DOI: 10.24996/ijs.2025.66.5.19

ISSN: 0067-2904

A New Iteration Process for Approximate Common Fixed Points for Three Non-Expansive Mapping

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Received: 25/2/2024 Accepted: 28/6/2024 Published: 30/5/2025

Abstract
A point that stays unchanged during a transformation is known as a fixed point(Fp). Such points play an important part in several mathematical domains. Applying Fp theory approaches allows for the efficient derivation of suitable solutions for operator equations that describe phenomena in several nonlinear scientific disciplines. The goal of solving these equations is to identify the Fp and its approximation. In this paper, a new iteration process is provided to approximate common fixed points (CFp) for non-expansive mapping (Non-exp map). Some convergence theorems have been established and the results obtained are confirmed with examples and tables. A numerical example is given to demonstrate that a novel process converges more rapidly than other existing iteration procedures.

Keywords: Banach space, Converge sequence, Fixed point, Iteration process, Non-expansive mapping.

طريقة تكرارية جديدة لتقريب النقاط الصامدة المشتركة لثلاث تطبيقات غير واسعة

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الخلاصة
تُعرف النقطة التي تظل دون تغيير أثناء التحويل بالنقطة الصامدة. تلعب مثل هذه النقاط دورًا مهمًا في العديد من المجالات الرياضية. يسمح تطبيق مناهج نظرية النقطة الصامدة بالاشتقاق الفعال للحلول المناسبة لمعادلات المؤثر التي تصف الظواهر في العديد من التخصصات العلمية غير الخطية. الهدف من حل هذه المعادلات هو تحديد النقطة الصامدة وتقريبها. في هذا البحث، يتم تقديم عملية تكرار جديدة لتقريب النقاط الصامدة المشتركة للدوال غير الموسعة. وقد تم التوصل إلى بعض نظريات التقارب وتأكيد النتائج التي تم الحصول عليها بالأمثلة والجداول. تم تقديم مثال عددي لإثبات أن الطريقة الجديدة تتقارب بسرعة أكبر من الطرق التكرارية الموجودة الأخرى.

1. Introduction

In nonlinear functional analysis, Fp theory is an exceptionally difficult and swiftly expanding field. Fp theorems explicitly refer to conclusions concerning the existence of Fps.

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Proving the existence and uniqueness of solutions to a variety of mathematical models is an essential utilization of these theorems. Fp theory is the subject of an extensive and expanding set of literature [1–6].

However, the goal of Fp theory is to approximate Fps in various domains for nonlinear mappings through the use of distinct iterative processes. Scientists from all over the world are drawn to Fp theory due to its significance, and in recent years, numerous iterative processes have been developed in various domains. Shahram Rezapour et al.[7] present a modified F-iterative procedure to determine the CFp of three generalized α- Non-exp maps. In [8] Chanchal Garodia et al. presented a three-step iteration approach for the purpose of approximating CFp for two Non-exp maps. A novel modified iterative approach is proposed by Iqbal et al [9] to approximate a CFp of two G- Non-exp maps. A new iteration approach and some convergence findings were obtained by Thakur et al. [10]. Their iteration is described as:
Consider $\mathcal{P}: \mathcal{B} \rightarrow \mathcal{B}$ be three Non-exp maps, then the sequence $\{u_n\}$ is produced iteratively by $u_1 \in \mathcal{B}$ and
$\omega_n = (1-\gamma_n)u_n + \gamma_n \mathcal{P} u_n$
$\mu_n = (1-\beta_n)\mathcal{P}\omega_n + \beta_n \mathcal{P}\omega_n$
$u_{n+1} = (1-\vartheta_n)\mathcal{P}_3\omega_n + \vartheta_n \mathcal{P} u_n$
where γ_n , ϑ_n *and* β_n are real sequences in (0,1).

Additional studies about this subject are presented in [11-13]. Influenced and prompted by ongoing research in this area, in this work a new iteration process for approximating CFp of two Non-exp maps to obtain a faster convergence rate is presented. Consider $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3: \mathcal{B} \rightarrow \mathcal{B}$ be three Non-exp maps, then the sequence $\{u_n\}$ is produced iteratively by $u_1 \in \mathcal{B}$ and

$$\begin{aligned} \omega_n &= (1-\gamma_n)u_n + \gamma_n \mathcal{P}_1 u_n \\ \mu_n &= (1-\beta_n)\mathcal{P}_1\omega_n + \beta_n \mathcal{P}_2\omega_n \qquad (1) \\ u_{n+1} &= \mathcal{P}_3 \mu_n \end{aligned}$$

where γ_n *and* β_n are real sequences in (0,1).

