(2\mu+2)(2\mu+3)\cdots((k-1)\mu+k)(k\mu+k)}{(d-c)^k(b-a)^{k\mu+k}}\right]^{1/k}$$

then the approximate solutions v_k generated for $k = 0,1,...$, gradually converges to the exact solution v.

Proof:

Recall the multidimensional fractional partial (IDE) (9):

$$\mathcal{D}_p^{\mu} v(p,q) = h(p,q) + \int_a^p \int_b^q \mathcal{K}(p,q,r,s,v(r,s))dsdr.$$

Multiplying both sides by the Lagrange multiplier λ, we get:

$$\lambda(r,s)\left\{\mathcal{D}_p^{\mu} v(p,q) - h(p,q) - \int_a^p \int_b^q \mathcal{K}(p,q,r,s,v(r,s))dsdr\right\} = 0 \quad (10)$$

where λ is to be determined.
Applying $\mathcal{D}_p^{\mu}$ (the conformable fractional integral) to both sides of Equation (4.2), we obtain:

$$I_p^{\mu}\left[\lambda(r,s)\left\{\mathcal{D}_p^{\mu} v(p,q) - h(p,q) - \int_a^p \int_b^q \mathcal{K}(p,q,r,s,v(r,s))dsdr\right\}\right] = 0.$$

The correction functional related to p can now be formulated as:

$$v_{k+1}(p,q) = v_k(p,q) + I_p^{\mu}\left[\lambda(w,q)\left\{\mathcal{D}_w^{\mu} v(w,q) - h(w,q) - \int_a^w \int_c^q \mathcal{K}(w,q,r,s,\tilde{v}_k(r,s))dsdr\right\}\right] \quad (11)$$

where $\tilde{v}_k$ is a restricted variation.
Since determining λ directly is challenging due to the fractional derivative and integral, we approximate $\mathcal{D}_p^{\mu}$ and I_p^{μ} for $0 < \mu \leq 1$ using first-order derivatives and integrals. Thus, the correction functional becomes:

$$v_{k+1}(p,q) = v_k(p,q) + \int_m^p \left[\lambda(w,q)\left\{\begin{matrix} \frac{\partial}{\partial w} v(w,q) - h(w,q) \\ - \int_a^w \int_c^q \mathcal{K}(w,q,r,s,\tilde{v}_k(r,s))dsdr \end{matrix}\right\}\right] dw \quad (12)$$

By applying the first variation of Equation (4.4) with respect to v_k gives:

$$\delta v_{k+1}(p,q) = \delta v_k(p,q) + \delta \int_m^p \left[\lambda(w,q)\left\{\begin{matrix} \frac{\partial}{\partial w} v(w,q) - h(w,q) \\ - \int_a^w \int_c^q \mathcal{K}(w,q,r,s,\tilde{v}_k(r,s))dsdr \end{matrix}\right\}\right] dw$$

Using integration by parts, we find:

$$\delta v_{k+1}(p,q) = \delta v_k(p,q) + [(1+\lambda)\delta v_k(w,q)]_{w=p} - \int_a^p \frac{\partial \lambda(w,q)}{\partial w} \delta v_k(w,q)dw$$

This leads to the following necessary conditions with the initial condition:

$$\frac{\partial}{\partial w}\lambda(w,q)\Big|_{w=p} = 0, \text{ and } 1+\lambda|_{w=p} = 0 \tag{13}$$

Solving these equations gives $\lambda(w,q) = -1$. Substituting this value into Equation (12) yields:

$$v_{k+1}(p,q) = v_k(p,q) - I_p^{\mu}\left[\mathcal{D}_w^{\mu}v(w,q) - h(w,q) - \int_a^w \int_c^q \mathcal{K}(w,q,r,s,v(r,s))dsdr\right]$$

5. Convergence Analysis of (MDIDEs) with Fractional Derivative:

After (VIM) derived and studying the existence and uniqueness, it's essential to study the problem convergence including approximation with presence of the exact solution and using Lagrange multiplier to get more than one iteration [2].
This section will used the constructed (VIM) to embed and prove a theorem on the convergence of the approximated solution achieved.

Theorem 2

Let $v, v_k \in C_p^d([a,b]\times[c,d])$ be the exact and approximate solutions of Equation (1). If $E_k(p,q) = v_k(p,q) - v(p,q)$, and the kernel $\mathcal{K}$ satisfies the Lipschitz condition with a constant L such that:

$$L < \left[\frac{(\mu+1)(\mu+2)(2\mu+2)(2\mu+3)\cdots((k-1)\mu+k)(k\mu+k)}{(d-c)^k(b-a)^{k\mu+k}}\right]^{1/k},$$

then the approximate solutions v_k generated for $k = 0,1,...$, gradually converges to the exact solution v.

