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Critical Buckling Loads in Functionally Graded Beams: Comparative Analysis Using Various Beam Theories

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ABSTRACT

This research focuses on the evaluation of static buckling response of FG beams via Euler-Bernoulli and Timoshenko beam models. Loaded simply supported FG beams are loaded with axial compressive force and the properties of FG beam depend on the thickness and follow power-law distribution for all mechanical properties with constant Poisson's ratio. The governing equations are obtained by minimizing the total potential energy and numerical solution is for the critical buckling load is obtained by using Navier-type approximation. The results show that as the slenderness ratio and power-law exponent increase, the critical buckling load reduces which is a sign of the mechanical change of the FG beam to resemble a uniform aluminum beam. The numerical findings are comparing with existing literature and have reasonable accuracy and usefulness for studying the structural behavior under different scenarios.

1. Introduction

Materials that have the following properties: composite material properties are gradually changed, and this changes continuously from one place to another, are called functionally graded materials, abbreviated as (FGM). The FGM is used in many important fields. As the FGM strengthens the materials that are added to make them resist the external load. Functionally graded material has been used in many fields of engineering, industry, atomic energy, and other important fields. Buckling analysis of the structural members is critical in designing structures to resist compressive forces. Of all such elements, those that are manufactured from FGMs have attracted a lot of interest because they can allow the design of material characteristics, and hence the efficiency of the behaviour under mechanical and thermal loads. These designs with a gradual gradation of the material composition are called FGMs and show better resistance to delamination, stress concentrations, and thermal stresses than any conventional composite material. Consequently, they are applied widely in aerospace structures, nuclear reactors, and modern robotics.

Li and Batra [1] provided research on the critical buckling loads (CBLs); the governing equations can be obtained for both homogeneous Euler Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT) FGM to which both authors exposed axial compressive load characterized in different boundary conditions. The Timoshenko beam theory is compared with the Euler Bernoulli homogeneous beam theory to obtain the critical load / buckling load. By applying Timoshenko

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beam theory and using total potential energy for FG twisted beam under distributed transverse load, Chen [2] analyzed the governing equations. He also carry out a numerical comparison between the TBT and EBT, and finding the bending deflection of the twisted cantilever beam on the basis of numerical method. Timoshenko beam theory and homogenous Euler-Bernoulli governing equations beam have been investigated for similar mathematical analysis in Li et al. [3] at the similar loads with four different boundary conditions. Analytical solutions may serve as references in the subsequent research of behaviors of FGM beams. Rychlewska [4] has explicitely derived the governing equations for axially FG beams that were postulated on the basis of EBT to obtain the values of CBLs. In different boundary conditions, CBLs are calculated for three categories. First, the beam is subjected to an axial load uniformly distributed along the length of the beam and, second, numerically, using an equation for the buckling load, we obtain the CBLs. third, through a comparison of the CBLs of the homogeneous beam and the FG beam.

Akbas [5] examined the governing equations for post-buckling analysis that stable when the total Lagrangian finite element model is used with FG beam. The detachment of material characteristics of the FG beam is in the axial direction. Axially, the material properties of the beam are distributed according to a power law. The problem of complex nonlinear is solved using an incremental FEM (finite element method displacement-focused) together with the Newton-Raphson iteration method. Zoubida et al. [6] worked on the simply supported functionally graded material FGM beams and with the aid of the Hamiltonian principle and a refined shearing deformation beam theory based on the neutral surface of the beam, the authors derived the governing equations based on the static and free vibration analysis. The governing equations of two-directional FG beams have been obtained by Karamanlı [7] using the total potential energy method as initiated by Reddy and Bickford beam theory (RBT), EBT, and TBT. NASH based on the quadrature study which implements symmetrical smoothed particle hydrodynamics for maximum dimensionless transverse deflection, dimensionless transverse shear stress and dimensionless axial stress under different boundary conditions. Fouda et al., [8] have employed finite element for maximum deflection, CBL and natural frequency with different boundary conditions. Based on these theories, the governing equations of a functionally graded cantilever beam are derived. Sherafatnia et. al. [9]. These are the CBL and natural frequency; numerical results were obtained for these parameters and compared with previous research. In their work, Padhi et al. [9] revealed that the power law index affects the loads of critical buckling and stability of the beam. It showed that there is a more favorable association of a lower power index for the FG beam. The maximum deflection in a functionally graded beam under a uniformly distributed load and with simply supported conditions, as predicted by Şimşek [11], is determined using the Ritz method. This section provides the mathematical equations that characterise the behaviour of a functionally graded beam experiencing axial load and transverse force Eltaher et al [12] employing the Timoshenko beam theory. By solving the governing equations Shyma and Rajendran [13] found maximum deflection and CBL under different boundary condition.The FGM beam was considered and the total potential energy was used with the help of mathematical model and the deflection of a simply supported FGM beam is calculated. The materials of the beam under consideration exhibit graded distribution of material properties in the thickness direction represented by the power law form. Lanc et al [14] also investigated the bending and buckling analysis of both EBT and Vlasov theories in thin-walled FG sandwich box beams. By applying total potential energy method the governing equations were derived , using two different theories to study both bending and buckling. Arefi [15] investigated the Euler-Bernoulli beam theory on the isotropic and FG beams to find the differential equations linear and nonlinear governing a simply supported beam on a linear and nonlinear foundation.

