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Monoform Modules Relative to a Pure Property

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Abstract

The monoform concept is defined as a module in which every nonzero submodule is rational. The primary goal of this research is to study this class of modules in terms of pure property, named purely monoform modules. It constitutes an extension of the monoform modules; in fact, the monoform modules are properly included in the class of purely monoform modules. Many characteristics of a purely monoform module have been offered as analogues to those in monoform modules. A discussion of how this class of modules relates to other related modules is considered, like almost monoform, purely uniform, purely quasi-Dedekind and purely prime modules. Besides that, other characterizations of the purely monoform module have been given similar to those known and satisfied in the monoform modules.

Keywords: Rational submodules, Pure submodules, Monoform modules, Purely monoform modules

المقاسات أحادية الصيغة المتعلقة بخاصية النقاء

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الخلاصة

تُعرّف المقاسات أحادية الصيغة على أنها تلك المقاسات التي كل مقاس جزئي غير صفري فيها يكون نسبياً. ان الهدف الأساسي من هذا البحث هو دراسة هذا الصنف من المقاسات بواسطة خاصية النقاء، وأطلقنا عليه بالمقاسات أحادية الصيغة النقية. انه يشكل امتداداً للمقاسات أحادية الصيغة، في الواقع ان المقاسات أحادية الصيغة النقية تحوي فعلياً المقاسات أحادية الصيغة. العديد من خواص المقاسات أحادية الصيغة النقية قُدّمت في هذا البحث والتي تكون مناظرة للصفات المتحققة في المقاسات أحادية الصيغة. كما تم مناقشة علاقة هذا النوع من المقاسات بمقاسات أخرى ذات العلاقة، مثل المقاسات أحادية الصيغة تقريباً، المقاسات المنتظمة النقية، المقاسات شبه الديديكاندية النقية والمقاسات الاولية النقية. الى جانب ذلك، تم اعطاء تشخيصات أخرى للمقاسات أحادية الصيغة النقية مناظرة لما هو متحقق ومعروف في مفهوم المقاسات الأحادية النقية.

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1. Introduction

Many researchers studied monoform modules such as H.H. Storrer, J. Zelmanowitiz, I.M.A. Hadi, A. Hajikarimi and A.R. Naghipour. This paper aims to generalize this class of modules by using the pure property. This new concept is named purely monoform modules. Rational submodule T in a module M (shortly, $T \leq_r M$) is defined as $\mathrm{Hom}_R(\frac{M}{T}, \mathcal{E}(M))=0$, $\mathcal{E}(M)$ is indicated to the injective envelope of M, [1, P.274]. Monoform is a module in which every nonzero submodule is rational, [2]. We denoted to the pure submodule T of M by $T \leq_{pu} M$, and it is defined as $T \cap HM = HT$ for every ideal H of R, [3].

There are three sections in this article. In section two, purely monoform modules are studied. Various properties and characterizations of this type of module are presented and discussed that are comparable to the results that are known in the polyform module, Among the main of these outcomes are listed below:

- Consider the following for any module M satisfying Condition 2.5.

i. All partial endomorphisms f: N⟶M with a nonzero pure submodule N≤M having zero kernels in their domains (i.e., f is monomorphism).

ii. $\mathrm{Hom}_R(\frac{M}{N}, \mathcal{E}(M))=0$ for each nonzero pure submodule N of M. (i.e., M is a purely monoform module).

Then **(i) ⟹ (ii)**.

See Proposition 2.6

- If a module M is purely monoform and satisfies Condition 2.7, then all partial endomorphisms f: N⟶M where N is a nonzero pure in M have zero kernels in their domains (i.e., f is monomorphism). See Proposition 2.8.

- If M is a purely monoform module then the kernel of f: M⟶ $\mathcal{E}(M)$ is equal to 0 for each nonzero homomorphism f. See Proposition 2.10.

In section three, another characterization and partial characterization of purely monoform modules are given such as the following theorems:

- A module M is purely monoform if and only if any one of the following is achieved:

i. M is a purely uniform and PF-polyform module.

ii. M is a purely P-uniform and PF-polyform module.