This work aims to provide some convergence findings for the new iteration process (1). A numerical example is provided, followed by graphs to confirm the conclusions. The corresponding results of [10] are expanded upon in this paper.

2. Preliminaries

A review of some lemmas and definitions that will be applied repeatedly throughout the work follows.
Suppose $\mathcal{U}$ is a subset of a Banach space ($BN-space$) $\mathcal{B}$. A mapping $\mathcal{P}: \mathcal{U} \rightarrow \mathcal{U}$ is termed as nonexpansive if $\|\mathcal{P}u - \mathcal{P}v\| \leq \|u - v\|$. A point $u \in \mathcal{U}$ is called a Fp of $\mathcal{P}$ if $\mathcal{P}u = u$. The set of Fps of $\mathcal{P}$ will be denoted by $\mathfrak{F}(\mathcal{P})$.

Definition 2.1[14]: A $BN-$space $\mathcal{B}$ is termed as uniformly convex if for every $\epsilon \in (0,2]$, there is $\lambda > 0$ such that for all $u, v \in \mathcal{B}$

$$\left.\begin{aligned} \|u\| &\leq 1 \\ \|v\| &\leq 1 \\ \|u - v\| &> \epsilon \end{aligned}\right\} \text{implies } \frac{\|u+v\|}{2} \leq \lambda.$$

Definition 2.2[15]: A $BN-$space $\mathcal{B}$ fulfills the Opial's condition if for any sequence $\{u_n\}$ in $\mathcal{B}$ that converges weakly to $u \in \mathcal{B}$ i.e. $u_n \rightarrow u$ it can be stated that,

$\lim_{n\to\infty} sup\|u_n - u\| < \lim_{n\to\infty} sup\|u_n - w\|$ for each $w \in \mathcal{B}$ with $w \neq u$.

The following two lemmas provided an important upcoming discussions.

Lemma 2.3[16]: Let $\mathcal{B}$ be uniformly convex Banach space (UCB −space)and $\{c_n\}$ is a real sequence with $0 < a \leq c_n \leq b < 1$ for each $n \geq 1$. Assume that $\{u_n\}$, $\{v_n\}$ are two sequences of $\mathcal{B}$ with $\lim_{n\to\infty} sup\|u_n\| \leq \vartheta$, $\lim_{n\to\infty} sup\|v_n\| \leq \vartheta$ and $\lim_{n\to\infty} sup\|c_n u_n + (1 - c_n)v_n\| = \vartheta$ hold for $\vartheta \geq 0$. Then $\lim_{n\to\infty} sup\|u_n - v_n\| = 0$.

Lemma 2.4 [17]: Let $\mathcal{U}$ be a subset of a UCB −space $\mathcal{B}$ where $\mathcal{U}$ closed and convex and if $\mathcal{P}$ a Non-exp map on $\mathcal{U}$. Then, $I - \mathcal{P}$ is demiclosed at zero.

Definition 2.5[18]: Three mappings $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3: \mathcal{B} \to \mathcal{B}$ fulfill the condition (A) if there is a function (non-decreasing) $h: [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(m) > 0$ for each $m \in (0, \infty)$ such that $\|w - \mathcal{P}_1 w\| \geq h(d(w, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))$ or $\|w - \mathcal{P}_2 w\| \geq h(d(w, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))$ or $\|w - \mathcal{P}_3 w\| \geq h(d(w, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))$ for each $w \in \mathcal{U}$.

The concept of demiclosed mapping is described as:

Definition 2.6[8]: A mapping $\mathcal{P}: \mathcal{U} \to \mathcal{B}$ is demiclosed at $w \in \mathcal{U}$ if for each sequence $\{u_n\}$ in $\mathcal{U}$ and each $u \in \mathcal{B}$, $u_n \to u$ and $\mathcal{P}u_n \to w$ imply $u \in \mathcal{U}$ and $\mathcal{P}u = w$.

3. Main Results

The convergence results for the new iteration procedure are established then an example is provided to show that a novel process (1) converges more rapidly than other existing iteration procedures.