Proof:

Recall the multidimensional fractional partial (IDE) (9):

$$\mathcal{D}_p^{\mu}v(p,q) = h(p,q) + \int_a^p \int_b^q \mathcal{K}(p,q,r,s,v(r,s))dsdr.$$

with the initial condition $v(0,q) = v_0(q)$.
The approximate solution using (VIM) for Equation (1) is given by:

$$v_{k+1}(p,q) = v_k(p,q) - I_p^{\mu}\begin{bmatrix} \mathcal{D}_w^{\mu}v_k(w,q) - h(w,q) \\ -\int_a^w \int_c^q \mathcal{K}(w,q,r,s,v_k(r,s))dsdr \end{bmatrix} \tag{14}$$

Since $v(p,q)$ is the exact solution, it satisfies:

$$v(p,q) = v(p,q) - I_p^{\mu}\left[\mathcal{D}_w^{\mu}v(w,q) - h(w,q) - \int_a^w \int_c^q \mathcal{K}(w,q,r,s,v(r,s))dsdr\right] \tag{15}$$

Subtracting the exact solution from the approximate solution gives:

$$v_{k+1}(p,q)-v(p,q)$$
$$=v_k(p,q)-I_p^{\mu}\left[\mathcal{D}_w^{\mu}v_k(w,q)-h(w,q)-\int_a^w\int_c^q \mathcal{K}(w,q,r,s,v_k(r,s))dsdr\right]$$
$$-\left[v(p,q)-I_p^{\mu}\left[\mathcal{D}_w^{\mu}v(w,q)-h(w,q)-\int_a^w\int_c^q \mathcal{K}(w,q,r,s,v(r,s))dsdr\right]\right]$$

Let $E_k(p,q)=v_k(p,q)-v(p,q)$. Then:

$$E_{k+1}(p,q)=E_k(p,q)-I_p^{\mu}\left[\mathcal{D}_w^{\mu}E_k(w,q)-\int_a^w\int_c^q\begin{pmatrix}\mathcal{K}(w,q,r,s,v_k(r,s))\\-\mathcal{K}(w,q,r,s,v(r,s))\end{pmatrix}dsdr\right] \quad (16)$$

Since $I_p^{\mu}\mathcal{D}_p^{\mu}E_k(p,q)=E_k(p,q)-E_k(0,q)$, we have:

$$E_{k+1}(p,q)=I_p^{\mu}\int_a^p\int_c^q(\mathcal{K}(p,q,r,s,v_k(r,s))-\mathcal{K}(p,q,r,s,v(r,s)))dsdr \quad (17)$$

Taking the supremum norm, we get:

$$\|E_{k+1}(p,q)\|\le\int_a^p\frac{1}{(w-a)^{1-\mu}}\left[\int_a^w\int_c^q\left\|\begin{matrix}\mathcal{K}(w,q,r,s,v_k(r,s))\\-\mathcal{K}(w,q,r,s,v(r,s))\end{matrix}\right\|dsdr\right]dw$$

Using the Lipschitz condition, this becomes:

$$\|E_{k+1}(p,q)\|\le L\int_a^p\frac{1}{(w-a)^{1-\mu}}\left[\int_a^w\int_c^q\|E_k(r,s)\|dsdr\right]dw \quad (18)$$

For $k=0$, it follows that:

$$\|E_1(p,q)\|\le L(q-c)\frac{(p-a)^{\mu+1}}{\mu+1}\|E_0(p,q)\| \quad (19)$$

Similarly, for $k=1$:

$$\|E_2(p,q)\|\le L^2\frac{(q-c)^2}{2}\frac{(p-a)^{2\mu+2}}{(\mu+1)(\mu+2)(2\mu+2)}\|E_0(p,q)\| \quad (20)$$

By induction:

$$\|E_k(p,q)\|\le L^k\frac{(q-c)^k}{k!}\frac{(p-a)^{k\mu+k}}{(\mu+1)(\mu+2)\cdots((k-1)\mu+k)(k\mu+k)}\|E_0(p,q)\|$$

Taking the supremum norm over all p and q, we have:

$$\|E_k(p,q)\|\le L^k\frac{(d-c)^k}{k!}\frac{(b-a)^{k\mu+k}}{(\mu+1)(\mu+2)\cdots((k-1)\mu+k)(k\mu+k)}\|E_0(p,q)\|$$

Since $L<\left[\frac{(\mu+1)(\mu+2)\cdots((k-1)\mu+k)(k\mu+k)}{(d-c)^k(b-a)^{k+k}}\right]^{1/k}$, it follows that:

Therefore, $\lim_{k\to\infty}\|E_k\|\to 0$, i.e., $v_k(p,q)\to v(p,q)$.