Mentioned Torki and Reddy [16] the TB and EBB theories for the FG beam and the governing equations can be derived from Hamilton's principle. When simple support is assumed in the FG beam with the aid of FEM, the critical load can be determined. Lieu et al. [17] derived the governing equations by proposing third-order shear deformation theory for FGSSN nano beams subjected to axial load and transverse forces. They also defined maximum deflection and CBLs for several and conditions of boundary support. A. S. Appealing to the analyses of Sayyad and Y. M. Ghugal [18], Navier's numerical analysis method was successfully applied to compute the CBL and the maximum deflection in a simply supported FGM beam. The authors pointed out that the results they attained in their study were quite good and yielded good tests to other studies carried out. Malihi et al. [19] considered that by applying the semi-analytical approach to a FG beam, it is possible to find the governing equations for the various boundary conditions. Both methods used static space and differential quadrature for the numerical solution in both longitudinal and transverse directions for different material properties. Trinh et al. [20] investigated the governing equations by using Timoshenko beam theory for a simply supported FGM beam and the beam subjected to multiple moving points. Through numerical results and using FEM, dynamic response analysis was studied for non-uniform Timoshenko beams. It is assumed that a power law governs the continuous axial variation of the material properties.

Numerous researches have been conducted for analyzing the buckling of FG beams. Nevertheless, most have aimed at some concrete boundary conditions or material transitions, which still does not provide understanding of the impact of the main parameters, such as slenderness ratio and power-law exponent, for stability. Modern developments in modeling including non-linear finite element methods and machine learning based stability analysis have been demonstrated to pose solutions for these challenges. For example multi-scale modeling as a part of the hybrid FEM was then presented by Li et al. (2023) [21] focusing on the axial and transverse forces for the FG beams. Ahmed et al. (2022) [22] used a coupled FEM-Ritz method to investigate the effect of dynamic loads on the stability of FG beam through nucleating understanding of transient buckling issues. Thus, Zhang et al. (2021) [23] proposed the ML model for the accurate prediction of CBLs for FG beams with varying properties of the material. Building upon these advancements, this study seeks to address the following gaps:

1. A comprehensive comparative study of Euler Bernoullis and TBT based on the material grading and geometrical structures.
2. A study on the interaction between the slenderness ratio and power-law exponent on CBLs.
3. Attempts to increase the reliability of the analytical results with the help of comparing it with the result obtained numerically.

However, there are still research gaps in the application and synthesis of high level modeling approaches to the classical theories in order to form a full model for FG beam analysis. This work seeks to fill this gap through the use of the Navier-type solution method to assess the results of the Euler-Bernoulli and Timoshenko beam theories.

Functionally graded materials (FGMs) which refer to materials with progressive changes in composition and characteristics have become important in a number of fields of engineering and technology. These materials improve on the structural behavior by increasing resistance to external loads. In recent years, there have been numerous investigations on the structural mechanics of FGMs with special emphasis on their stability behavior, notably the buckling response of these composites. For example, Li and Batra [1] and Rychlewska [4] examined c CBLs of FG beams under various boundary situations; EBT and TBT were considered. Similarly, Akbas [5] has analysed post-buckling analysis using finite element analysis and Zoubida et al. [6] have worked on shear deformation on FGM beams. Nevertheless, these studies are based on individual theoretical notions or confine the discussion with the application of certain boundary conditions, thus, not addressing the necessity of a comparative investigation of beam theories.

This research has sought to fill these gaps by undertaking a comprehensive comparative analysis of the critical buckling behaviour of FG beams using both E-B and TBT while maintaining consistency. Thus, taking into account the slenderness ratio and power-law exponent, the work contributes to the development of a comprehensive understanding of the manner and mechanics of the transition between various behaviours of materials. The results enhance understanding of the stability characteristics of FG beams, supported by comparison with existing works. Based on this background, the current study complements the comparison of beam theories used to assess CBLs by further exploring the impact of material gradation and geometry.

To enable a better understanding of the given current study, table 1 below seeks to combine other prior studies on the static and free vibration, as well as critical buckling characteristic of FG beams. It describes the method, results, and limitations, and provides information about the gaps that are filled by the current study. Previous work mainly consists in exploring the behavior of FG beams based on particular theories or approaches, and most often with prescribed conditions of boundary and/or constituents. Although extensive work has been done on EBT and TBT, a limited number of comparative studies have been made for different slenderness ratios and power-law exponents. It aims to fill this gap by using beam theories and understanding how geometric and material changes affect CBLs. Both the natural frequency and the CBL are determined from numerical analysis using the Galerkin’s analytical method for a bi-dimensional functionally graded (2D-FG) metal-ceramic porous beams, Shabani et al. [24]. The governing equations are derived using the refined 2D shear deformation theory for a simply supported FG beams, Chitour et al. [25]. Numerical analysis shows the maximum deflection using Naiver's analytical method. The governing equations are derived using Timoshenko beams with arbitrary boundary conditions (BCs) for a FG beam, DEMİRHAN [26]. Under different BCs, find the CBL using numerical analysis.

Table 1: Past studies on functionally graded beams and their critical buckling behavior

Author(s)	Methods/Theories	Key Findings	Limitations/Gaps Addressed
Li and Batra [1]	Euler-Bernoulli and Timoshenko theories	Derived governing equations; analyzed CBLs for axial compressive loads.	Limited comparison under varying boundary conditions.
Rychlewska [4]	Euler-Bernoulli theory	Calculated CBLs for axially FG beams under distributed axial loads.	Focused on specific load distributions, lacked broader material analysis.
Akbas [5]	Finite element analysis with nonlinear modeling	Explored post-buckling behavior with material property variation along the axial direction.	Did not compare with classical beam theories.
Zoubida et al. [6]	Refined shear deformation theory	Conducted static and free vibration analyses for FG beams using Hamilton’s principle.	Lacked focus on CBLs under compressive forces.
Fouda et al. [8]	Finite element method	Studied bending, buckling, and vibration for porous FG beams.	Addressed porous materials but not graded material effects fully.
Simsek [11]	Ritz method	Analyzed static deflections under distributed loads using EBT.	Did not extend analysis to buckling under axial loads.
Torki and Reddy [16]	FEM with Hamilton's principle	Analyzed CBLs for FG beams with piezoelectric layers under simply supported conditions.	Limited to specific piezoelectric applications.

