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Best Proximity Point for Some Type of Cyclic Mapping in Strong b̲-Metric Space

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Abstract

The best proximity point generalization of fixed point that is beneficial when the contraction map is not a self-map. The aim of this paper is to introduce new types of proximal contraction for cyclic mapping in strong b̲-metric space that. Let (V,ρ) be complete strong b̲-metric space and let Γ,Λ be two nonempty closed subsets of V. Take the cyclic mapping $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$. If A is satisfies following condition $\rho(Ac,Ae)\le k(\rho(c,Ac)+\rho(e,Ae))+(1-2k)\rho(\Gamma,\Lambda)$ for all $c\in\Gamma$and $e\in\Lambda$, where $k\in(0,\frac{1}{2})$ then A is Kannan cyclic, if A is satisfies following condition $\rho(Ac,Ae)\le kmax\{\rho(c,e),\rho(Ac,c),\rho(Ae,e)\}+(1-k)\rho(\Gamma,\Lambda)$ for all $c\in\Gamma$ and $e\in\Lambda$, where $k\in(0,1)$ then A is Ćirić cyclic, if A is satisfies following condition $\rho(Ac,Ae)-\rho(\Gamma,\Lambda)\le\emptyset(\frac{\rho(c,Ac)+\rho(e,Ae)+\rho(c,Ae)+\rho(e,Ac)}{4}-\rho(\Gamma,\Lambda))$ for all $c\in\Gamma$ and $e\in\Lambda$, where $\emptyset$ Meir-Keeler mapping then A is Meir-Keeler -Kannan-Ghatterjee. The existeness and uniqueness of the best proximity point for thes types of mapping are proven, furthermore, some examples are offered to show the results usefulness.

Keywods: Strong b̲-metric space, best Proximity Point, Kennan mapping, Ćirić mapping, Meir-Keeler-Kannan-Ghatterjee mapping Contraction.

افضل نقطة تقارب لبعض الدوال الدائرية في الفضاء المتري القوي من نوع− b̲

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الخلاصة:

افضل نقاط تقارب هي تعميم النقطة الثابتة حيث يكون مفيدا عندما لا تكون دالة الانكماش دالة ذاتية في هذا البحث قدمنا انواع جديدة من نقاط التقارب للدوال الدائرية في الفضاء المتري القوي من نوع b̲. وليكون (V,ρ) هو فضاء متري قوي متكامل من نوع b̲ ولتكون Γ,Λ هي مجموعتين مغلقتين وجزئيتين من V وتسمى الدالة الدائرية $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$بدالة كنان الدائرية اذا حققت الشرط التالي

$\rho(Ac,Ae)\le k(\rho(c,Ac)+\rho(e,Ae))+(1-2k)\rho(\Gamma,\Lambda)$ for all $c\in\Gamma$and $e\in\Lambda$, where $k\in(0,\frac{1}{2})$

او بدالة كيرك الدائرية اذا حققت الشرط التالي

$\rho(Ac,Ae)\le kmax\{\rho(c,e),\rho(Ac,c),\rho(Ae,e)\}+(1-k)\rho(\Gamma,\Lambda))$ for all $c\in\Gamma$ and $e\in\Lambda$, where $k\in(0,1)$

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اوبدالة ميئر–كلير كنان– كاتريجي الدائرية اذا حققت الشرط التالي
$(Ac, Ae) - \rho(\Gamma, \Lambda) \leq \emptyset(\frac{\rho(c,Ac)+\rho(e,Ae)+\rho(c,Ae)+\rho(e,Ac)}{4} - \rho(\Gamma, \Lambda))$ for all $c \in \Gamma$ and $e \in \Lambda$, where $\emptyset$ Meir–Keeler mapping
وبرهنا وجود ووحدانية افضل نقطة تقارب لهذا نوع من الدوال وعلاوة عله ذلك اعطينا بعض الامثلة لاثبات امكانية تطبيق النتائج