6. Illustrative Examples:

Two illustrative examples are presented to demonstrate the utilization of the (VIM) in addressing both linear and nonlinear fractional order (IDE) with conformable fractional derivatives.

Example 1

Consider the linear (IDE) of fractional order problem and trying to find the solution:

$$\mathcal{D}_p^{\mu} v(p,q) = h(p,q) + \int_a^p \int_c^q (r-s)^2 v(r,s)dsdr$$

where $h(p,q) = 2p^{3-\mu}q + \frac{p^4q^3}{30} - \frac{p^3q^2}{15}$.
The exact solution for comparison purposes is $v(p,q) = p^2q$.
we begin with the initial approximation guess for the (VIM):

$$v_0(p,q) = h(p,q) = 2p^{3-\mu}q + \frac{p^4q^3}{30} - \frac{p^3q^2}{15}$$

Using the following iterative formula of (VIM) to find $v_1(p,q), v_2(p,q)$ and $v_3(p,q)$

$$v_{k+1}(p,q) = v_k(p,q) - I_p^{\mu}\left[\mathcal{D}_w^{\mu} v(w,q) - h(p,q) - \int_a^w \int_c^q (r-s)^2 v_k(r,s)dsdr\right],$$

We get:

$$v_1(p,q) = 1.0p^2q + 0.005p^{3.5}q^3 - 0.0025p^{4.5}q^2 + 0.00075p^{5.5}q^4$$

$$v_2(p,q) = 1.0p^2q + 0.0045p^{3.5}q^3 - 0.0018p^{4.5}q^2 + 0.0008p^{5.5}q^4 + 0.00015p^{6.5}q^5 - 0.000025p^{7.5}q^6$$

$$v_3(p,q) = 1.0p^2q + 0.0044p^{3.5}q^3 - 0.00175p^{4.5}q^2 + 0.00085p^{5.5}q^4 + 0.00016p^{6.5}q^5 - 0.000026p^{7.5}q^6 - 0.000004p^{8.5}q^7 + 0.0000008p^{9.5}q^8$$

(Table 1) below show the numerical exact solution v and approximate solutions v_0, v_1, v_2 and v_3:

p	q	Exact solution $v(p,q)$	Approximate solutions $v_0(p,q)$	Approximate solutions $v_1(p,q)$	Approximate solutions $v_2(p,q)$	Approximate solutions $v_3(p,q)$
0.2	0	0	0	0	0	0
0.2	0.2	8×10^{-3}	7.134×10^{-3}	8.00007×10^{-3}	8.00007×10^{-3}	8.00007×10^{-3}
0.2	0.4	16×10^{-3}	14.22891×10^{-3}	16.00086×10^{-3}	16.00082×10^{-3}	16.00081×10^{-3}
0.2	0.6	24×10^{-3}	21.28577×10^{-3}	24.00323×10^{-3}	24.00302×10^{-3}	24.00296×10^{-3}
0.2	0.8	32×10^{-3}	28.30764×10^{-3}	32.00805×10^{-3}	32.00746×10^{-3}	0.03200730974079
0.2	1.0	40×10^{-3}	35.29708×10^{-3}	40.01620×10^{-3}	40.01493×10^{-3}	40.01461×10^{-3}
0.4	0	0	0	0	0	0
0.4	0.2	32×10^{-3}	40.3133×10^{-3}	32×10^{-3}	32.00029×10^{-3}	32.00030×10^{-3}
0.4	0.4	64×10^{-3}	80.32625×10^{-3}	64.00660×10^{-3}	64.00713×10^{-3}	64.00700×10^{-3}
0.4	0.6	96×10^{-3}	0.12007978215046570	0.096029773051615	0.0960295525902	0.09602901360777
0.4	0.8	128×10^{-3}	159.61485×10^{-3}	128.07970×10^{-3}	128.07685×10^{-3}	128.07543×10^{-3}