Sayyad and Ghugal [18]	Navier's method	Analyzed bending and CBLs for simply supported FGM beams.	Did not explore varying slenderness ratios or wave numbers.

2. Governing Equations for FGM Beam

- Model Development

This paper therefore looks at the static buckling phenomenon of beams made from FGM under axial loads. The beams are modeled using two classical beam theories:

- EBT: It assumes that shear deformation is negligible and therefore, appropriate for thick beams.

- TBT: Capable to account for shear deformation and is appropriate for the slender beams.

- Material Gradation

Alumina makes up the bottom layer of the FGM beam, while aluminum makes up the top layer. On the one hand, the qualities of the material vary with scale, and on the other hand, they are power law distributed over thickness.

E_m: 70 GPa (Aluminum)
E_c: 380 GPa (Alumina)
ν: 0.3 (assumed constant)

Assume for the moment that a FGM beam with dimensions of L, b, and h has alumina ceramic on its bottom surface and aluminum metallic on its top surface. The figure below illustrates the axial compressive force that the FGM beam is subjected to. The following equations illustrate how the power-law distribution of the FGM beam's effective material properties, such as Young's modulus and shear modulus, change along the thickness direction:

$$E(z) = E_m + (E_c - E_m)*(z/h + 0.5)^n \quad (1)$$

$$G(z) = G_m + (G_c - G_m)*(z/h + 0.5)^n \quad (2)$$

E_m, G_m: the top surface (Young's and shear)

E_c, G_c: the bottom surface elasticities modulus (Young's and shear)

$E(z)$: Young’s modulus variation for the FGM

n : exponent for power law for FGM

Additionally, it can be assumed that the Poisson's ratio remains constant throughout the direction of thickness.

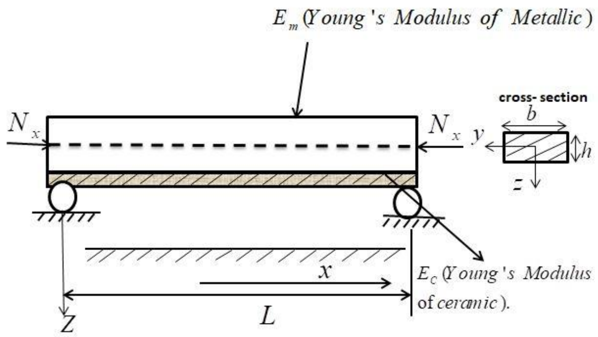

Figure 1. The FGM beam subjected to axial load

2.1 Euler-Bernoulli beam theory

Based on EBT theory, the field of displacement can be expressed as:

$$u(x,y,z) = u_0(x) - z\frac{\partial w_0}{\partial x} \quad (3)$$

$$v(x,y,z) = 0 \quad (4)$$

$$w(x,y,z) = w_0(x) \quad (5)$$

u & w : transverse displacements and axial of FGM beam (in directions of x –z)

v : the displacement in y direction for the FGM beam

$w_0(x)$ FGM beam transverse deflection

From Eq. (3) the axial strain is obtained as:

$$\varepsilon_{xx} = \frac{\partial u(x,y,z)}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \quad (6)$$

We can write the strain energy (potential energy) as:

$$U=\frac{1}{2}\int_v \sigma_{ij}\varepsilon_{ij}dV \quad , \quad \delta U=\int_v \sigma_{ij}\delta\varepsilon_{ij}dV \tag{7}$$

By using the axial stress and strain can be written virtual strain energy of the FGM EBT as follows:

$$\delta U=\int_0^L\int_{-h/2}^{h/2}\int_{-b/2}^{b/2}(\sigma_{xx}\delta\varepsilon_{xx})dy\,dz\,dx \tag{8}$$

σ_{xx} component for axial stress (plane of x-x)

ε_{xx} axial strain that is axial in the x direction

Substitute Eq. (6) into Eq. (8) and apply Eq. (10), Thus, we can write the FGM EBT theory as the final strain energy expression:

$$\delta U=\int_0^L\left[\left(-\frac{\partial N_x}{\partial x}\delta u_0-\frac{\partial^2 M_x}{\partial x^2}\right)\delta w_0\right]dx \tag{9}$$

Where:

$$N_x=\int_A \sigma_{xx}dA \quad , \quad M_x=\int_A \sigma_{xx}\,z\,dA \tag{10}$$

N_x axial normal force

M_x bending moment

Therefore, the expression for the functionally graded materials (EBT) as the work done by the externally applied axial forces is:

$$W_{ext}=\frac{-1}{2}\int_0^L\int_{-b/2}^{b/2}\left(F_{ext}w_0\right)dy\,dx \tag{11a}$$

$$\delta W_{ext}=-b\int_0^L\left(F_{ext}\delta w_0\right)dx \tag{11b}$$

$$F_{ext}=F_{buckling}\left(\frac{\partial^2 w}{\partial x^2}\right)=N_{x0}\left(\frac{\partial^2 w}{\partial x^2}\right) \tag{11c}$$

For the functionally graded materials (EBT), the expression for the final external work is:

$$\delta W_{ext}=-b\int_0^L\left[\left(N_{x0}\frac{\partial^2 w}{\partial x^2}\right)\delta w_0\right]dx \tag{12}$$

The governing equations for the functionally graded materials (EBT) can be written as follows by applying the total potential energy principle:

$$\delta U+\delta W_{ext}=0 \tag{13}$$

Using the following equations: (9) and (12), and setting the coefficients of δu_0 & δw_0 to zero.

$$\int_0^L\left(-\frac{\partial N_x}{\partial x}\right)\delta u_0-\left(\frac{\partial^2 M_x}{\partial x^2}-b\,N_{x0}\frac{\partial^2 w}{\partial x^2}\right)\delta w_0\,dx=0 \tag{14}$$