Lemma 3.1. Let $\mathcal{U}$ be a subset of a UCB −space $\mathcal{B}$ where $\mathcal{U}$ closed and convex and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3: \mathcal{B} \to \mathcal{B}$ be three Non-exp maps with $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3) \neq \emptyset$. Let $\{u_n\}$ be specified by the iteration process (1). Then

(1) $\lim_{n\to\infty} \|u_n - b\|$ exists for each $b \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

(2) $\lim_{n\to\infty} \|\mathcal{P}_1 u_n - u_n\| = \lim_{n\to\infty} \|\mathcal{P}_2 u_n - u_n\| = \lim_{n\to\infty} \|\mathcal{P}_3 u_n - u_n\| = 0$.

Proof: Consider $b \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$. Then, by using (1) we ge

$$\begin{aligned} \|\omega_n - b\| &= \|((1 - \gamma_n)u_n + \gamma_n \mathcal{P}_1 u_n) - b\| \\ &\leq (1 - \gamma_n)\|u_n - b\| + \gamma_n \|\mathcal{P}_1 u_n - b\| \\ &\leq (1 - \gamma_n)\|u_n - b\| + \gamma_n \|u_n - b\| \\ &= \|u_n - b\| \end{aligned} \tag{2}$$

and

$$\begin{aligned} \|\mu_n - b\| &= \|((1 - \beta_n)\mathcal{P}_1 \omega_n + \beta_n \mathcal{P}_2 \omega_n) - b\| \\ &\leq (1 - \beta_n)\|\mathcal{P}_1 \omega_n - b\| + \beta_n \|\mathcal{P}_2 \omega_n - b\| \\ &\leq (1 - \beta_n)\|\omega_n - b\| + \beta_n \|\omega_n - b\| \\ &= \|\omega_n - b\| \leq \|u_n - b\|. \end{aligned} \tag{3}$$

By using (3) we obtain

$$\begin{aligned}\|u_{n+1} - b\| &= \|\mathcal{P}_3\mu_n - b\| \\ &\leq \|\mu_n - b\| \\ &= \|\omega_n - b\| \leq \|u_n - b\|.\end{aligned} \tag{4}$$

Thus, $\{\|u_n - b\|\}$ is bounded and non-increasing for each $b \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$. Which gives that $\lim_{n\to\infty} \|u_n - b\|$ exists for each $b \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

(2) Consider $\lim_{n\to\infty} \|u_n - b\| = \alpha.$ (5)

So, from (2) and (3), we have

$$\lim_{n\to\infty} sup\|\omega_n - b\| \leq \alpha, \tag{6}$$

and

$$\lim_{n\to\infty} sup\|\mu_n - b\| \leq \alpha. \tag{7}$$

Now, $\alpha = \lim_{n\to\infty} \big\|\|u_{n+1} - b\|\big\| = \lim_{n\to\infty} \big\|\|\mathcal{P}_3\mu_n - b\|\big\|$.

Thus $\lim_{n\to\infty} \|\mathcal{P}_3\mu_n - b\| = \alpha$.

Also, $\|\mathcal{P}_3\mu_n - b\| \leq \|\mu_n - b\|$ which gives $\alpha \leq \lim_{n\to\infty} inf\|\mu_n - b\|$.

Now, using (7) we obtain

$$\lim_{n\to\infty} \|\mu_n - b\| = \alpha. \tag{8}$$

From (3) we have,

$$\|\mu_n - b\| \leq \|\omega_n - b\|$$

Which gives

$$\alpha \leq \lim_{n\to\infty} inf\|\omega_n - b\|. \tag{9}$$

By virtue of (6) and (9), we obtain

$$\lim_{n\to\infty} \|\omega_n - b\| = \alpha \tag{10}$$

and therefore in light of Lemma 2.3,

$$\lim_{n\to\infty} \|\mathcal{P}_1 u_n - u_n\| = 0. \tag{11}$$

Now, $\|u_n - b\| = \|((1-\gamma_n)u_n + \gamma_n\mathcal{P}_1u_n) - u_n\| = \|\gamma_n(\mathcal{P}_1u_n - u_n)\|$.