0.4	1.0	160×10^{-3}	198.97243×10^{-3}	160.16676×10^{-3}	160.15854×10^{-3}	160.15565×10^{-3}
0.6	0	0	0	0	0	0
0.6	0.2	72×10^{-3}	111.00048×10^{-3}	71.99672×10^{-3}	71.99887×10^{-3}	71.99894×10^{-3}
0.6	0.4	144×10^{-3}	221.05632×10^{-3}	144.01454×10^{-3}	144.02056×10^{-3}	144.02037×10^{-3}
0.6	0.6	216×10^{-3}	330.37488×10^{-3}	216.09620×10^{-3}	216.10421×10^{-3}	216.10282×10^{-3}
0.6	0.8	288×10^{-3}	439.16352×10^{-3}	288.28620×10^{-3}	288.29121×10^{-3}	288.28719×10^{-3}
0.6	1.0	360×10^{-3}	547.62960×10^{-3}	360.63076×10^{-3}	360.62527×10^{-3}	360.61686×10^{-3}
0.8	0	0	0	0	0	0
0.8	0.2	128×10^{-3}	$227.717254 \times 10^{-3}$	127.98203×10^{-3}	127.99049×10^{-3}	127.99088×10^{-3}
0.8	0.4	256×10^{-3}	453.35920×10^{-3}	256.00562×10^{-3}	256.03272×10^{-3}	256.03311×10^{-3}
0.8	0.6	384×10^{-3}	677.58120×10^{-3}	384.19334×10^{-3}	384.24062×10^{-3}	384.23938×10^{-3}
0.8	0.8	512×10^{-3}	901.03861×10^{-3}	512.6762×10^{-3}	512.73939×10^{-3}	512.73428×10^{-3}
0.8	1.0	640×10^{-3}	1124.3868×10^{-3}	641.59365×10^{-3}	641.66626×10^{-3}	641.65509×10^{-3}
1.0	0	0	0	0	0	0
1.0	0.2	2×10^{-1}	3.97×10^{-1}	1.99941×10^{-1}	1.99965×10^{-1}	1.99966×10^{-1}
1.0	0.4	4×10^{-1}	7.91466×10^{-1}	3.99939×10^{-1}	4.00021×10^{-1}	4.00024×10^{-1}
1.0	0.6	6×10^{-1}	11.832×10^{-1}	6.00277×10^{-1}	6.00438×10^{-1}	6.00441×10^{-1}
1.0	0.8	8×10^{-1}	15.744×10^{-1}	8.01267×10^{-1}	8.01522×10^{-1}	8.01525×10^{-1}
1.0	1.0	10×10^{-1}	19.66666×10^{-1}	10.0325×10^{-1}	10.03625×10^{-1}	10.036308×10^{-1}

Table 1: The exact and the approximate solutions, $\mu = 0.5$.

p	q	$\lvert v - v_0\rvert$	$\lvert v - v_1\rvert$	$\lvert v - v_2\rvert$	$\lvert v - v_3\rvert$
0.2	0	0	0	0	0
0.2	0.2	8.65×10^{-4}	7.17×10^{-8}	7.74×10^{-8}	7.60×10^{-8}
0.2	0.4	1.77×10^{-3}	8.61×10^{-7}	8.27×10^{-7}	8.10×10^{-7}
0.2	0.6	2.71×10^{-3}	3.23×10^{-6}	3.02×10^{-6}	2.96×10^{-6}
0.2	0.8	3.69×10^{-3}	8.05×10^{-6}	7.46×10^{-6}	7.30×10^{-6}
0.2	1.0	4.70×10^{-3}	1.62×10^{-5}	1.49×10^{-5}	1.46×10^{-5}
0.4	0	0	0	0	0
0.4	0.2	8.31×10^{-3}	7.77×10^{-9}	2.99×10^{-7}	3×10^{-7}
0.4	0.4	1.63×10^{-2}	6.60×10^{-6}	7.13×10^{-6}	7×10^{-6}
0.4	0.6	2.4×10^{-2}	2.97×10^{-5}	2.95×10^{-5}	2.90×10^{-5}
0.4	0.8	3.16×10^{-2}	7.97×10^{-5}	7.68×10^{-5}	7.54×10^{-5}
0.4	1.0	3.89×10^{-2}	1.67×10^{-4}	1.59×10^{-4}	1.56×10^{-4}
0.6	0	0	0	0	0
0.6	0.2	3.9×10^{-2}	3.27×10^{-6}	1.12×10^{-6}	1.05×10^{-6}
0.6	0.4	7.7×10^{-2}	1.45×10^{-5}	2.05×10^{-5}	2.03×10^{-5}