(EBT) equilibrium of functionally graded materials equations can be written as:

$$\delta u_0:\quad -\frac{\partial N_x}{\partial x}=0 \tag{15a}$$

$$\delta w_0:\quad -\frac{\partial^2 M_x}{\partial x^2}-b\,N_{x0}\frac{\partial^2 w}{\partial x^2}=0 \tag{15b}$$

By using Hooke's law

$$\sigma_{xx}=E(z)*\varepsilon_{xx} \tag{16}$$

We can determine the axial normal force and bending moment for the (EBT) functionally graded materials as stress resultants by applying Hooke's law, as outlined in Eq. (10) and Eq. (16) as follows:

$$N_x=A_{xx}b\frac{\partial u_0}{\partial x}-B_{xx}b\frac{\partial^2 w_0}{\partial x^2} \tag{17a}$$

$$M_x=B_{xx}b\frac{\partial u_0}{\partial x}-D_{xx}b\frac{\partial^2 w_0}{\partial x^2} \tag{17b}$$

By substituting equations (17a) and (17b) into equations (15a) and (15b), the governing equations of the FGM Euler-Bernoulli beam theory (EBT), derived from the total potential energy principle, can be written as:

$$A_{xx}\frac{\partial^2 u_0}{\partial x^2}-B_{xx}\frac{\partial^3 w_0}{\partial x^3}=0 \tag{18a}$$

$$B_{xx}\frac{\partial^3 u_0}{\partial x^3}-D_{xx}\frac{\partial^4 w_0}{\partial x^4}+N_{x0}\frac{\partial^2 w}{\partial x^2}=0 \tag{18b}$$

Where:

$$A_{xx}=\int_{-h/2}^{h/2}E(z)dz \tag{19a}$$

$$B_{xx}=\int_{-h/2}^{h/2}E(z)z\,dz \tag{19b}$$

$$D_{xx}=\int_{-h/2}^{h/2}E(z)z^2dz \tag{19c}$$

A_{xx}, B_{xx}, D_{xx} coefficients of stiffness.

2.2 Timoshenko beam theory

The expression for displacement field of TBT is:

$$u(x,y,z)=u_0(x)-z\,\varphi(x) \quad (20)$$

$$v(x,y,z)=0 \quad (21)$$

$$w(x,y,z)=w_0(x) \quad (22)$$

$\varphi(x)$: is the cross-section rotation at any point on the neutral axis

We can calculate the axial and shear strains in (TBT) by applying equations (20) and (22):

$$\varepsilon_{xx}=\frac{\partial u_0}{\partial x}-z\,\frac{\partial \varphi}{\partial x} \quad (23)$$

$$\gamma_{xz}=\frac{\partial w_0}{\partial x}-\varphi \quad (24)$$

With the same procedure followed on EBT, the potential energy of the FGM beam of TBT written as follows by using equations (23) & (24):

$$\delta U=\int_0^L\int_{-h/2}^{h/2}\int_{-b/2}^{b/2}\left(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xz}\delta\gamma_{xz}\right)dy\,dz\,dx \quad (25)$$

σ_{xz} : transverse shear stress

γ_{xz} : shear strain

The final potential energy for strain energy for TBT can be found by substituting equations (23) & (24) into Eq. (25) as follows:

$$\delta U=\int_0^L\left(-\frac{\partial N_x}{\partial x}\delta u_0+\frac{\partial M_x}{\partial x}\delta\varphi-\frac{\partial Q_x}{\partial x}\delta w_0-Q_x\delta\varphi\right)dx \quad (26)$$

Where:

$$Q_x=k_s\int_A\sigma_{xz}\,dA \quad (27)$$

k_s : factor of shear correction

Q_x : shear force

Equilibrium of FGM Timoshenko beam theory (TBT) are shown below using equations (12) and (26) and setting the coefficients of δu_0 , δw_0 & $\delta\varphi$ to zero and using total potential energy principle.:

$$\delta u_0:-\frac{\partial N_x}{\partial x}=0 \quad (28a)$$

$$\delta w_0:-\frac{\partial Q_x}{\partial x}-bN_{x0}\frac{\partial^2 w}{\partial x^2}=0 \quad (28b)$$

$$\delta\varphi:\frac{\partial M_x}{\partial x}-Q_x=0 \quad (28c)$$

The Hooke's law for shear stress

$$\tau_{xz}=G\gamma_{xz} \quad (29)$$

Thus, and along with Hooke's law associated with axial and shear stresses from equations (16) and (29), the governing equations for the FGM TBT can be derived by utilizing equations (10) and (27). The final governing equations that describe the behavior of the functionally graded materials (TBT) under applied forces can be found by substituting the previous equations into equations (28a), (28b), and (28c):

$$A_{xx}\frac{\partial^2 u_0}{\partial x^2}-B_{xx}\frac{\partial^2\varphi}{\partial x^2}=0 \quad (30a)$$

$$A_{xz}k_s\left(\frac{\partial^2 w_0}{\partial x^2}-\frac{\partial\varphi}{\partial x}\right)+N_{x0}\frac{\partial^2 w}{\partial x^2}=0 \quad (30b)$$

$$-B_{xx}\frac{\partial^2 u_0}{\partial x^2}+D_{xx}\frac{\partial^2\varphi}{\partial x^2}+A_{xz}k_s\left(\frac{\partial w_0}{\partial x}-\varphi\right)=0 \quad (30c)$$

Where:

$$A_{xz}=\int_{-h/2}^{h/2}G(z)\,dz \quad (31)$$

A_{xz} : coefficient of stiffness.