See Theorem 3.11.

This kind of module relationship with other relevant concepts is taken into consideration, including the following theorems:

- The statements below are equivalent to any module M on a regular ring R.

i. M is almost monoform;

ii. M is purely monoform;

iii. M is monoform;

iv. M is a purely P-uniform and PF-polyform module.

See Theorem 3.13.

- Take a quasi-Dedekind ring R and the below statements:

1. R is polyform.

2. R is QI-monoform.

3. R is monoform.

4. R is purely monoform.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

See Theorem 3.21.

It is worth noting that any ring mentioned in this work has an identity and commutative, all modules are unitary left R-modules. They are indicated by R and M respectively.

2. Purely monoform modules

This section introduces an extension of the monoform module, called a purely monoform module. We start with its definition.

Definition 2.1: A nonzero module M is termed purely monoform if every nonzero T$\leq_{pu}$M is rational. If R is a purely monoform R-module, then R is named a purely monoform ring.

Note: From now on, we will write mono instead of monoform to avoid plagiarism.

Remarks 2.2: Each mono module is purely mono because, in the mono module, every nonzero submodule of M is rational, hence every nonzero pure is rational. The reverse is not true, for instant Z_{p^∞} as Z-module, is a purely mono Z-module since Z_{p^∞} itself is the only nonzero pure submodule and $Z_{p^\infty} \leq_r Z_{p^\infty}$. In contrast, Z_{p^∞} is not mono since the nonzero submodule $< \frac{1}{p}$+Z$>$ of Z_{p^∞} is not rational in Z_{p^∞}.

Remarks and examples 2.3:

1. In the class of F-regular modules, there is no difference between purely mono and mono modules, where an R-module M is termed F-regular if every submodule of M is pure, [4].

2. The Z-module Z_4 is purely mono, since Z_4 contains only one nonzero pure submodule which is Z_4, and it is rational in itself.

3. As the only nonzero pure in a simple module is itself, Besides that it is a rational submodule, so every simple module is purely mono.

4. A semisimple module is not purely mono, since in a semisimple module say M all submodules are direct summands, and clearly, the only rational submodule in M is itself.

A pure simple is a nonzero module in which the only pure submodules are only (0) and itself, [5]. Examples of such modules are the Z-modules Z and Q, where Z is the integers and Q is the set of rational numbers.

5. Any pure simple module is purely mono. Indeed, the only nonzero pure submodule of this kind of module is itself which is rational.

6. Every purely mono module is indecomposable.

Proof: Take a purely mono M, and assume the contrary, so ∃ T$\lneqq$M and G$\lneqq$M such that M=T⊕G. Now, $\mathrm{Hom}_R(\frac{M}{T}, \mathcal{E}(M)) \cong \mathrm{Hom}_R(G, \mathcal{E}(M)) \neq 0$, which contradicts the assumption.

7. Any integral domain is a purely mono ring. In fact, it is known that any integral domain is a mono ring, hence it is purely mono.

The next result concerns the hereditary property of a purely mono module.

Proposition 2.4: A nonzero pure submodule of a purely mono is itself purely mono.

Proof: Take a nonzero pure submodule N of M and a pure submodule T of N. By [3], T is pure in M. Since M is a purely mono module, then T$\leq_r$M, hence T$\leq_r$N, [1, Proposition 8.7, P.274].

Condition 2.5: For any submodules T≤ B≤ U. If T$\leq_{pu}$U, then B $\leq_{pu}$U.

In view of [2], a mono module is defined as all nonzero partial endomorphisms of M are monomorphisms. For a purely mono module, the following is obtained.

Proposition 2.6: If M satisfies Condition 2.5 and:

i. All partial endomorphisms f: N⟶M with a nonzero pure submodule N of M have zero kernels in their domains (i.e., f is monomorphism).

ii. $\mathrm{Hom}_R(\frac{M}{N}, \mathcal{E}(M))=0$, for each nonzero pure submodule N of M. (i.e., M is a purely mono module).