1.Introduction

As we all know that the fixed point theory in metric space is a significant branch of nonlinear analysis. And it is a simplified method to consider the existence and uniqueness of solution of the operator equations $Ac = c$. In addition, it provides an approach for researching the solutions of some integral and differential equations in fields of mathematics and physics [1]. In order to solve the more complex nonlinear analysis problems, the concept of metric space has been extended in many aspects. In particular, the concept of b̲-metric space was reported by Bakhtin 1989, [2]. In 1993, Gzerwik [3] extended the results of b̲-metric space. It is clear that every metric space is b̲-metric space, but convers not true for example $\rho(c, e) = (c - e)^2$, let $c, e \in V$ where V is non-empty sets [3,4]. Note that b̲-metric may not continuous [5]. To remedy this deficiency kirik and Shahazad [5] introduced a strong b̲-metric space. On other hand, in 2003, Kirik et al. [6] introduced the cyclic contraction as a generalization of the usual contraction, which did not have to be continuous as Banach type contractions. In [7] Eldered and Veeramani are concerned with the case when $\Gamma \cap \Lambda = \emptyset$ where Γ, Λ are nonempty subsets of strong b̲-metric space and in this case they did not seek for the existence of fixed point of $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$ but for the existence of a best proximity point that is point defined by letting (V, ρ) be strong b̲-metric space and let Γ, Λ be two nonempty subsets of V, as well as by letting $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$. When $c \in \Gamma \cap \Lambda$ is said a best proximity point if $(c, Ac) = \rho(\Gamma, \Lambda)$, where $\rho(\Gamma, \Lambda) = \inf\{\rho(c, e): c \in \Gamma, e \in \Lambda\}$. In [8] Thagafi and Shahazad introduced a new class of mappings know as cyclic $\emptyset$ −contraction and they proved some convergence and existence results for best proximity points in 2011,Sadiq Basha [9] stated the best proximity point results, see [10-25]. We are concerned with the existence of best proximity points for cyclic Kannan and Ćirić, Meir-Keeler – Kannan-Ghatterjee contractions in the class of strong b̲-metric space. In what follows, one recall some notation and definitions that we will need in the sequel.

Definition 1.1: [4] Suppose that $V \neq \emptyset$ and λ be given real number. A function $\rho: V \times V \to (0, \infty^+)$ is called a strong b̲-metric if and only if for any c,e,$z \in V$ and $\lambda \geq 1$ the following conditions are satisfied:
W1) $\rho(c, e) = 0$ if and only if $c = e$,
W2) $\rho(c, e) = \rho(e, c)$,
W3) $\rho(c, z) \leq \rho(c, e) + \lambda\rho(e, z)$,
The pair (V, ρ) is said strong b̲-metric space (Sb̲M-S).
Definition 1.2:[5] Suppose that (V, ρ) be an Sb̲M-S. A sequence $\{c_n\}$ in V converges to $c \in V$ if and only if $\lim_{n\to\infty} \rho(c_n, c) = \rho(c, c)$.
Definition 1.3:[5] Suppose that (V, ρ) be an Sb̲M-S. A sequence $\{c_n\}$ in V is said to be Cauchy if and only if $\lim_{n.m\to\infty} \rho(c_n, c_m)$ exists and is finite.
Definition 1.4:[26] Let(V, ρ) be an Sb̲M-S and let Γ, Λ be two non-empty subsets of V and let $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$. $c \in \Gamma \cap \Lambda$ is said a best proximity point (bpp) if $\rho(c, Ac) = \rho(\Gamma, \Lambda)$, where $\rho(\Gamma, \Lambda) = \inf\{\rho(c, e): c \in \Gamma, e \in \Lambda\}$.

Definition 1.5:[6] Let (V,ρ) be a complete Sb̲M-S and let Γ,Λ be two non-empty closed subsets of V and $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ is called cyclic mapping (c.m) if $A(\Gamma)\subseteq\Lambda$ and $A(\Lambda)\subseteq\Gamma$.