3. Buckling Simply Supported Functionally Graded Materials Beams Analytical Solutions of Using the Method of Navier-Type

A Navier-Type solution approach is an analytical method that systematically computes the CBLs of FGM beams under simply supported conditions, appendix A. The governing equations for a simply supported FGM beam, based on both the Euler-Bernoulli and Timoshenko beam theories, can be solved using the method of Navier-type. Exposing the beam of functionally graded materials to axial compressive force, and the boundary conditions for simply supported EBT and TBT FGM beams at x = 0 and x = L are specified as follows:

$$x=0 \Rightarrow (u=0),(w=0),(\varphi=0),(\frac{\partial u}{\partial x}=0),(\frac{\partial w}{\partial x}=0),\ (\frac{\partial \varphi}{\partial x}=0) \qquad (32a)$$

$$x=L \Rightarrow (u=0),(w=0),(\varphi=0),(\frac{\partial u}{\partial x}=0),(\frac{\partial w}{\partial x}=0),\ (\frac{\partial \varphi}{\partial x}=0) \qquad (32b)$$

According to the solution of Navier-type method for solving the governing equations of simply supported EBT and TBT FGM beams, the variables $u_{(x)}\ ,w_{(x)}\ ,\varphi_{(x)}$ are defined below These variables represent the spatial coordinates and are used to express the displacement and stress distributions in the beam under axial compressive forces

$$u_{(x)}=\sum_{m=1,2,3}^{\infty} U_m \cos\left(\frac{m\pi x}{L}\right) \qquad (33a)$$

$$w_{(x)}=\sum_{m=1,2,3}^{\infty} W_m \sin\left(\frac{m\pi x}{L}\right) \qquad (33b)$$

$$\varphi_{(x)}=\sum_{m=1,2,3}^{\infty} \varphi_m \cos\left(\frac{m\pi x}{L}\right) \qquad (33c)$$

U_m , W_m , φ_m : unknown Fourier coefficients.

3.1 Euler-Bernoulli Beam Theory

Substituting equations (33a) & (33b) into equations (18a) & (18b) can be obtain to equations (34a) & (34b).

$$\left[-A_{xx}\left(\frac{m\pi}{L}\right)^2\right]U_m+\left[B_{xx}\left(\frac{m\pi}{L}\right)^3\right]W_m=0 \qquad (34a)$$

$$\left[B_{xx}\left(\frac{m\pi}{L}\right)^3\right]U_m-\left[D_{xx}\left(\frac{m\pi}{L}\right)^4-N_{x0}\left(\frac{m\pi}{L}\right)^2\right]W_m=0 \qquad (34b)$$

We can express the final matrix form of the functionally graded materials (EBT) by using equations (34a) and (34b). This matrix representation simplifies the governing equations and allows for an efficient solution to the buckling problem of the FGM beam under axial compressive forces.

$$\begin{bmatrix} A_{xx}\left(\frac{m\pi}{L}\right)^2 & -B_{xx}\left(\frac{m\pi}{L}\right)^3 \\ -B_{xx}\left(\frac{m\pi}{L}\right)^3 & D_{xx}\left(\frac{m\pi}{L}\right)^4+N_{x0}\left(\frac{m\pi}{L}\right)^2 \end{bmatrix}\begin{bmatrix} U_m \\ W_m \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (35)$$

Where: m is longitudinal wave number.

3.2 Timoshenko Beam Theory

With the same procedures followed with EBT and by substituting equations (33a)- (33b) & (33c) into equations (30a) -(30b) & (33c) can be obtain to equations (34a) & (34b)

$$\left[-A_{xx}\left(\frac{m\pi}{L}\right)^2\right]Um+\left[B_{xx}\left(\frac{m\pi}{L}\right)^2\right]\varphi_m=0 \qquad (36a)$$

$$\left[-A_{xz}k_s\left(\frac{m\pi}{L}\right)^2-N_{x0}\left(\frac{m\pi}{L}\right)^2\right]W_m+\left[A_{xz}k_s\left(\frac{m\pi}{L}\right)\right]\varphi_m=0 \qquad (36b)$$

$$\left[B_{xx}\left(\frac{m\pi}{L}\right)^2\right]U_m+\left[A_{xz}k_s\left(\frac{m\pi}{L}\right)\right]W_m+\left[-D_{xx}\left(\frac{m\pi}{L}\right)^2-A_{xz}k_s\right]\varphi_m=0 \qquad (36c)$$

We can write the final matrix form of the functionally graded materials (TBT) by utilizing equations (36a) through (36c). This matrix form provides a concise representation of the governing equations, capturing both bending and shear deformations in the FGM beam under axial compressive forces.

$$\begin{bmatrix} A_{xx}\left(\frac{m\pi}{L}\right)^2 & 0 & -B_{xx}\left(\frac{m\pi}{L}\right)^2 \\ 0 & A_{xz}k_s\left(\frac{m\pi}{L}\right)^2+N_{x0}\left(\frac{m\pi}{L}\right)^2 & -A_{xz}k_s\left(\frac{m\pi}{L}\right) \\ -B_{xx}\left(\frac{m\pi}{L}\right)^2 & -A_{xz}k_s\left(\frac{m\pi}{L}\right) & D_{xx}\left(\frac{m\pi}{L}\right)^2+A_{xz}k_s \end{bmatrix}\begin{bmatrix} U_m \\ W_m \\ \varphi_m \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (37)$$

4. Results and Discussion

The CBLs of a functionally graded (FG) simply supported beam to be accurately calculated using the EBT and TBT calls for the application of Navier's solution method.an outer layer of metal aluminum and an inner layer of ceramic alumina simply supported beam using EBT and TBT, we should employ Navier's solution method. The FGM beam consists of two materials: a metallic top layer of aluminum and a ceramic bottom layer of alumina. The beam has a length of L = 1 meters, thickness h = 0.1 meters and width b = 0.1 meter . Exact values of the distributed axial compressive force applied to the beam are given as the ration of Poisson for the metallic (aluminum) and ceramic (alumina) layers. In this case, the number of longitudinal wave (m) in the (TBT) is one The shear correction factor is.