So, by using (11), we obtain

$$\lim_{n\to\infty} \|u_n - b\| = 0. \tag{12}$$

Additionally, the nonexpansiveness of $\mathcal{P}_2$ and (10) yields

$$\lim_{n\to\infty} sup\|\mathcal{P}_2\omega_n - b\| \leq \alpha \tag{13}$$

and from (8), (10), (13) and Lemma 2.3 obtain

$$\lim_{n\to\infty} \|\mathcal{P}_2\omega_n - \omega_n\| = 0. \tag{14}$$

Consider

$$\|\omega_n - u_n\| = \|((1-\gamma_n)u_n + \gamma_n\mathcal{P}_1u_n) - u_n\| = \gamma_n\|\mathcal{P}_1u_n - u_n\|. \tag{15}$$

Using (11) offers

$$\lim_{n\to\infty}\|\omega_n - u_n\| = 0. \tag{16}$$

Consider

$$\|\mathcal{P}_2 u_n - u_n\| \leq \|\mathcal{P}_2 u_n - \mathcal{P}_2\omega_n\| + \|\mathcal{P}_2\omega_n - \omega_n\| + \|\omega_n - u_n\|$$
$$\leq \|\mathcal{P}_2\omega_n - \omega_n\| + 2\|\omega_n - u_n\|.$$

Using (14) and (16), it yields

$$\lim_{n\to\infty}\|\mathcal{P}_2 u_n - u_n\| = 0. \tag{17}$$

Now, since $\mathcal{P}_3$ is Non-exp map then

$$\lim_{n\to\infty} sup\|\mathcal{P}_3\mu_n - \mathscr{b}\| \leq \lim_{n\to\infty} sup\|\mu_n - \mathscr{b}\| = \alpha. \tag{18}$$

Applying (5), (13), (18) and Lemma 2.3 obtain

$$\lim_{n\to\infty}\|\mathcal{P}_2\omega_n - \mathcal{P}_3\mu_n\| = 0. \tag{19}$$

Furthermore,

$$\|\mu_n - u_n\| = \|((1-\beta_n)\mathcal{P}_1\omega_n + \beta_n\mathcal{P}_2\omega_n) - u_n\|$$
$$\leq (1-\beta_n)\|\mathcal{P}_1\omega_n - u_n\| + \beta_n\|\mathcal{P}_2\omega_n - u_n\|$$
$$\leq (1-\beta_n)\|\mathcal{P}_1\omega_n - u_n\| + \beta_n(\|\mathcal{P}_2\omega_n - \omega_n\| + \|\omega_n - u_n\|).$$

This, combined with (14) and (16), leads to

$$\lim_{n\to\infty}\|\mu_n - u_n\| = 0. \tag{20}$$

Now,

$$\|\mathcal{P}_3 u_n - u_n\| \leq \|\mathcal{P}_3 u_n - \mathcal{P}_3\mu_n\| + \|\mathcal{P}_3\mu_n - \mathcal{P}_2\omega_n\| + \|\mathcal{P}_2\omega_n - \omega_n\| + \|\omega_n - u_n\|$$
$$\leq \|u_n - \mu_n\| + \|\mathcal{P}_3\mu_n - \mathcal{P}_2\omega_n\| + \|\mathcal{P}_2\omega_n - \omega_n\| + \|\omega_n - u_n\|.$$

Using (14), (16), (19) and (20), obtain $\lim_{n\to\infty}\|\mathcal{P}_3 u_n - u_n\| = 0.$

Theorem 3.2. Let $\mathcal{U}$ be a subset of a BN −space $\mathcal{B}$ where $\mathcal{U}$ closed and convex that meets Opial's condition and if $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3: \mathcal{B} \to \mathcal{B}$ be three Non-exp maps with $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3) \neq \emptyset$ and let $\{u_n\}$ be specified by the iteration (1). Then $\{u_n\}$ converges weakly to a CFp of $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$.

Proof: Consider $\mathscr{b} \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$. According to Lemma 3.1, $\lim_{n\to\infty}\|u_n - \mathscr{b}\|$ exists. To establish that the iteration process (1) approaches a CFp among $\mathcal{P}_1, \mathcal{P}_2,$ and $\mathcal{P}_3$, it is necessary to demonstrate that $\{u_n\}$ possesses a unique weak subsequential limit in $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$. In light of this, assume that $\{u_{n_i}\}$, $\{u_{n_j}\}$ are two subsequences of $\{u_n\}$ that converge to q_1 and q_2 .
According to Lemma 3.1, deduce that
$\lim_{n\to\infty}\|\mathcal{P}_1 u_n - u_n\| = \lim_{n\to\infty}\|\mathcal{P}_2 u_n - u_n\| = \lim_{n\to\infty}\|\mathcal{P}_3 u_n - u_n\| = 0$ and applying Lemma 2.4 obtain $I - \mathcal{P}_1$, $I - \mathcal{P}_2$ and $I - \mathcal{P}_3$ are demiclosed at zero. So, q_1 , $q_2 \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