Then **(i) ⟹ (ii):**

Proof: Consider (i), and take a nonzero pure submodule N of M. Assume that f: $\frac{M}{N} \longrightarrow \mathcal{E}(M)$ is a homomorphism. If f≠0, then ∃m+N∈$\frac{M}{N}$ with f(m+N)=ḿ≠0, ḿ ∈ $\mathcal{E}(M)$. Since $M \leq_e \mathcal{E}(M)$, so ∃r∈R with 0≠rḿ∈M. Put rm′=x. Define φ: N+Rm ⟶ Rx⊆M by φ(n+rm)=rx ∀n∈N, r∈R. To prove that φ is well-defined, assume that n_1+r_1m=n_2+r_2m where n_1, n_2∈N, r_1,r_2∈R, that is n_1-n_2=(r_1-r_2)m∈N. But

f [(r_1-r_2)(m+N)]=f [(r_1-r_2)m+N]=0. ………...(1)

Also,

f(r_1-r_2)(m+N)=(r_1-r_2)f(m+N)= (r_1-r_2)ḿ. …….(2)

From (1) and (2) we get (r_1-r_2)ḿ=0, that is r_1ḿ=r_2ḿ, then $r_1 r$ḿ=$r_2 r$ḿ, hence r_1x=$r_2 x$. This implies that $\varphi(n_1$+r_1m)=r_1x = $\varphi(n_2$+r_2m)= r_2x, therefore, φ is well-defined. Also, φ is a nonzero homomorphism. It remains to prove that N⊆kerφ, let n∈N that is n=n+0m, so that φ(n)=0x=0, that is N⊆kerφ. Now, since N⊆kerφ ⊆M and $N \leq_{pu} M$, by Condition 2.5, kerφ $\leq_{pu}$M. Now, kerφ ⊆ N+Rm ⊆ M, again by Condition 2.5, we have N+Rm $\leq_{pu}$M. From (i), kerφ =0, therefore, N=0, which is a contradiction, thus f=0.

The following condition is useful to prove the other direction of Proposition 2.6.

Condition 2.7: For all submodules T, D and H with T≤ D≤ H. If $D \leq_{pu} H$, then $T \leq_{pu}$ H.

Proposition 2.8: If M is a purely mono module and satisfies Condition 2.7, then all partial endomorphisms f: N⟶M with $0 \neq N \leq_{pu} M$, have zero kernels in their domains (i.e., f is monomorphism**).**

Proof: Take a purely mono module M and a nonzero pure N≤ M, 0≠f: N→M. If kerf=0 then we are done, so assume that kerf≠0. By the first isomorphism theorem $\frac{N}{\mathrm{kerf}} \cong$f(N), so there is an isomorphism Ψ: $\frac{N}{\mathrm{kerf}} \to$ f (N). It is clear that Ψ ≠0. Take the inclusion homomorphism f(N) $\xrightarrow{i}$ M, so we have the following sequence:

$\frac{N}{\mathrm{kerf}} \xrightarrow{\Psi}$ f(N) $\xrightarrow{i}$ M

. Since i∘Ψ≠0, then (i∘Ψ)($\frac{N}{\mathrm{kerf}}$)≠0. Now, kerf ≤ N ≤ M, and N is pure in M, so by Condition 2.7, kerf is pure in M. In contrast, M is purely mono, which implies that kerf is a rational submodule. That means $\mathrm{Hom}_R(\frac{N}{\mathrm{kerf}}$, M)=0, [1, Proposition 8.6, P.274]. But this is impossible because i∘Ψ≠0, therefore kerf=0.

From Proposition 2.6 and Proposition 2.8, a partial characterization of the definition of a purely mono module is given as follows.

Corollary 2.9: If a module M satisfies Condition 2.5 and Condition 2.7, then M is a purely mono module if and only if all partial endomorphisms f: N⟶M with N is a nonzero pure in M, having zero kernels in their domains.

Proposition 2.10: For every nonzero f∈Hom(M, $\mathcal{E}(M)$), kerf equals zero whenever M is a purely mono module.