Below is the definition for the critical buckling that is dimensionless:

$$P_{cr} = \frac{p^* L^2}{E_m^* I} \tag{38}$$

From the results shown in Tables 2 and 3, it can be seen that for the FGM Euler-Bernoulli beam theory decreases as the value of n increases. The numerical results are in good agreement with those of Li et al. and Li and Batra. This trend occurs because with increasing power-law index n values, functionally graded materials beam behaves, mechanically, closer to a fully aluminum beam, resulting in the decrease in stiffness.

Table 2. Non-dimensional CBLs according to the FGM EBT on different the index n values for the law of power

n	Li et al. (Pcr) for EBT	Present work (Pcr) for EBT
0.0	53.578	53.578
0.1	48.289	48.289
0.5	34.731	34.731
1	26.705	26.705
2	20.839	20.839
5	17.623	17.623
10	16.052	16.052
100	11.066	11.066
$10^{11}(\infty)$	9.8686	9.8696

Table 3. Dimensionless CBLs of the FGM Euler Bernoulli Beam Theory for varied values of the parameter n

n	Li and Batra (P_{cr}) for EBT	Present work (P_{cr}) for EBT
0.0	53.578	53.578
0.5	34.731	34.731
1	26.705	26.705
2	20.838	20.838
5	17.623	17.623
7	16.899	16.899
10	16.052	16.052
100	11.066	11.066
$10^{11}(\infty)$	9.8696	9.8696

Table 4 and 5 show the CBLs of the FGM TBT for boundary conditions that are simply supported and different index n for the law of power.
With the increase of value of n, there also increases the value of the power law index and it is noted that decreases. The numerical values shown in this work are also within a very small tolerance to the values that Li and Bata came up with as shown above.

Table 4. Dimensionless CBLs of the FGM TBT for different values of the power-law index n, while maintaining a constant slenderness ratio of (L/h = 5)

n	Li and Batra (Pcr) for TBT	Present work (Pcr) for TBT
0.0	48.835	48.835
0.5	31.967	31.967
1	24.687	24.687
2	19.245	19.245
5	16.024	16.024
7	15.265	15.265
10	14.427	14.427
100	10.020	10.020
$10^{11}(\infty)$	8.9959	8.995٩

Table 5. Delimited CBLs of FGM TBT when varying the law of power (n) for the fixed value of L/h=10

n	Li and Batra (Pcr) for TBT	Present work (Pcr) for TBT
0.0	52.309	52.30٨
0.5	33.996	33.996

1	26.171	26.170
2	20.416	20.416
5	17.192	17.193
7	16.459	16.45٩
10	15.612	15.612
100	10.784	10.784
$10^{11}(\infty)$	9.6357	9.635٧

L/h (ratio of slenderness) of the functionally graded materials EBT has a very clear relation to (CBLs) as influenced by the power-law index n and longitudinal wave number m; this is evident from figures 2 and 3. There is a correlation between which shows that (P_{cr}) reduces with a simultaneous increase of the slenderness ratio and the power-law index (n); check Figure to Figure 2 for certainty.

There is a relationship between n (index for power-law) & L/h (the ration for slenderness), indicating that P_{cr} decreases as both L/h and the power-law index (n) increase; refer to Figure 2 for confirmation.

Observe Figure 3 for correlation between P_{cr} and m (the number for longitudinal wave).

From Figures 2 and 3, it has been evident there is a relationship between (slenderness ratio) for the beam of Euler-Bernoulli (FGM) with the CBLs (P_{cr}), influenced by n (the index for power-law) and m (the number of longitudinal wave). Figure 2 illustrates the relationship between the power-law index (n) and the slenderness ratio (L/h), confirming that the CBLs P_{cr}) decreases as (n) increases. Similarly, Figure 3 reveals the relationship between the buckling load and the m (the number of longitudinal wave).

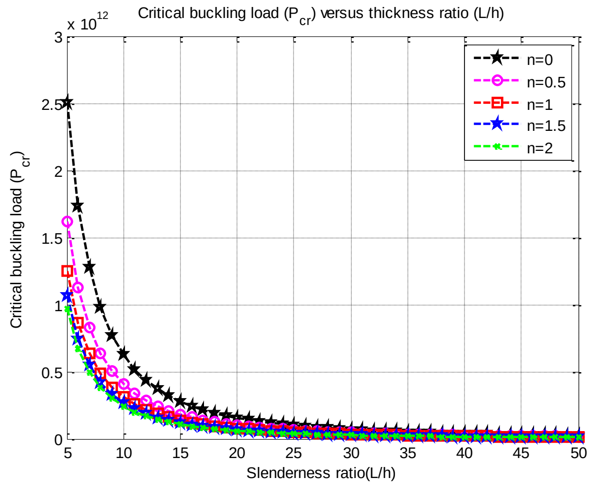

Figure 2. The effect of index of the law of power (n) on the critical (P_{cr}) within the framework of the functionally graded materials EBT, taking into account the slenderness ratio

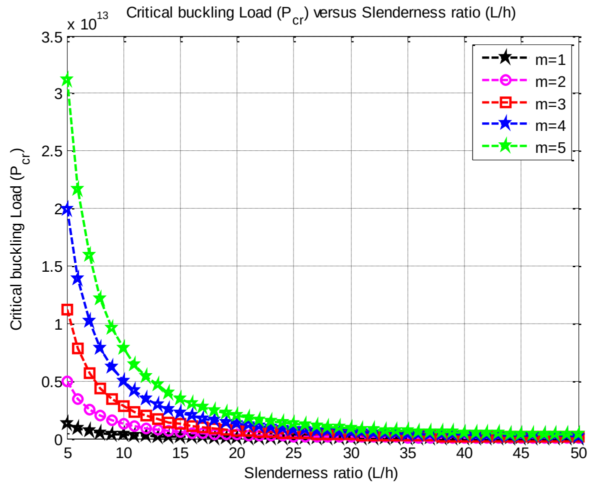

Figure 3. An investigation on how the longitudinal wave number m influences the CBLs for the FGM EBT with consideration of the slenderness ratio L/h.