Theorem 1.6:[12] Let (V,ρ) be a complete Sb̲M-S and Γ,Λ be two non-empty closed and convex subsets of V and let $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ be (c.m) suppose that for any $c\in\Gamma$ and $e\in\Lambda$, $\rho(Ac,Ae)\le\alpha\rho(c,e)+(1-\alpha)\rho(\Gamma,\Lambda)$, where $\alpha\in(1,0)$ and for $c_0\in\Gamma$ define $c_{n+1}=Ac_n$ for all $n\ge 0$. Then, there exists a unique $c\in\Gamma$ such that $c_{2n}\to c$ and

$$\rho(c,Ac)=\rho(\Gamma,\Lambda).$$

Here c is called bpp of A.

Definition 1.7:[27] Let (V,ρ) be a complete Sb̲M-S and let Γ,Λ be two non-empty closed subsets of V. Take the c.m $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$. A is said a cyclic Kannan type contraction if $\rho(Ac,Ae)\le k(\rho(c,Ac)+\rho(e,Ae))+(1-2k)\rho(\Gamma,\Lambda)$ for all $c\in\Gamma$and $e\in\Lambda$, where $k\in(0,\frac{1}{2})$.

Definition 1.8:[27] Let (V,ρ) be an Sb̲M-S and Γ,Λ be two non-empty closed subsets of V. Take the c.m $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$. A is said be a cyclic Ćirić type contraction if $\rho(Ac,Ae)\le kmax\{\rho(c,e),\rho(Ac,c),\rho(Ae,e)\}+(1-k)\rho(\Gamma,\Lambda)$ for all $c\in\Gamma$ and $e\in\Lambda$, where $k\in(0,1)$.

Definition 1.9:[27] Let (V,ρ) be a complete Sb̲M-S and Γ,Λ be two non-empty closed subsets of V. Take the c.m $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$. A is called a Ghatterjee type contraction if $\rho(Ac,Ae)\le k\big(\rho(Ac,e)+\rho(Ae,c)\big)+(1-4k)\rho(\Gamma,\Lambda)$. for all $c\in\Gamma$ and $e\in\Lambda$, where $k\in(0,\frac{1}{4})$.

Definition 1.10:[28] A function $\emptyset:(0,\infty^+)\to(0,\infty^+)$ is said to be a Meir-Keeler mapping, if $\emptyset$ is satisfies the following requirement:

$\forall\eta>0\ \exists\delta>0\ \forall t\epsilon R^+\ (\eta\le t<\eta+\delta\Rightarrow\emptyset(t)<\eta)$.

Definition 1.11:[28] Let (V,ρ) be complete Sb̲M-S, let $\Gamma.\Lambda$ be two nonempty sets and $\emptyset:(0,\infty^+)\to(0,\infty^+)$ be Meir-Keeler. A mapping $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ is said Meir-Keeler -Kannan-Ghatterjee contraction if the following conditions hold:

W1)$A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ is cyclic mapping.

W2) for any $c\in\Gamma$ and $\epsilon\Lambda$ $\rho(Ac,Ae)-\rho(\Gamma,\Lambda)\le\emptyset(\frac{\rho(c,Ac)+\rho(e,Ae)+\rho(c,Ae)+\rho(e,Ac)}{4}-\rho(\Gamma,\Lambda))$.

Lamme 1.12:[28] Let $(V.\rho)$ be a complete Sb̲M-S and let $\Gamma.\Lambda$ be two non-empty closed subsets of V and $\emptyset:(0,\infty^+)\to(0,\infty^+)$ be an increasing Meir-Keeler, and let $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ Meir-Keeler-Kannan-Ghatterjee contraction. For $c_0\in\Gamma\cup\Lambda$, define $c_{n+1}=Ac_n$ for each $n\in N\cup\{0\}$. Then $\rho(c_n,c_{n+1})\to\rho(\Gamma,\Lambda),n\to\infty$.

2. Main results

In the following theorem we prove the existence and uniqueness of bpp for Cyclic Kannan Mapping in an Sb̲M-S.