Proof: Assume that M is a purely mono module and let f: M ⟶ $\mathcal{E}(M)$ be a homomorphism with kerf ≠0. We have to show that f=0. Define g: $\frac{M}{\mathrm{kerf}} \longrightarrow \mathcal{E}(M)$ by g(m+kerf)=f(m) for all m∈$\mathcal{E}$(M). To verify that g is well-defined, assume that x_1+kerf=x_2+kerf, x_1, x_2∈M. This implies that (x_1-x_2)∈kerf that is f(x-x_2)=0. Since f is a homomorphism, then f(x_1)-f(x_2)=0, hence f(x_1)=f(x_2). Moreover, since f≠0, then g≠0. This means $\mathrm{Hom_R}(\frac{M}{\ker f}, \mathcal{E}(M))\neq 0$ which is a contradiction, therefore, f=0. Thus, kerf=0.

A module M is named Artinian if every descending chain of submodules is terminated, [2].

Proposition 2.11: Every nonzero Artinian module has a purely mono submodule.

Proof: Take a nonzero Artinian module M, and N≤M. Our goal is to prove that N is a purely mono submodule. If N is purely mono then there is nothing to prove, otherwise, there is a nonzero pure submodule N_1 of N such that $N_1 \nleq_r N$. Next, if N_1 is purely mono then we are done, if not then there exists a nonzero pure $N_2 \leq N_1$ with $N_1 \nleq_r N_2$. If N_2 is mono then we are through, otherwise, there is a nonzero pure submodule N_3 of N_2 such that $N_3 \nleq_r N_2$. By continuing in this process, the descending chain of submodules is deduced:

$$N \geq N_1 \geq N_2 \geq N_3 \geq N_4 \geq \ldots\ldots$$

After a finite number of steps, we must find a submodule in M which is a purely mono, otherwise, we would have a contradiction because M is an Artinian module. Thus, N is a purely mono submodule.

3. Purely mono modules and related concepts

This part of our work is devoted to examining the connection of the purely mono module with other related modules such as polyform, P-polyform, P-uniform, essentially quasi-Dedekind, purely quasi-Dedekind and SQD modules.

P-rational submodule N of M (shortly, $N \leq_{pr} M$) is pure and satisfies $\mathrm{Hom_R}(\frac{M}{N}, \mathcal{E}(M)) = 0$, [6]. An R-module M is called almost mono if every 0≠N≤M is P-rational in M, [7]. Note that the class of purely mono modules contains the class of almost mono modules properly, as shown in the following.

Remark 3.1: Every almost mono module is purely mono.

Proof: Since every almost mono module is mono, the last concept implies the purely mono module. Therefore, the outcome is achieved.

The opposite side of Remark 3.1 is not necessarily true, as shown in Example 2.3(2), we verify that Z_4 as Z-module, is purely mono. In contrast, that Z_4 is not almost mono, [7, Example 2.4(7)]. For the same reason, the Z-module Z_{p^∞} is a purey mono module as we saw in Remark 2.2, while Z_{p^∞} not almost mono module, [7, Example 2.4(8)].

Remark 3.2: As a consequence of the above argument, one can deduce that the mono module lies between purely mono and almost mono modules.

However, the two concepts coincide under certain conditions, as shown in the following.

Proposition 3.3: A module M is almost mono if and only if M is purely mono, provided that M is F-regular.

Proof: Take 0≠N≤M. because M is F-regular, then N is pure. In addition, M is purely mono, therefore, $N \leq_r M$. So, N is rational and pure, hence $N \leq_{pr} M$. The first direction is straightforward.

A nonzero submodule T of M is said to be essential (shortly, $T\leq_e M$) if T∩H≠0 for each 0≠ H ≤ M, ([8], P.15). A nonzero module M is called purely uniform if every nonzero pure submodule of M is essential in M, [7].

Proposition 3.4: All purely mono are purely uniform modules.
Proof: Since in the purely mono every nonzero pure submodule is rational, and every rational submodule is essential, therefore, M is a purely uniform module.

