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Solution of Euler's Partial Differential Equations Using the General Polynomial Transform: A Novel Approach to First- and Second-Order PDEs

Murat Düz*[A], Emad Kuff [B], Ahmad Issa [A]

[A] Department of Mathematics/Faculty of Sciences/Karabuk University-Karabuk, Turkey

[B] Department of Mathematics/College of Basic Education/Mustansiriyah University-Iraq

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*Corresponding author:

Ahmad Issa

Department of Mathematics/Faculty of Sciences/Karabuk University-Karabuk, Turkey

Abstract: In this paper, Euler's Partial Differential Equations are solved by general polynomial transform. In addition, a general form of Euler's partial differential equation of the first and second order is obtained, and these formulas are applied to some initial value problems. Three examples of first and second order partial differential equations are solved to demonstrate the accuracy and efficiency of the proposed method.

حل معادلات اويلر التفاضلية الجزئية باستخدام التحويل العام لمتعددة الحدود: اسلوب جديد للمعادلات التفاضلية الجزئية من الرتبة الاولى والثانية

مراد دوز	عماد عباس كوفي	احمد عيسى
قسم الرياضيات/كلية العلوم	قسم الرياضيات/كلية التربية الاساسية	قسم الرياضيات/كلية العلوم
جامعة كارابوك-تركيا	الجامعة المستنصرية-العراق	جامعة كارابوك-تركيا

المستخلص

في هذا البحث، تم حل معادلات اويلر التفاضلية الجزئية باستخدام التحويل العام لمتعددات الحدود، بالإضافة الى ذلك تم الحصول على صيغة عامة لمعادلة اويلر التفاضلية الجزئية من الرتبة الاولى والثانية ويتم تطبيق هذه الصيغ على بعض مسائل القيم الابتدائية. تم حل ثلاث امثلة للمعادلات التفاضلية الجزئية من الرتبة الاولى والثانية لإثبات دقة وكفاءة الطريقة المقترحة.

الكلمات المفتاحية: التحويل متعدد الحدود العام، المعادلات التفاضلية الجزئية الخطية، معادلة أويلر، مشكلة القيمة الابتدائية، تجانس غير تجانس، المعادلات التفاضلية الجزئية من الدرجة الأولى والثانية.

1. Introduction

In mathematics, a partial differential equation is a type of differential equation or a relationship that includes an unknown function or functions that have several independent variables as well as the partial derivatives of these variables. Partial differential equations are used to formulate and solve problems related to functions of several variables, such as those found in sound, heat, static, electricity, fluid flow, elasticity, etc., as it possible to express different physical phenomena using mathematical equations to similar form [3-6]. Euler's equations are a set of partial differential equations that describe the motion of fluids and gases in mathematical physics. These equations are named after the SWISS physicist Leonhard Euler who developed them in the 8^{th} century. Euler's equations are used in various fields such as marine engineering, aviation, and atmosphere control [7-9].

Many papers and researchers were interested in solving partial differential equations using integral transforms such as the Laplace, Sumudu, Elzaki, SEE, complex SEE, Aboodh, Kamal, Shehu, Altememe, and others [10-22]. In [2], researchers generalized reference [1] to solve partial differential equations with variable coefficients and initial conditions. In addition, what was presented in the two references differs from what we presented in our paper, as our paper was presented in a more general method.

2. Apply to partial differential equations of the General Polynomial Transform.

2.1 Definitions, Properties, and Theorems of General Polynomial Transform

Definition 1:[1] The General Polynomial transform of f(t) denoted by F is given by

$P_g f(t) = \int_1^\infty t^{-(q(p)+1)} f(t)dt = F(q(p)).$

Theorem 1: The General Polynomial transform is linear. That is

$P_g(af(t) + bg(t)) = aI_s(f(t)) + bI_s(g(t)).$

Where a, b are constants.