Theorem2.1. Let (V,ρ) be a complete Sb̲M-S and $\Gamma.\Lambda$ be two non-empty closed subsets of Vsuch that $\Gamma\cap\Lambda=\emptyset$ .Suppose $A:\Gamma\cup\Lambda\to\Gamma\cup\Lambda$ is a cyclic Kannan mapping then A has unique bpp.

Proof : Suppose that $c_0\in\Gamma\cup\Lambda$. Define $c_{n+1}=Ac_n$ for any $n\ge 0$. By Definition 1.7, we have

$$\rho(c_{n+2},c_{n+1})=\rho(Ac_{n+1},Ac_n)$$
$$\le k\big(\rho(c_{n+1},Ac_{n+1})+\rho(c_n,c_{n+1})\big)+(1-2k)\rho(\Gamma,\Lambda)$$
$$\le k\big(\rho(c_{n+1},c_{n+2})+\rho(c_n,c_{n+1})\big)+(1-2k)\rho(\Gamma,\Lambda)$$

$\leq k\rho\big((c_{n+1}, c_{n+2}) + \rho(c_n, c_{n+1})\big) + (1-2k)\rho(c_n, c_{n+1})$
$\leq k\rho(c_{n+1}, c_{n+2}) + (1-k)\rho(c_n, c_{n+1})$.
Thus,
$\rho(c_{n+2}, c_{n+1}) \leq \rho(c_{n+1}, c_n)$ for all $n \geq 0$.
That is, $\{\rho(c_{n+1}, c_n)\}$ is non-increasing and is bounded below, so there exists $t \geq 0$ such that $\lim_{n\to\infty} \rho(c_{n+1}, c_n) = t$. We know that
$\rho(\Gamma, \Lambda) \leq \rho(c_{n+1}, c_{n+2}) \leq k\big(\rho(c_{n+2}, c_{n+1}) + \rho(c_n, c_{n+1})\big) + (1-2k)\rho(\Gamma, \Lambda)$,
So, letting $n \to \infty$, we conclude that $t = \rho(\Gamma, \Lambda)$, i.e., $\lim_{n\to\infty} \rho(c_{n+1}, c_n) = \rho(\Gamma, \Lambda)$.
Suppose that $c_0 \in \Gamma$. Since A is cyclic, so $\{c_{2n}\} \in \Gamma$ and $\{c_{2n+1}\} \epsilon \Lambda$ for any $n \geq 0$. Now, if $\{c_{2n}\}$ has a subsequence $\{c_{2ni}\}$ converging to $u \in \Gamma$ with $\rho(u.u) = 0$ then
$\lim_{i\to\infty} \rho(c_{2ni}, u) = \rho(u.u) = 0$.
We have
$\rho(\Gamma, \Lambda) \leq \rho(u, Au) \leq \lambda\rho(u, c_{2ni}) + \rho(c_{2ni}, Au)$
$= \lambda\rho(u, c_{2ni}) + \rho(Ac_{2ni-1}, Au)$
$\leq \lambda\rho(u, c_{2ni}) + k\big(\rho(c_{2ni}, c_{2ni-1}) + \rho(Au, u)\big) + (1-2k)\rho(\Gamma, \Lambda)$.
Letting $\to \infty$,$\rho(c_n, c_{n+1}) = \rho(\Gamma, \Lambda)$ we obtain
$\rho(\Gamma, \Lambda) \leq \rho(u, Au) \leq k\rho(u, Au) + (1-k)\rho(\Gamma, \Lambda)$.
We have, $\rho(u, Au) = \rho(\Gamma, \Lambda)$, that is, u is bpp of A.
□

The Second main result is proving a mapping A has bpp when A is cyclic Ćirić type mapping in a SḇM-S.

Theorem2.2. Let (V, ρ)be a complete SḇM-S and Γ,Λ be two non-empty closed subsets of V such that $\Gamma \cap \Lambda = \emptyset$. Suppose that $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$ is cyclic Ćirić type mapping. Then A has bpp.