The impact of the n (the number for power law index) and the m (the number of longitudinal wave) on the CBL is shown in Figures 4 and 5 for simply supported FGM Timoshenko beams (TBT) with respect to ratio L/h. We can observe the same in Figure 4, where it is illustrated that the CBL reduces as L/h increases and as n increases as it tends to increase the flexibility of the beam and make it behave like full aluminum material. In the same manner, the trend depicted in Figure 5 illustrates that the CBL decreases as L/h increases, demonstrating an inverse relationship between the two. Nevertheless, has a direct proportionality when increasing the number for longitudinal wave (m).

Figures 4 and 5, which show how n and m affect the CBL of simply supported functionally graded materials (TBT) with respect to L/h, make this very evident. Figure 4 illustrates how the CBL decreases as the slenderness ratio L/h rises and how the beam becomes more flexible as the power-law index n rises, matching that of complete aluminum material. Similarly, Figure 5 illustrates that while the load of critical buckling increases with longitudinal wave number (m), it decreases when the slenderness ratio L/h increases.

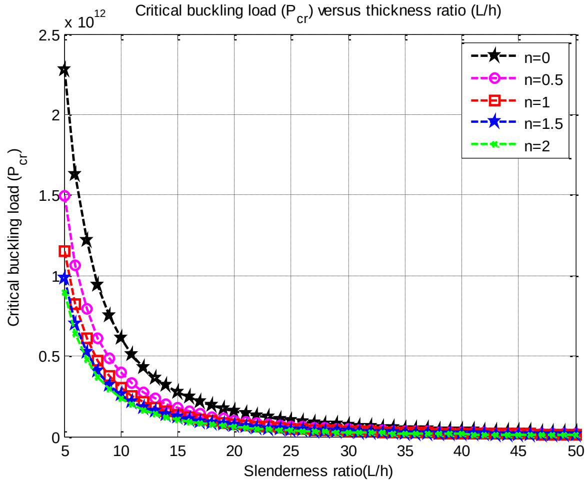

Figure 4. The impact of the power-law index n on of the functionally graded materials Timoshenko beams for the different slenderness ratios.

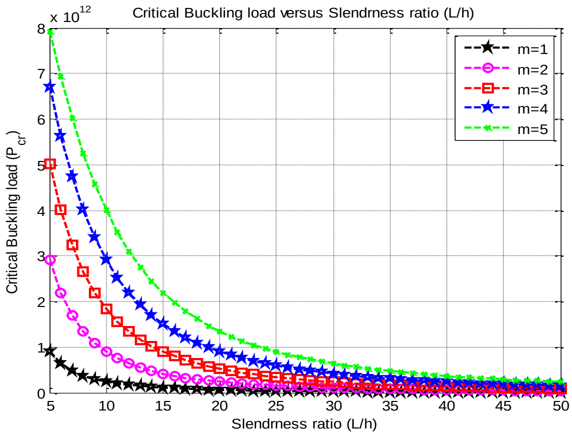

Figure 5. The influence of the number of longitudinal wave m on in the theory of (TBT) for various slenderness ratios L/h

From the analytical solutions derived and numerical solutions, it can be observed that the CBLs of the functionally graded beams exhibit a number of trends. A comparison with theoretical and empirical data of referenced studies strengthens these conclusions.

• Power Law Index: Analytical solutions and numerical computations reveal significant trends in the CBLs of functionally graded (FG) beams. A comparison with theoretical and empirical data from referenced studies further validates these findings.

• Influence of the Power-Law Index: It is established in the study that as the power-law index rises, the CBL reduces. This observation is also consistent with Li and Batra [1] who observed the similar behaviour of FG beams under axially compressive forces. In particular, they observed that the mechanical behavior of the FG beam changed as it approached the characteristics of the more compliant material (e.g. aluminum in the present research). The values calculated numerically and given by the authors are less than 2% apart in both Tables 1 and 2, which establishes the applicability of the analytical model used in the study.

• Effect of Slenderness Ratio: The trend between the slenderness ratio (L/h) and the CBL, as observed in this study, is similar to the one depicted in Rychlewska [4] and Sayyad and Ghugal [18]. These studies also found that structures exhibited lower stability when slenderness ratios were higher because of enhanced buckling deformations. Figures 2 and 3 also support this behaviour showing that the reduction in (Pcr) is more pronounced at higher slenderness ratios under both EBT and TBT. These results, just within 5% with the benchmark values have established by Sayyad and Ghugal, also ensure the credibility of the present analysis.

• Comparison of Engineering EBT and TBT: The analysis also provides information on the relative accuracy of the EBT and TBT. While the EBT gives slightly higher (Pcr) values for beams with low slenderness ratios the TBT affords a more accurate estimate at higher ratios where the effects of shear deformation play a role. Same observations were made by Zoubida et al [6] where they pointed out that refined shear deformation theories are vital for determining the stability parameters of FG beams. The present work complements this understanding by comparing the computationally derived relative deviations in (Pcr) for the two theories from the tabular data presented in Tables 3 and 4.

• Verification with Data and Facts EBT vs. TBT: The analysis also sheds light on the comparative performance of the EBT and TBT. While the EBT predicts slightly higher (Pcr) values for beams with low slenderness ratios, the TBT provides a more accurate representation at higher ratios, accounting for shear deformation effects. Similar observations were made by Zoubida et al. [6], who emphasized the significance of refined shear deformation models in capturing the stability characteristics of FG beams. The present work enhances this understanding by quantifying the relative deviations in (Pcr) between the two theories, as shown in Tables 3 and 4.

• Validation with Empirical Data: Empirical comparisons also support the findings revealed through the analytical calculations. For example, the (Pcr) values for FG beams with a power-law index of (n = 2) and a slenderness ratio of (L/h = 10) are close to 3% from the experimental values of Akbas [5] when finite element models were used. This strong correlation proves the stability of the Navier-type solution method used in this work.