0.6	0.6	1.14×10^{-1}	9.62×10^{-5}	1.04×10^{-4}	1.03×10^{-4}
0.6	0.8	1.51×10^{-1}	2.86×10^{-4}	2.91×10^{-4}	2.87×10^{-4}
0.6	1.0	1.88×10^{-1}	6.31×10^{-4}	6.25×10^{-4}	6.17×10^{-4}
0.8	0	0	0	0	0
0.8	0.2	9.97×10^{-2}	1.79×10^{-5}	9.50×10^{-6}	9.11×10^{-6}
0.8	0.4	1.97×10^{-1}	5.62×10^{-6}	3.27×10^{-5}	3.31×10^{-5}
0.8	0.6	2.94×10^{-1}	1.93×10^{-4}	2.4×10^{-4}	2.39×10^{-4}
0.8	0.8	3.89×10^{-1}	6.76×10^{-4}	7.39×10^{-4}	7.34×10^{-4}
0.8	1.0	4.84×10^{-1}	1.59×10^{-3}	1.67×10^{-3}	1.66×10^{-3}
1.0	0	0	0	0	0
1.0	0.2	1.98×10^{-1}	5.87×10^{-5}	3.46×10^{-5}	3.33×10^{-5}
1.0	0.4	3.91×10^{-1}	6.08×10^{-5}	2.19×10^{-5}	2.48×10^{-5}
1.0	0.6	0.5832	2.77×10^{-4}	4.38×10^{-4}	4.41×10^{-4}
1.0	0.8	0.7744	1.27×10^{-3}	1.52×10^{-3}	1.53×10^{-3}
1.0	1.0	9.67×10^{-1}	3.25×10^{-3}	3.62×10^{-3}	3.63×10^{-3}

Table 2: The absolute errors for the approach solutions proportional to the exact solution

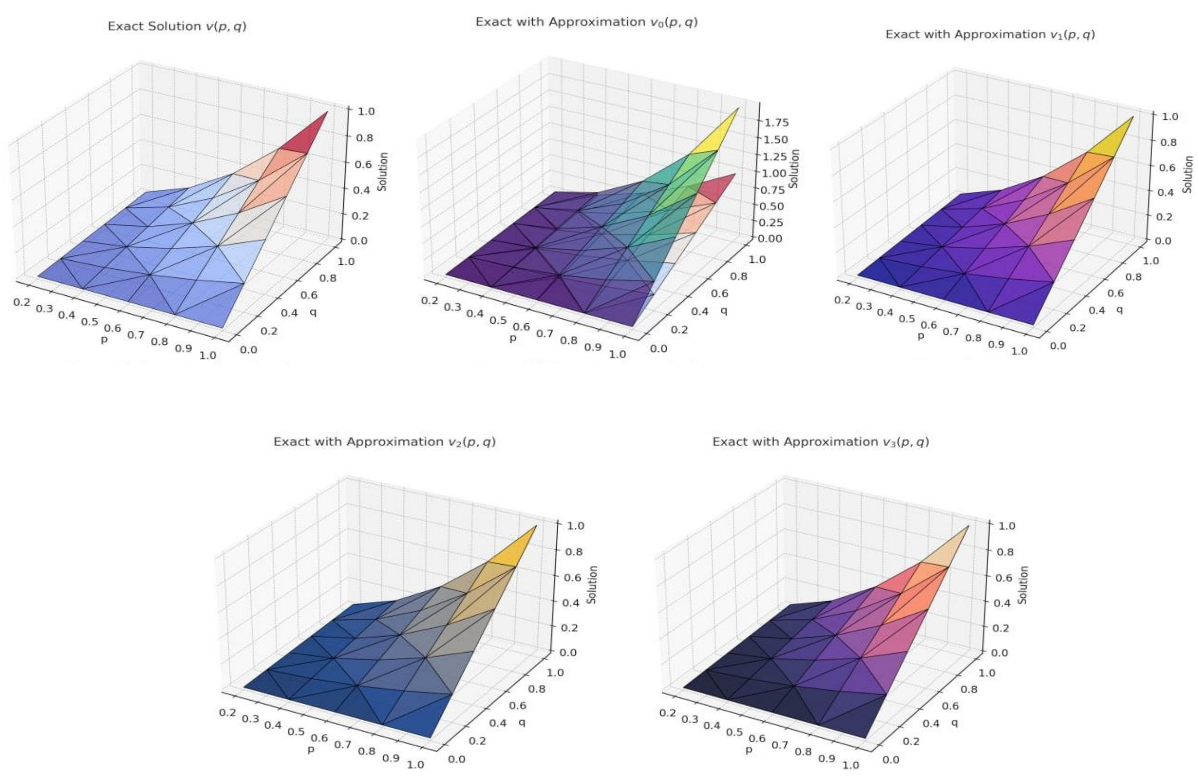

Fig. 1 Exact and approach solutions

Example 2

Consider the nonlinear (IDE) of fractional order:

$$\mathcal{D}_p^{\mu} v(p,q) = h(p,q) + \int_a^p \int_c^q (r^2+s^2)(1+2v_k(r,s))dsdr,$$

when:

$$h(p,q) = p^2\mathrm{q} + \frac{p^3q^2}{2} - \frac{p^4q^3}{3}$$

and the exact solution is:

$$v(p,q) = p^2\mathrm{q} + p^3\mathrm{q}^2.$$

Using (VIM), the iterations are given by:

$$v_{k+1}(p,q) = v_k(p,q) - I_p^{\mu}\left[D_p^{\mu} v(p,q) - h(p,q) - \int_a^p \int_c^q (r^2+s^2)(1+2v_k(r,s))dsdr\right].$$