Proof: Suppose that $c_0 \in \Gamma \cup \Lambda$. Define $c_{n+1} = Ac_n$ for all $n \geq 0$. Since $\Gamma \cap \Lambda = \emptyset$, we have $\rho(\Gamma, \Lambda) > 0$. Then $\rho(c_{n+2}, c_{n+1}) > 0$ for any $n \geq 0$.
By Definition 1.8 we have

$$\rho(c_{n+2}, c_{n+1}) = \rho(Ac_{n+1}, Ac_n)$$
$$\leq k max\{\rho(c_{n+1}, c_n), \rho(Ac_{n+1}, c_{n+1}), \rho(Ac_n, c_n)\} + (1-k)\rho(\Gamma, \Lambda)$$

$= k max\{\rho(c_{n+1}, c_n), \rho(c_{n+2}, c_{n+1}), \rho(c_{n+1}, c_n)\} + (1-k)\rho(\Gamma, \Lambda)$
$= k max\{\rho(c_{n+1}, c_n), \rho(c_{n+2}, c_{n+1}), \rho(c_{n+1}, c_n)\} + (1-\mathrm{k})\rho(c_n, c_{n+1})$.
If for some n, we have $\max\{\rho(c_{n+1}, c_n), \rho(c_{n+2}, c_{n+1})\} = \rho(c_{n+2}, c_{n+1})$. Then,
$0 < \rho(c_{n+2}, c_{n+1}) \leq k\rho(c_{n+2}, c_{n+1}) + (1-2k)\rho(\Gamma, \Lambda) \leq (1-k)\rho(c_{n+2}, c_{n+1})$.
It is contradiction. Thus $\rho(c_{n+2}, c_{n+1}) \leq \rho(c_{n+1}, c_n)$ for any $n \geq 0$.
So, there exist $t \geq 0$ such that $\lim_{n\to\infty} (c_{n+1}, c_n) = t$. We know that
$\rho(\Gamma, \Lambda) \leq \rho(c_{n+2}, c_{n+1}) \leq k\rho(c_{n+1}, c_n) + (1-k)\rho(\Gamma, \Lambda)$.
So, letting $n \to \infty$, we conclude that $t = \rho(\Gamma, \Lambda)$, i.e., $\lim_{n\to\infty} \rho(c_{c+1}, c_n) = \rho(\Gamma, \Lambda)$.
Suppose that $c_0 \in \Gamma$. Again, Γ is cyclic, so $\{c_{2n}\} \in \Gamma$ and $\{c_{2n+1}\} \in \Lambda$ for any $n \geq 0$. Now, if $\{c_{2n}\}$ has subsequence $\{c_{2ni}\}$ converging to $u \in \Gamma$ with $\rho(u.u) = 0$ then

$$\lim_{i\to\infty} \rho(c_{2ni}, u) = \rho(u, u) = 0.$$

We have

$$\rho(\Gamma, \Lambda) \leq \rho(u, Au) \leq \lambda\rho(u, c_{2ni}) + \rho(c_{2ni}, Au)$$
$$= \lambda\rho(u, c_{2ni}) + \rho(Ac_{2ni-1}, Au)$$
$$\leq \lambda\rho(u, e_{2ni}) + k max\{\rho(e_{2n_i-1}, u), \rho(e_{2n_i}, u), \rho(Au, u)\} + (1-k)\rho(\Gamma, \Lambda)$$

Letting $i \to \infty$, $\rho(c_n, c_{n+1}) = \rho(\Gamma, \Lambda)$, we obtain
$\rho(\Gamma, \Lambda) \leq \rho(u, Au) \leq k\rho(u, Au) + (1-k)\rho(\Gamma, \Lambda)$.

We have, $\rho(u, Au) = \rho(\Gamma, \Lambda)$, thus, u is bpp of A.
□

The next theorem shows that a mapping A has bpp when A is a cyclic Meir-Keeler-Kannan-Ghatterjee.