T is a P-essential submodule of M (briefly $T\leq_{pe}M$) if ∀ $U\leq_{pu}M$ with T∩U=(0) implies U=(0), [9]. A module M is called P-uniform if every nonzero submodule of M is P-essential, [6]. In the following, we give a restriction of P-uniform modules.

Definition 3.5: A purely P-uniform is a module in which all nonzero pure submodules are P-essential.

Remark 3.6: Each purely mono module is a purely P-uniform.

Proof: Take a purely mono module M, that is every nonzero pure submodule of M is rational. But any rational submodule implies essential, hence P-essential. Thus, the result is achieved.

Recall that a module M is named polyform if all essential submodules of M are rational, [2]. As a special kind of this known concept, we define the following.

Definition 3.7: A module M is named purely polyform if all nonzero pure and essential submodules of M are rational.

Remark 3.8: Any purely mono module M is purely polyform.

Proof: Since M is a purely mono module, then every nonzero pure (hence nonzero pure and essential) submodule of M is rational. Thus, the result is obtained.

An R-module M is said to be fully polyform if each P-essential submodule of M is rational. That is $\mathrm{Hom_R}(\frac{M}{N}, \mathcal{E}(M) = 0$ for any P-essential N≤M, [6]. There is no direct implication between purely mono and fully polyform, this motivates the authors to define the following.

Definition 3.9: A PF-polyform M is a module in which every nonzero pure and P-essential submodule of M is rational.

Remark 3.10. Each purely mono module is a PF-polyform.

Proof: Take $N\leq_e M$. Since M is purely mono then every nonzero (especially each P-essential) submodule of M is rational, so the result is done.

The following gives characterizations of purely mono

Theorem 3.11: M is purely mono module if and only if any one of the following is achieved:
1. M is a purely uniform and PF-polyform module.
2. M is a purely P-uniform and PF-polyform module.
Proof:

1. Take a purely mono module M. By Remarks 3.4 and 3.10, point (1) is obtained. For sufficiency, taking 0≠N≤M with N is pure. Now, M is a purely uniform implying that $N\leq_e M$, hence $N\leq_{pe} M$, [9]. Besides that, M is fully polyform, so $N\leq_r M$, this means M is purely mono.
2. For the necessity, suppose that M is purely mono, by Remarks (3.6) and (3.10), point (2) is followed. Conversely, taking 0≠N≤M with N is pure. because M is P-uniform then $N\leq_{pe} M$. In addition, M is fully polyform implying $N\leq_r M$. Thus, M is purely mono.

Next, the two modules purely mono and purely uniform are equivalent under certain conditions, as follows, previously, a module M is named multiplication if each N≤M can be written as N = HM for a certain ideal H of R, [10].

Theorem 3.12: For any multiplication module M with a prime annihilator, M is a purely mono module iff M is a purely uniform.

Proof: The direction one is clear. For sufficiency, take a nonzero pure submodule N of M. If $N\nleq_r M$, then there exists V≤M, N≤V≤M such that $\mathrm{Hom}_R(\frac{V}{N}, M)\neq 0$, i.e., there is $0\neq f\in \mathrm{Hom}_R(\frac{V}{N}, M)$. This implies that there is $x+N\in \frac{V}{N}$, x∉N such that f(x+N)=m≠0. Now, M is purely uniform implying that $N\leq_e M$, so there exists r≠0 with 0≠rx∈N, ([1], Definition 3.26). It follows that rm=rf(x+N) = f(rx+N)=0, hence rm=0. Besides that, M is multiplication means Rm=KM for some an ideal K of R. Hence rKM=(0). Thus, $rK\subseteq \mathrm{ann}_R M$. But $\mathrm{ann}_R M$ is a prime ideal of R, so we have two cases: either $r\in \mathrm{ann}_R M$ or $K\subseteq \mathrm{ann}_R M$. If $r\in \mathrm{ann}_R M$, then rM=0, and since x∈M, thus rx=0, which is a contradiction. The other case is $K\subseteq \mathrm{ann}_R M$, hence Rm=KM=(0), a contradiction since m≠0. Therefore, $\mathrm{Hom}_R(\frac{V}{N}, M)=0$, that is $N\leq_r M$, hence M is a purely mono.