The initial approximate solution which will be used in the VIM is assumed to be:

$$v_0(p,q) = h(p,q) = p^2\mathrm{q} + \frac{p^3q^2}{2} - \frac{p^4q^3}{3}$$

Now, using the (VIM), the first few iterations are found to be:

First Iteration $v_1(p,q)$:

$$v_1(p,q) = p^2q + \frac{p^3q^2}{2} + \frac{p^4q^3}{4} - \frac{p^5q^4}{5}.$$

Second Iteration $v_2(p,q)$:

$$v_2(p,q) = p^2q + \frac{p^3q^2}{2} + \frac{p^4q^3}{4} + \frac{p^5q^4}{8} - \frac{p^6q^5}{6}.$$

Third Iteration $v_3(p,q)$:

$$v_3(p,q) = p^2q + \frac{p^3q^2}{2} + \frac{p^4q^3}{4} + \frac{p^5q^4}{8} - \frac{p^6q^5}{6} + \frac{p^7q^6}{7}.$$

(Table 3) below show the numerical exact solution v and approximate solutions v_0, v_1, v_2 and v_3:

p	q	**Exact solutions** $v(p,q)$	**Approximate solutions** $v_0(p,q)$	**Approximate solutions** $v_1(p,q)$	**Approximate solutions** $v_2(p,q)$	**Approximate solutions** $v_3(p,q)$
0.2	0	0	0	0	0	0
0.2	0.2	8.32×10^{-3}	8.15573×10^{-3}	8.16309×10^{-3}	8.16326×10^{-3}	8.16326×10^{-3}
0.2	0.4	17.28×10^{-3}	16.60586×10^{-3}	16.66396×10^{-3}	16.66651×10^{-3}	16.66652×10^{-3}
0.2	0.6	2.688×10^{-2}	2.53248×10^{-2}	2.55181×10^{-2}	2.55308×10^{-2}	2.55308×10^{-2}
0.2	0.8	3.712×10^{-2}	3.42869×10^{-2}	3.47386×10^{-2}	3.47777×10^{-2}	3.47782×10^{-2}
0.2	1.0	4.8×10^{-2}	4.34667×10^{-2}	4.4336×10^{-2}	4.44293×10^{-2}	4.44312×10^{-2}
0.4	0	0	0	0	0	0
0.4	0.2	3.456×10^{-2}	3.32117×10^{-2}	3.33279×10^{-2}	3.3333×10^{-2}	3.3333×10^{-2}
0.4	0.4	7.424×10^{-2}	6.85739×10^{-2}	6.94772×10^{-2}	6.95554×10^{-2}	6.95563×10^{-2}
0.4	0.6	1.1904×10^{-1}	1.056768×10^{-1}	1.08637×10^{-1}	1.09015×10^{-1}	1.09026×10^{-1}
0.4	0.8	1.6896×10^{-1}	1.44111×10^{-1}	1.50918×10^{-1}	1.52057×10^{-1}	1.52119×10^{-1}

0.4	1.0	2.24×10^{-1}	1.83467×10^{-1}	1.96352×10^{-1}	1.98997×10^{-1}	1.99231×10^{-1}
0.6	0.0	0	0	0	0	0
0.6	0.2	8.064×10^{-2}	7.59744×10^{-2}	7.65543×10^{-2}	7.65923×10^{-2}	7.65925×10^{-2}
0.6	0.4	1.7856×10^{-1}	1.58515×10^{-1}	1.62955×10^{-1}	1.63523×10^{-1}	1.63539×10^{-1}
0.6	0.6	2.9376×10^{-1}	2.45549×10^{-1}	2.59863×10^{-1}	2.62533×10^{-1}	2.6272×10^{-1}
0.6	0.8	4.2624×10^{-1}	3.35001×10^{-1}	3.67339×10^{-1}	3.75142×10^{-1}	3.7619×10^{-1}
0.6	1.0	5.76×10^{-1}	4.248×10^{-1}	4.84848×10^{-1}	5.02344×10^{-1}	5.06343×10^{-1}
0.8	0	0	0	0	0	0
0.8	0.2	1.4848×10^{-1}	1.37148×10^{-1}	1.38954×10^{-1}	1.39111×10^{-1}	1.391127×10^{-1}
0.8	0.4	3.3792×10^{-1}	2.88229×10^{-1}	3.01836×10^{-1}	3.04115×10^{-1}	3.04237×10^{-1}
0.8	0.6	5.6832×10^{-1}	4.4667×10^{-1}	4.89785×10^{-1}	5.00189×10^{-1}	5.01587×10^{-1}
0.8	0.8	8.3968×10^{-1}	6.05935×10^{-1}	7.01425×10^{-1}	7.30729×10^{-1}	7.38583×10^{-1}
0.8	1.0	1.152	7.59467×10^{-1}	9.32864×10^{-1}	9.95669×10^{-1}	1.02563
1.0	0	0	0	0	0	0
1.0	0.2	2.4×10^{-1}	2.17333×10^{-1}	2.2168×10^{-1}	2.22147×10^{-1}	2.22156×10^{-1}
1.0	0.4	5.6×10^{-1}	4.58667×10^{-1}	4.9088×10^{-1}	4.97493×10^{-1}	4.98078×10^{-1}
1.0	0.6	9.6	7.08×10^{-1}	8.0808×10^{-1}	8.3724×10^{-1}	8.43905×10^{-1}
1.0	0.8	1.44	9.49333	1.16608	1.24459	1.28204
1.0	1.0	2.0	1.16667	1.55	1.70833	1.85119