Theorem 2.3. Let $(V.\rho)$ be a complete SḇM-S and let $\Gamma.\Lambda$ be two non-empty closed subsets of V and $\emptyset: (0, \infty^+) \to (0, \infty^+)$ be an increasing Meir-Keeler, and let $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$ Meir-Keeler-Kannan-Ghatterjee. Then A has bpp.
Proof: Suppose that $c_0 \in \Gamma$ since A is cyclic $c_{2n} \in \Gamma$ and $c_{2n+1} \in \Lambda$ for all $n \in N$.Now, if $\{c_{2n}\}$ has subsequence $\{c_{2nk}\}$ converge to $u \in \Gamma$ with $\rho(u, u) = 0$, then
$\lim_{n\to\infty} \rho(c_{2n}, u) = \rho(u, u) = 0.$
Since A is cyclic Meir-Keeler-Kannan-*G*hatterjee contraction and $\emptyset$ is increasing Meir-Keeler mapping. We have

$$\rho(u, Au) - \rho(\Gamma, \Lambda) \leq \lambda\rho(u, c_{2nk}) + \rho(c_{2nk}, Au) - \rho(\Gamma, \Lambda)$$
$$\leq \lambda\rho(u, c_{2nk}) + \rho(Ac_{2n-1}, Au) - \rho(\Gamma, \Lambda)$$
$$\leq \lambda\rho(u, c_{2nk}) + \emptyset\left(\frac{\rho(c_{2nk-1}, c_{2nk}) + \rho(u, Au) + \rho(c_{2nk-1}, Au) + \rho(u, Ac_{2nk-1})}{4} - \rho(\Gamma, \Lambda)\right)$$
$$\leq \lambda\rho(u, c_{2nk}) + \emptyset\left(\frac{\rho(c_{2nk-1}, c_{2nk}) + \rho(u, Au) + \rho(c_{2nk-1}, Au) + \rho(u, c_{2nk})}{4} - \rho(\Gamma, \Lambda)\right)$$
$$\leq \lambda\rho(u, c_{2nk}) + (\frac{2\rho(c_{2nk-1}, c_{2nk}) + 2\rho(u, Au) + 2\rho(u, c_{2nk})}{4} - \rho(\Gamma, \Lambda))$$
$$< \lambda\rho(u, c_{2nk}) + \left(\frac{\rho(c_{2nk-1}, c_{2nk}) + \rho(u, Au) + \rho(u, c_{2nk})}{2} - \rho(\Gamma, \Lambda)\right),$$

Letting $k \to \infty$, by Lamme1.12. we obtain

$$\rho(u, Au) - \rho(\Gamma, \Lambda) < \frac{\rho(\Gamma, \Lambda) + \rho(u, Au)}{2} - \rho(\Gamma, \Lambda) = \frac{\rho(u, Au) - \rho(\Gamma, \Lambda)}{2},$$

Thus, we conclude that $\rho(u, u) = \rho(\Gamma, \Lambda)$ that is, u is bpp of A.
□

3. Applications

The example below satisfies the Theorems 2.1
Example 3.1. Let $V = \{0,1,2,3\}$ be an endowed with the SḇM-S ρ
$\rho(0,0) = \rho(1,1) = \rho(2,2) = \rho(3,3) = 0$ and $\rho(1,2) = \rho(2,1) = 2, \rho(1,3) = \rho(3,1) = 6, \rho(2,3) = \rho(3,2) = 1, \rho(2,0) = \rho(0,2) = 2, \rho(3,0) = \rho(0,3) = 6, \rho(1,0) = \rho(0,1) = 1.$
(V, ρ) is complete SḇM-S. Take $\Gamma = \{0\}$ and $\Lambda = \{1,2\}$, we have $\rho(\Gamma, \Lambda) = 1$ choose $A: \Gamma \cup \Lambda \to \Gamma \cup \Lambda$ since $A(0) = 1. A(1) = A(2) = 0$. We have $A(\Gamma) = \{1\} \subset \Lambda$ with $A(\Lambda) = \{0\} = \Gamma$. Assume that $k\epsilon(0, \frac{1}{2})$, and $c\epsilon\Gamma$ and $e\epsilon\Lambda$ then $c = 0$ and $e\epsilon\{1,2\}$ in this case