Now, to show the uniqueness, since q_1 , $q_2 \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$ *so* $\lim_{n\to\infty} \|u_n - q_1\|$ and $\lim_{n\to\infty} \|u_n - q_2\|$ exists. Consider $q_1 \neq q_2$ then by Opial's condition one get

$$\begin{aligned}\lim_{n\to\infty} \|u_n - q_1\| &= \lim_{n\to\infty} \|u_{n_i} - q_1\| \\ &< \lim_{n\to\infty} \|u_{n_i} - q_2\| \\ &= \lim_{n\to\infty} \|u_n - q_2\| \\ &= \lim_{n\to\infty} \|u_{n_j} - q_2\| \\ &< \lim_{n\to\infty} \|u_{n_i} - q_1\| = \lim_{n\to\infty} \|u_n - q_1\|,\end{aligned}$$

that is a contradiction, so $q_1 = q_2$. As a result, $\{u_n\}$ weakly convergence to a CFp of $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$.

Theorem 3.3. Let $\mathcal{U}$ be a subset of a BN −space $\mathcal{B}$ where $\mathcal{U}$ closed and convex and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 : \mathcal{B} \to \mathcal{B}$ be three Non-exp maps with $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3) \neq \emptyset$. Let $\{u_n\}$ be specified by the iteration (1). Then $\{u_n\}$ converges to a point of $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$ if and only if $\lim_{n\to\infty} inf d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$.

Proof: If $\{u_n\}$ converges to $\ell \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$ then it is evident that $\lim_{n\to\infty} inf d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$. Regarding the converse, suppose that $\lim_{n\to\infty} inf d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$. From Lemma 3.1 $\lim_{n\to\infty} \|u_n - \ell\|$ exists which gives

$$\|u_{n+1} - \ell\| \leq \|u_n - \ell\|,$$

and this provides

$$d(u_{n+1},\ \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) \leq\ d(u_n,\ \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) \tag{21}$$

Therefore, $\{d(u_n,\ \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))\}$ constitutes a decreasing that is bounded below by zero, so it may be obtained that $\lim_{n\to\infty} d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))$ exists. Since $\lim_{n\to\infty} inf d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$ so $\lim_{n\to\infty} d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$. Now, to prove that $\{u_n\}$ is a Cauchy in $\mathcal{U}$. Assume $\xi > 0$. Because $\lim_{n\to\infty} d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$ there is n_0 such that for every $n \geq n_0$,
d$(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) < \frac{\xi}{4}$.
Especially, $\inf\{\|u_{n_0} - \ell\| : \ell \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)\} < \frac{\xi}{4}$.
Hence, it must exist $r \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$ with $\|u_{n_0} - r\| < \frac{\xi}{2}$.
Thus, for $n, m \geq n_0$

$$\|u_{n+m} - u_n\| \leq \|u_{n+m} - r\| + \|u_n - r\| < 2\|u_{n_0} - r\| < \xi,$$

thus indicating that $\{u_n\}$ is Cauchy. Because $\mathcal{U}$ is a closed subset of $\mathcal{B}$ then $\{u_n\}$ is converges in $\mathcal{U}$. Consider $\lim_{n\to\infty} u_n = b$ for any $b \in \mathcal{U}$. Applying $\lim_{n\to\infty} \|\mathcal{P}_1 u_n - u_n\| = 0$, one obtains

$$\|b - \mathcal{P}_1 b\| \leq \|b - u_n\| + \|u_n - \mathcal{P}_1 u_n\| + \|\mathcal{P}_1 u_n - \mathcal{P}_1 b\|$$
$$\leq \|b - u_n\| + \|u_n - \mathcal{P}_1 u_n\| + \|u_n - b\|.$$
$$\to 0 \; as \; n \to \infty$$

Thus $b = \mathcal{P}_1 b$. In a similar manner, it may be shown that $b = \mathcal{P}_2 b$ and $b = \mathcal{P}_3 b$, thus $b \in \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