Theorem 3.13: For any module M on a regular ring R, the statements below are equivalent:
1. M is an almost mono;
2. M is purely mono;
3. M is mono;
4. M is a purely P-uniform and PF- polyform.

Proof:
(1)⇒(2): It is clear by Remark 3.1.
(2)⇔(3): The regularity of R implies M is F-regular, [3]. Additionally, by Remark 2.3(1), the two concepts, purely mono and mono modules are identical.
(3)⇒(4): Let 0≠N≤M, By assumption, $N\leq_r M$, hence $N\leq_e M$, so that $N\leq_{pe} M$, [9]. Thus, M is P-uniform. Now, assume that $K\leq_{pe} M$, since M is a mono module then each nonzero pure (hence each nonzero pure and essential) submodule of M is rational, thus, $K\leq_r M$, and the proof is completed.
(4)⇒(1): Since M is a purely P-uniform and PF-polyform module, then by Theorem 3.11(2), M is a purely mono module. But R is a regular ring, then M is an F-regular module, [3]. And by Proposition 3.3, M is an almost mono module.

Next, point (4) of Theorem 3.13 can be replaced with (M is a purely uniform and PF-polyform R-module) as follows:
Theorem 3.14: The below assertions are identical for any module M over a regular ring R,
1. M is almost mono.
2. M is purely mono.

3. M is mono.
4. M is a purely uniform and PF-polyform.

Proof: Similarly to the proof of Theorem 3.13, the only difference is in depending on Theorem 3.11(1) instead of Theorem 3.11(2).

The following lemma is needed, it is appeared in ([1], Exc 8.4, P.284).

Lemma 3.15: Let M be a nonsingular uniform module, then any nonzero submodule of M is rational in M.

Proposition 3.16: Any nonsingular module M is a purely mono iff M is uniform.
Proof: The direction one is evident. To prove sufficiency, Since M is nonsingular and uniform, then according to Lemma 3.15, M is mono, and by Remark 2.2, M is a purely mono module.

A quasi-invertible submodule T of M (briefly, $T\leq_{pq}M$) is defined as $Hom_R(\frac{M}{T}, M) = 0$. A quasi-Dedekind is a module in which every nonzero submodule is quasi-invertible, [11]. T is termed a purely quasi-invertible submodule of M (briefly, $T\leq_{pqu}M$) if T is pure and satisfies $Hom_R(\frac{M}{T}, M) = 0$. A module M is said to be purely quasi-Dedekind if every proper nonzero pure submodule of M is quasi-invertible, [12].

Proposition 3.17: Any purely mono module is purely quasi-Dedekind.
Proof: Take a purely mono module M and a nonzero pure $N< M$. Because M is purely mono then $N<_{r}M$, hence $N<_{qu}M$, [11]. In addition, N is pure, thus M is purely quasi-Dedekind.

We think the reverse of Proposition 3.17 is not generally true, but we didn't find an example. But as the following illustrates, that is valid when M is a multiplication module as shown below. Firstly, the following lemma is required.

Lemma 3.18: ([11], Theorem 3.11, P.18)

Taking a multiplication module M with its annihilator is a prime ideal. $N\leq M$ is quasi-invertible if and only if N is an essential submodule of M.

Proposition 3.19: M is purely mono if and only if it is a purely quasi-Dedekind module, provided that M is multiplication.
Proof: The direction one is just Proposition 3.17. Conversely, assume that M is purely quasi-Dedekind and $0\neq T\leq_{pu}M$. By assumption, $N\leq_{pqu}M$, hence $N\leq_{qu}M$. Moreover, M is a multiplication, so by Lemma 3.18, $N\leq_{r}M$. hence M is purely mono.

Following [12], if a module M is satisfied $ann_R(M) = ann_R(N)$ for each nonzero pure submodule N of M, then M is called purely prime.

Proposition 3.20: For any R-module M, the implications (1)⇒(2)⇒(3) ⇒(4) hold, where:
1. M is mono
2. M is purely mono.
3. M is purely quasi-Dedekind.
4. M is purely prime.