Table 3: The exact and the approximate solutions, $\mu = 0.5$

p	q	$\lvert v-v_0\rvert$	$\lvert v-v_1\rvert$	$\lvert v-v_2\rvert$	$\lvert v-v_3\rvert$
0.2	0	0	0	0	0
0.2	0.2	5.2066×10^{-2}	5.207×10^{-2}	5.2066×10^{-2}	5.2066×10^{-2}
0.2	0.4	1.8×10^{-2}	8.068×10^{-6}	5.577×10^{-6}	5.577×10^{-6}
0.2	0.6	2.7×10^{-2}	5.933×10^{-5}	4.651×10^{-5}	4.651×10^{-5}
0.2	0.8	3.5×10^{-2}	2.398×10^{-4}	2.046×10^{-4}	2.046×10^{-4}
0.2	1	4.2×10^{-2}	7.135×10^{-4}	6.404×10^{-4}	6.404×10^{-4}
0.4	0	0	0	0	0
0.4	0.2	4.8×10^{-2}	2.073×10^{-5}	$2.56\times\times10^{-6}$	$2.558\times\times10^{-6}$
0.4	0.4	9.6×10^{-2}	6.647×10^{-6}	1.712×10^{-5}	1.711×10^{-5}
0.4	0.6	1.43×10^{-1}	2.683×10^{-4}	2.121×10^{-4}	2.119×10^{-4}
0.4	0.8	1.89×10^{-1}	1.3×10^{-3}	1.008×10^{-3}	1.008×10^{-3}

0.4	1	2.29×10^{-1}	4.003×10^{-3}	3.259×10^{-3}	3.256×10^{-3}
0.6	0	0	0	0	0
0.6	0.2	1.24×10^{-1}	1.954×10^{-4}	3.178×10^{-5}	3.174×10^{-5}
0.6	0.4	2.48×10^{-1}	4.618×10^{-4}	4.416×10^{-5}	4.409×10^{-5}
0.6	0.6	3.72×10^{-1}	5.01×10^{-5}	3.578×10^{-4}	3.569×10^{-4}
0.6	0.8	4.92×10^{-1}	2.482×10^{-3}	2.269×10^{-3}	2.263×10^{-3}
0.6	1	6.03×10^{-1}	9.669×10^{-3}	7.905×10^{-3}	7.882×10^{-3}
0.8	0	0	0	0	0
0.8	0.2	2.41×10^{-1}	8.996×10^{-4}	1.695×10^{-4}	1.69×10^{-4}
0.8	0.4	4.83×10^{-1}	2.666×10^{-3}	4.651×10^{-4}	4.633×10^{-4}
0.8	0.6	7.24×10^{-1}	3.427×10^{-3}	8.922×10^{-5}	9.023×10^{-5}
0.8	0.8	9.61×10^{-1}	1.516×10^{-5}	3.088×10^{-3}	3.065×10^{-4}
0.8	1	1.184	1.3×10^{-2}	1.3×10^{-2}	1.3×10^{-2}
1	0	0	0	0	0
1	0.2	0.4	2.883×10^{-3}	6.073×10^{-4}	6.044×10^{-4}
1	0.4	8.01×10^{-1}	9.344×10^{-4}	1.938×10^{-3}	1.92×10^{-4}
1	0.6	1.203	1.5×10^{-2}	2.549×10^{-3}	2.532×10^{-3}
1	0.8	1.601	1.5×10^{-2}	1.155×10^{-4}	1.116×10^{-3}
1	1	1.983	1.636×10^{-3}	1.6×10^{-2}	1.6×10^{-2}

Table 4: The absolute errors for the approach solutions proportional to the exact solution