Theorem 3.4. Let $\mathcal{U}$ be a subset of a UCB −space $\mathcal{B}$ where $\mathcal{U}$ closed and convex and if $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 : \mathcal{B} \to \mathcal{B}$ be three Non-exp maps that meets condition(A) with $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3) \neq \emptyset$. Let $\{u_n\}$ be specified by the iteration (1). Then $\{u_n\}$ strongly convergence to a point of $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

Proof: According to (21), one can get $\lim_{n\to\infty} d(u_n, \mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3))$ exist and also by Lemma 3.1 we obtain that
$\lim_{n\to\infty} \|\mathcal{P}_1 u_n - u_n\| = \lim_{n\to\infty} \|\mathcal{P}_2 u_n - u_n\| = \lim_{n\to\infty} \|\mathcal{P}_3 u_n - u_n\| = 0$. It can be deduced from condition (A) that

$$\lim_{n\to\infty} h(d(u_n, (\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) \leq \lim_{n\to\infty} \|\mathcal{P}_1 u_n - u_n\| = 0,$$

or

$$\lim_{n\to\infty} h(d(u_n, (\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) \leq \lim_{n\to\infty} \|\mathcal{P}_2 u_n - u_n\| = 0,$$

or

$$\lim_{n\to\infty} h(d(u_n, (\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) \leq \lim_{n\to\infty} \|\mathcal{P}_3 u_n - u_n\| = 0.$$

Therefore, $\lim_{n\to\infty} h(d(u_n, (\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$. Because h is a non-decreasing function meets $h(0) = 0$ and $h(m) > 0$ for each $m \in (0, \infty)$ hence $\lim_{n\to\infty} d(u_n, (\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)) = 0$ and by Theorem 3.3 $\{u_n\}$ strongly convergence to a point of $\mathfrak{F}(\mathcal{P}_1) \cap \mathfrak{F}(\mathcal{P}_2) \cap \mathfrak{F}(\mathcal{P}_3)$.

4. Numerical Results

This part provides an example to verify the convergence result discussed in the preceding section. This example demonstrates via numerical and graphical analysis that the iterative technique provided in equation (1) converges to a CFp across different cases.

Example 4.1: Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{U} = [1,40]$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 : \mathcal{B} \to \mathcal{B}$ be three Non-exp maps defined by

$\mathcal{P}_1(u) = \sqrt{u^2 - 9u + 27}$
$\mathcal{P}_2(u) = \sqrt{u^2 - 7u + 21}$
$\mathcal{P}_3(u) = \sqrt{u^2 - 3u + 9}$

for each $u \in \mathcal{U}$. Clearly, $u = 3$ is the CFp of $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$. Set $\gamma_n = \beta_n = 0.9$. Then, the following tables and figures are produced utilizing different beginning values.

Table 1: Values of the iteration

step	New iteration when u=0.2	New iteration when u=0.6	New iteration when u=1

1	0.2	0.6	1
2	3.174908725126	3.116540246627	3.072082771438
3	3.000391584660	3.000227388986	3.000122807234
4	3.000000490514	3.000000284584	3.000000153610
5	3.000000000613	3.000000000355	3.000000000192
6	3.000000000000	3.000000000004	3.000000000002
7	3.000000000001	3.000000000000	3.000000000000
8	3	3	3
9	3	3	3
10	3	3	3
11	3	3	3