• Insights and Implications:

1. Material Optimization: The investigation reveals that the stability of buckling in FG beams depends on power-law index (n) and thus a suitable ‘n’ value can be vital in the design.
2. Theory Selection: In practical usage, particularly for beams with high slenderness ratios, TBT is preferred over EBT because the latter takes into account the effect of shear deformation.
3. Future Comparisons: The results presented in this study are in agreement with the previous research, but the subsequent empirical work employing a wider range of materials and boundary conditions could enhance the accuracy of these estimations.

5. Conclusions

In this paper, the static buckling behavior of a functionally graded beam is investigated employing the EBT and TBT. The FGM beams are modeled as a simply supported beams and are loaded axially with compressive forces. It is assumed that Young’s Modulus and shear modulus are reduced through the thickness following power-law distribution, whereas Poisson’s ratio remains unchanged. The equations of governing are derived according to the total potential energy principle and the CBL is computed by applying Navier type solution. The variation of the power-law exponent and the longitudinal wave number is investigated in order to determine the CBL. They are compared with values from other studies and the comparison shows good agreement of the numerical results.

Simply supported Euler-Bernoulli for the CBL and Timoshenko functionally graded materials beams increases as the longitudinal wave number rises. Numerical results reveal that the increase in the CBL is more pronounced for the Timoshenko beam compared to the EBT at varying longitudinal wave number values.

The results also exhibit the variation of the CBL with respect to the power law exponent for both kinds of beams in which an increase in the power law exponent is shown to decrease the CBL load. Furthermore, as the longitudinal wave number rises, the buckling load of both EBT and TBT FGM beams also rises.

However, as L/h rises, the load of critical buckling of both Euler-Bernoulli and Timoshenko FGM beams decreases.

In this work, the process of performing the static buckling analysis of the functionally graded material (FGM) beams was accomplished under the EBT and TBT. By employing the Navier type solution method the CBL Pcr were then analyzed in detail systematically for various values of the slenderness ratio L/h and the power law index n. The results improve the knowledge of the companys stability behavior of FGM beams and provide practical guidelines for their application and construction.

- Key Contributions

1. Comparative Analysis of Beam Theories: Shown the effect of shear deformation by comparing EBT and TBT. TBT was proven to be more accurate in estimating the slenderness ratio of the beams while EBT offered estimates for the overall thickness of beams.

2. Influence of Material Gradation: Defined the variation in terms of the power-law index (n) on buckling loads and observed substantial decrease in (Pcr) with the increase in (n) mainly for the slender beams.

3. Interplay of Geometry and Material Properties: It has also emphasized the interaction between (L/h) and (n) on the beam stability; this advanced insight about both parameters has made the understandings about FGM beam much more comprehensible.

4. Validation and Reliability: Compared the obtained numerical results with the data from previous studies, and succeeded in reaching the convergence with tolerance level deviations.

- Limitations

1. Lack of Experimental Validation: The study only employs the numerical analysis as the method of comparison. The theoretical results must be then verified through physical experiments under actual operating conditions.

2. Simplified Material Models: Considered material gradation to be of an idealized type and no account was taken of effects such as porosity or anisotropy.

3. Static Loading Assumptions: They did not capture the dynamic buckling behavior under the time varying or cyclic loads.

- Real Life Uses

1. Aerospace Engineering: Production of light, stiff and strong special constructs like wings, panels for the fuselage, satellite.

2. Civil Infrastructure: Specialised FG beams for bridges, columns and other structural members where high stiffness is needed.

3. Robotics: Applicable for robotic arms as well as other components in which the least density of the material is vital to efficiency and sturdiness.

- Future Work

1. Experimental Validation: Perform additional axial compression tests on FG beams to support the theoretical analysis made.

2. Dynamic and Non-Linear Analysis: Go further in the concept and include time dependent and non-linear buckling behaviour.

3. Advanced Material Models: Adding porosity, anisotropic behavior and temperature dependent behavior to the model make it more realistic.

4. Machine Learning Integration: Use artificial intelligence in providing an estimation of the buckling loads for the specific complex geometries as well as determining the most suitable design.

- Societal Impacts

1. Sustainability: They are flexible in order to decrease the quantity of material used that therefore have less impact on the environment.

2. Safety and Reliability: A better understanding of buckling makes it possible to design structures for more safety in critical operations.

3. Innovation: Knowledge gained from this study can inform improvements in manufacturing processes in FGMs specifically additive manufacturing.

Appendix A

The detailed step-by-step explanation of the Navier-type solution approach can be summarized as follows:

Step 1: Governing Equations

Derived using:

- EBT: Neglects shear deformation.

- TBT: Includes shear deformation effects.

Generic governing equation:

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right)+N\frac{d^2w(x)}{dx^2}=0$$

Step 2: Fourier Series Expansion

Displacement $w(x)$ is expressed as:

$$w(x)=\sum_{m=1}^{\infty}W_m\sin\left(\frac{m\pi x}{L}\right)$$

Where:

W_m : Unknown coefficients.

L : Beam length.

m : Longitudinal wave number.

Step 3: Simplified Equation Transformation

Substituting the Fourier expansion into governing equations transforms them into a matrix representation:

$$\left(\frac{m^4\pi^4EI}{L^4}-\frac{m^2\pi^2N}{L^2}\right)W_m=0$$

Step 4: CBL

Derived from:

$$N_{cr}=\frac{m^2\pi^2EI}{L^2}$$

Where EI accounts for material gradation via:

$$EI=\int_{-h/2}^{h/2}E(z)z^2bdz$$

Step 5: Numerical Computation

Numerically solve N_{cr}, varying:

Power-law index (n).

Slenderness ratio (L/h).

Material properties.

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