Proof:
(1)⇒(2): By Remark 2.2

(2)⇒(3): By Proposition 3.17.
(3)⇒(4): [12].

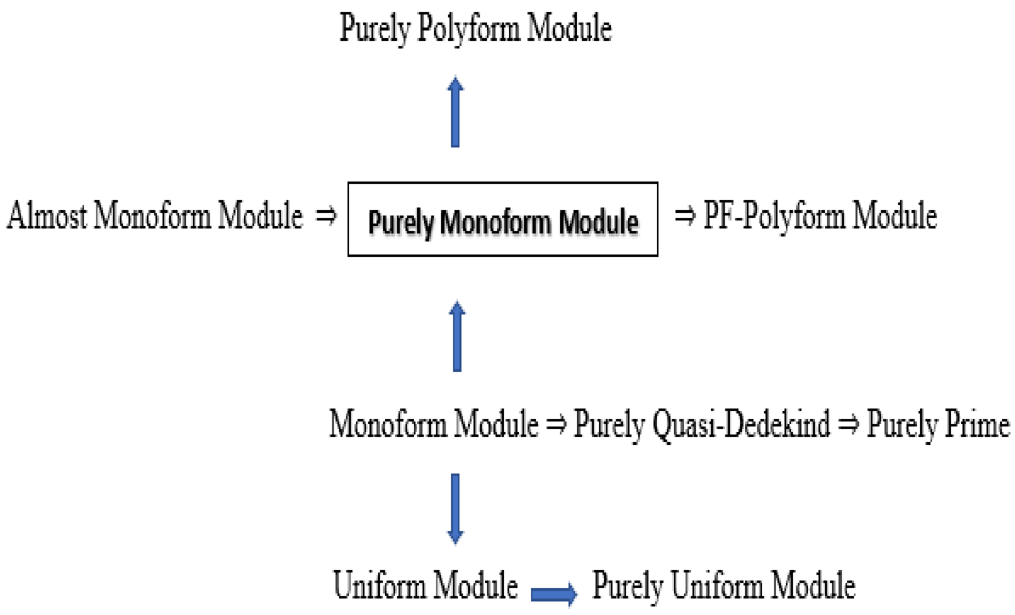
The following result is satisfied only in the category of rings. Before that, a quasi-invertibility mono or QI-mono is a module in which every nonzero quasi-invertible submodule is rational, [6].

Proposition 3.21: Let R be a quasi-Dedekind ring. Consider the following:
1. R is a polyform ring.
2. R is QI-mono.
3. R is mono.
4. R is purely mono.
Then (1) ⇒ (2) ⇒ (3) ⇒ (4).
Proof:
(1) ⇒ **(2):** Let I be a nonzero quasi-invertible ideal of a ring R. By ([11], Corollary 2.3, P.12), $I \leq_e R$. Since R is polyform, then $I \leq_r R$. So that R is a QI-mono ring.
(2) ⇒ **(3):** Let R be a QI-mono ring, and I a nonzero ideal of R. Since R is quasi-Dedekind, then $I \leq_{qu} R$. Besides that, R is a QI-mono ring, therefore, $I \leq_r R$. Thus, R is a mono ring.
(3) ⇒ **(4)**: It is obvious.

The following are achieved since every integral domain is a quasi-Dedekind ring.
Corollary 3.22: For any integral domain R, the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) hold, where:.
1. R is polyform.
2. R is QI-mono.
3. R is mono.
4. R is purely mono.

4. Conclusions

This work extends the class of mono into a new class of modules. The term for it is " purely mono module". This paper's main finding can be summed up as follows:
1. Several useful properties of purely mono modules have been shown that are analogous to those found in the concept of mono modules.
2. Other characterizations of purely mono modules are given.
3. Discuss appropriate conditions for the equivalence of mono and mono modules.
4. The connection of the purely mono module with other related concepts has been established such as almost mono, purely polyform, PF-polyform, purely quasi-Dedekind and purely prime modules. Nevertheless, the following figure can depict each of these relationships:



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