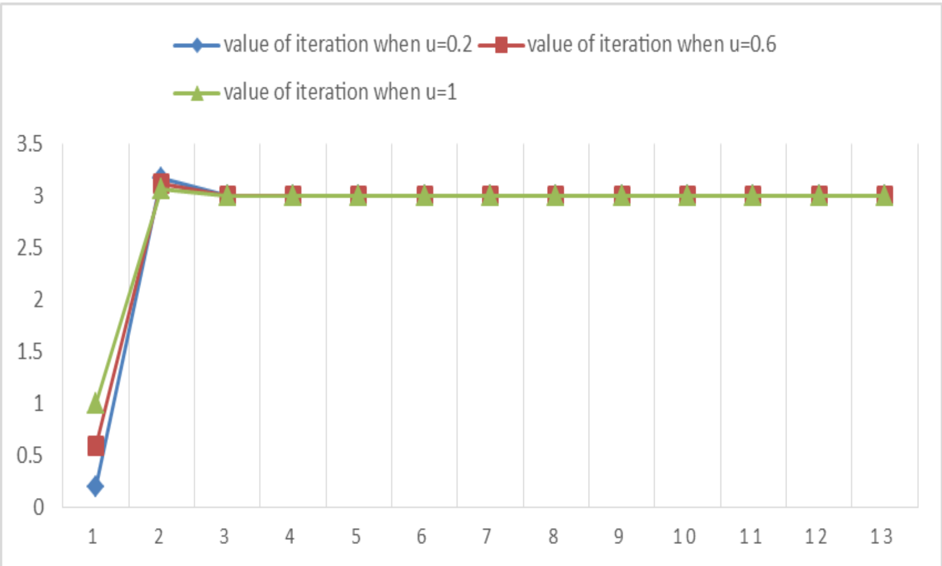

Figure 1: Graph corresponding to Table 1

Table 2: Values of the iteration

step	New iteration when u=5	New iteration when u=10	New iteration when u=15	New iteration when u=20
1	5	10	15	20
2	2.992252062771	3.881479501051	7.499266494533	12.042616352441
3	2.999990722936	3.002276510287	3.082968616473	5.141461318404
4	2.999999988404	3.000002880522	3.000146569750	2.990091716079
5	2.999999999985	3.000000003600	3.000000183357	2.999988282987
6	2.999999999999	3.000000000004	3.000000000229	2.999999985354
7	3	3.000000000000	3.000000000001	2.999999999981
8	3	3	3.000000000000	2.999999999999
9	3	3	3	3
10	3	3	3	3
11	3	3	3	3

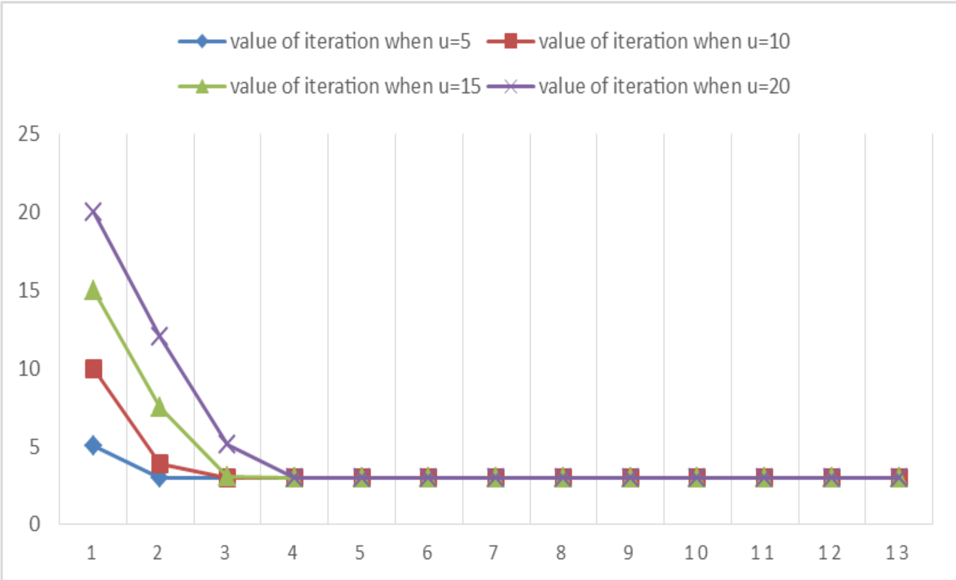

Figure 2: Graph corresponding to Table 2

Now, to demonstrate the superior convergence rate for approximating Fp of the new iterative technique compared to other iterative procedures namely: SP-iteration[19], Ishikawa [20], Mann [21], S-iteration[22], and Normal S-iteration[23] an illustrative example is displayed below.

Example 4.2: Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{U} = [0,6]$. Consider $\mathcal{P}: \mathcal{B} \to \mathcal{B}$ be a mapping specified by

$$\mathcal{P} = \begin{cases} \frac{u}{4} & if\ u \in [0,3) \\ \frac{u}{8} & if\ u \in [3,6] \end{cases}$$

for each $u \in \mathcal{U}$. Clearly, $u = 0$ is Fp of $\mathcal{P}$. $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$. Then, Table 1 and Figure 3 are generated with the initial value 4.