$$\rho(Ac, Ae) - \rho(1,0) - 1 - 2k + 1 - 2k$$
$$\leq k(e + 1) + (1 - 2k) = k(0 + 1 + e + 0) + (1 - 2k)\rho(\Gamma, \Lambda)$$
$$= k\big(\rho(c, Ac) + \rho(e, Ae)\big) + (1 - 2k)\rho(\Gamma, \Lambda).$$

Thus by Definition 1.7 the result hold for any $c \in \Gamma$ and $e \in \Lambda$. Now, choose $c_0\epsilon\Gamma$ then $c_{2n} = 0$ and $c_{2n+1=}1$ for any $n \geq 0$. While if $c_0 \in \Lambda$ then $c_{2n} = 1$ for any $n \geq 1$. And $c_{2n+1} = 0$ for any $n \geq 0$. We conclude that for any $n \geq 1$

$$\rho(c_n, c_{n+1}) = \rho(1,0) = 1 = \rho(\Gamma, \Lambda),$$

That is Theorem 2.1 is verified. In the case $c_0 \in \Gamma$, we have $c_{2n} = 0$ so it has a subsequence $\{c_{2ni}\}$ converge to $u = 0 \in \Gamma$. Here $\rho(0, A_0) = 1 = \rho(\Gamma, \Lambda)$ in fact for $c = 0$ and $e = 2$ .On other hand, we have $\rho(Ac, Ae) = 1 > 3\alpha = \alpha(\rho(Ac, c) + \rho(Ae, e))$ for any $\alpha \in (0, \frac{1}{3})$.

□

The next example fulfills the condition of Theorems 2.2.

Example 3.2. Let $V = [0,\infty) \times [0,\infty)$ endowed with the Sb̲M-S $\rho: V \times V \to [0,\infty)$ given as $\rho\big((c_1, c_2), (e_1, e_2)\big) = |c_1 - e_1| + |c_2 - e_2|$ if $(c_1, c_2), (e_1, e_2) \in R$

It is easy to prove that (V, ρ) is a complete Sb̲M-S. Take $\Gamma = \{0\} \times [0,1]$ and $\Lambda = \{1\} \times [0,1]$. Remark that

$\rho(\Gamma, \Lambda) = \inf\{\big(\rho(c_1, c_2), \rho(e_1, e_2)\big) : (c_1, c_2) \in \Gamma, (e_1, e_2) \in \Lambda$.

Defined by

$$A(c, e) = \begin{cases} \left(1, \frac{e}{4}\right), & (c,e) \in \Gamma; \\ \left(0, \frac{e}{4}\right), & (c,e) \in \Lambda, \end{cases}$$

In this case, we have

$$\rho\big(A(0, c), A(1, e)\big) = 1 + \left|\frac{c}{4} - \frac{e}{4}\right|,$$

Moreover,

$$k max\{\rho\big((0, c), (1, e)\big), \rho\big((0, c), (0, c)\big), \rho\big((1, e), (1, e)\big)\} + (1 - k)\rho(\Gamma, \Lambda)$$
$$= k max\left\{1 + |c - e|, 1 + \left|c - \frac{c}{4}\right|, 1 + \left|e - \frac{e}{4}\right|\right\} + (1 - k)$$

$= 1 + k max\{|c - e|, \frac{3c}{4}, \frac{3e}{4}\}$,

If we have $\max\left\{|c - e|, \frac{3c}{4}, \frac{3e}{4}\right\} = \frac{3c}{4}$ or $\frac{3e}{4}$ contradiction. Then $\max\left\{|c - e|, \frac{3c}{4}, \frac{3e}{4}\right\} = |c - e|$. It is clear that Definition 1.8 is A cyclic Ćirić type contraction. By theorem 2.2 A has unique bpp .

□

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