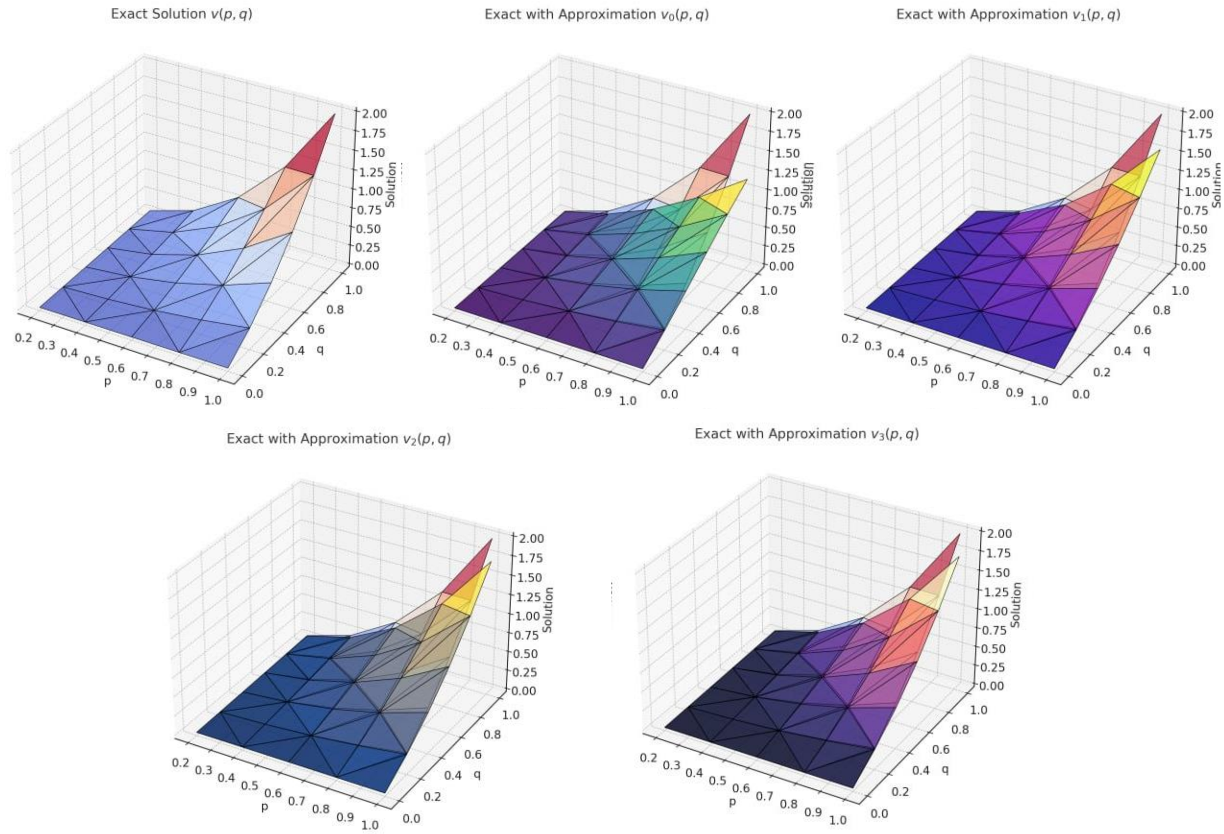

Fig. 2 Exact and approach solutions

7. Conclusions

The approximate solutions in the results tables, which obtained by using (VIM) on solving the (FOIDEs), the error tables and illustrative charts of the two given examples (linear and nonlinear) confirm the following fact:

- The iterative solutions $v_0(p,q)$, $v_1(p,q)$, $v_2(p,q)$ and $v_3(p,q)$ progressively converge to the exact solution $v(p,q) = p^2qv$.
- The numerical results confirm that the absolute errors decrease with each iteration, demonstrating the efficiency of (VIM).
- The approximate solution is close to the exact solution with reduce the errors to negligible levels for most points, showing the high precision achievable using VIM.
- For all tested values of p and q, the (VIM) method accurately approximates the solution with maintaining small absolute errors. The only note on the larger values of p and q is the slightly higher errors, but the iterative process effectively minimizes these discrepancies.
- The current work demonstrates that (VIM) is a precise, powerful and efficient method that provides the precise solve in a minimal number of iterations and proving the ability to handle fractional-order problems effectively and its potential as a reliable and robust tool for solving complex integral-differential equations.

However, a few circumstances, it necessitates extra computations, that will make the situation at hand more challenging. For future work, by considering fractional-order derivatives exceeding 1, this study could be enhanced in the time ahead integrating (IDEs) using kernels' that take into account fractional order derivatives of the unknown function.

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