Table3: Comparison of convergence rates for various iteration approaches

step	SP-iteration	Ishikawa	Mann	New iteration
1	4	4	4	4
2	1.983642578125	3.097656250000000	3.125000000000000	0.437500000000
3	1.063980162143	2.474494934082031	2.441406250000000	0.022216796875
4	0.570694437556	1.981529146432877	1.983642578125000	0.001128196716
5	0.306107343581	1.586771386791952	1.611709594726563	$5.729123950004\times 10^{-5}$
6	0.164188924279	1.270656774579493	1.309514045715332	$2.909320755861\times 10^{-6}$
7	0.088067154941	1.017518120268734	1.063980162143707	$1.477389446336\times 10^{-7}$
8	0.047237192238	0.814809432246448	0.864483881741762	$7.502368282175\times 10^{-9}$
9	0.025336941246	0.652484115666101	0.702393153915182	$3.809796393292\times 10^{-10}$
10	0.013590151347	0.522497045748245	0.570694437556085	$1.934662230968\times 10^{-11}$
11	0.007289443972	0.418405837415586	0.463689230514319	$9.824456641637\times 10^{-13}$

Table 4: Comparison of convergence rates for various iteration approaches

step	S-iteration	Normal S-iteration	New iteration
1	4	4	4
2	2.441406250000	0.390625000000	0.437500000000
3	1.611709594726	0.079345703125	0.022216796875
4	1.063980162143	0.016117095947	0.001128196716
5	0.702393153915	0.003273785114	$5.729123950004\times 10^{-5}$
6	0.463689230514	$6.649876013398\times 10^{-4}$	$2.909320755861\times 10^{-6}$
7	0.306107343581	$1.350756065221\times 10^{-4}$	$1.477389446336\times 10^{-7}$
8	0.202078676036	$2.743723257481\times 10^{-5}$	$7.502368282175\times 10^{-9}$
9	0.133403500977	$5.573187866758\times 10^{-6}$	$3.809796393292\times 10^{-10}$
10	0.088067154941	$1.132053785435\times 10^{-6}$	$1.934662230968\times 10^{-11}$
11	0.058138082754	$2.299484251665\times 10^{-7}$	$9.824456641637\times 10^{-13}$

The newly introduced iteration process converges to Fp, u=0 more rapidly than a number of the existing iteration processes, as shown in the tables.

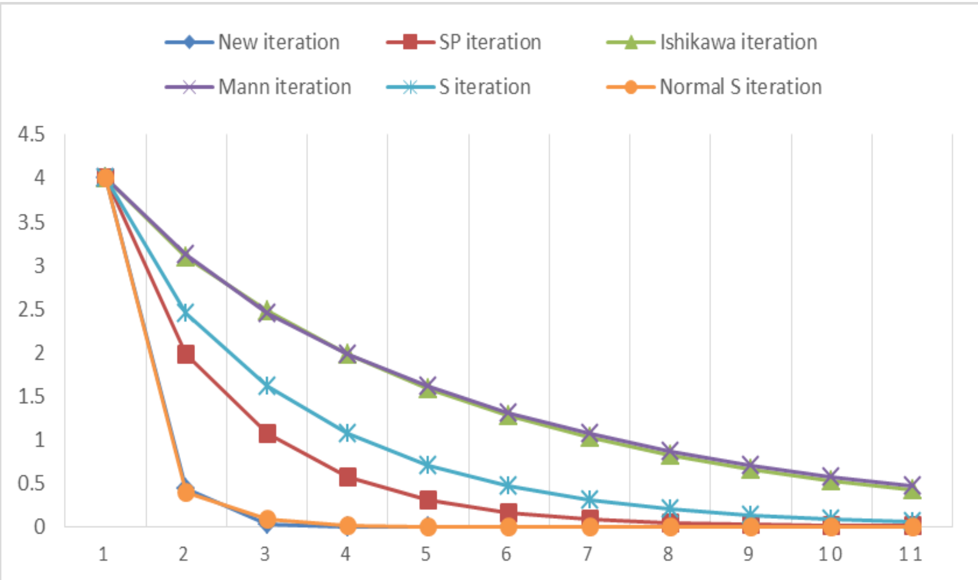

Figure 3: Graph corresponding to Table 3 and Table 4

5. Conclusions

This paper offers a novel iteration approach that approximates the CFp of three non-exp maps. In addition, a numerical example is given to demonstrate the convergence of some findings. Moreover, the tables and graphs make it abundantly evident that the proposed iteration technique converges at a far faster rate than a number of the other iteration procedures. It should be emphasized that $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = \mathcal{P}$ is a particular case of (1). Thus the major results of this paper expand the findings of [10] from one non-exp map to three non-exp maps.

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