Theorem 2:

1.$P_g(1) = \frac{1}{q(p)}, (p(\alpha) > 0).$

2. $P_g(t^n) = \frac{1}{q(p)-n}, (q(p) > n).$

3. $P_g(\ln t) = \frac{1}{[q(p)]^2}, (q(p) > 0).$

4. $P_g(t^n \ln t) = \frac{1}{[q(p)-n]^2}, (q(p) > n).$

5. $P_g(\sin(a\ln t)) = \frac{a}{(q(p))^2 + a^2}.$

6. $P_g(\cos(a\ln t)) = \frac{q(p)}{(q(p))^2 + a^2}, (q(p) > n).$

Theorem 3: If $P_g(f(t)) = F(q(p))$, then

$P_q(t^n f(t)) = F(q(p) - n),$

$P_q(t^n f^{(n)}(t)) = -f^{(n-1)}(1) - (q(p) - (n-1))f^{(n-2)}(1) - (q(p) - (n - 1))(q(p) - (n-2))f^{(n-3)}(1) - \cdots - (q(p) - (n-1)) \ldots (q(p) - 1)f(1) + \frac{q(p)!}{(q(p)-n)!}F(q(p)).$

Now let us give the necessary theorems for the application of the P_q transformation to partial differential equations.

Definition 2: The P_q transform of $z(x, y)$ denoted by $F(q(p), y)$is given by

$P_q[z(x,y)] = \int_1^\infty x^{-(q(p)+1)} z(x,y)dx = F(q(p), y).$

Theorem 4: If $P_q(z(x,y)) = F(q(p), y)$, then

$$P_q(xz_x) = -z(1,y) + q(p)F(q(p), y).$$

Proof: Let us apply P_q transform to function xz_x

$P_q(xz_x) = \int_1^\infty x^{-(q(p)+1)} xz_x dx, \quad z_x dx = dv$

$= \int_1^{\infty} x^{-(q(p)+1)}\, xz_x dx = \int_1^{\infty} x^{-q(p)}\, z_x dx,$

$$x^{-q(p)} = u, \ \ z_x dx = dv$$

$$-q(p)x^{-(q(p)+1)}dx = du, \qquad z = v$$

$P_q(xz_x) = -z(1, y) + \ q(p)F(q(p), y)$

Theorem 5: If $P_q(z(x, y)) = F(q(p), y)$, then

$$P_q(yz_y) = y\frac{\partial F(q(p), y)}{\partial y}$$

Proof: Let us apply P_q transform to function yz_y

$P_q(yz_y) = \int_1^{\infty} x^{-(q(p)+1)}\, yz_y dx,$

$$= y\int_1^{\infty} x^{-(q(p)+1)}\, z_y dx = y\frac{\partial}{\partial y}\left(\int_1^{\infty} x^{-(p(\alpha)+1)}\, zdx\right) = y\frac{\partial F(q(p), y)}{\partial y}$$

Theorem 6: If $P_q(z(x, y)) = F(q(p), y)$, then

$$P_q(x^2 z_{xx}) = -z_x(1, y) + (1 - q)z(1, y) + q(q - 1)F(q(p), y).$$

Proof: Let us apply P_q transform to function $x^2 z_{xx}$

$$P_q(x^2 z_{xx}) = \int_1^{\infty} x^{-(q(p)+1)}\, x^2 z_{xx} dx$$

$$= \int_1^{\infty} x^{-(q(p)-1)}\, z_{xx} dx$$

If we apply two times partial integration, then we obtained

$$P_q(x^2 z_{xx}) = -z_x(1, y) + (1 - q)z(1, y) + q(q - 1)F(q(p), y)$$

Theorem 7: If $P_q(z(x, y)) = F(q(p), y)$, then

$$P_q(y^2 z_{yy}) = y^2 \frac{\partial^2}{\partial y^2}\big(F(q(p), y)\big).$$

Proof: Let us apply P_q transform to function $y^2 z_{yy}$

$$P_q(y^2 z_{yy}) = \int_1^{\infty} x^{-(q(p)+1)}\, y^2 z_{yy} dx$$

$$= y^2 \frac{\partial^2}{\partial y^2}\left(\int_1^{\infty} x^{-(p(\alpha)+1)}\, zdx\right) = y^2 \frac{\partial^2 F(q(p), y)}{\partial y^2}$$

Theorem 8: If $P_q(z(x, y)) = F(q(p), y)$, then

$$P_q(xyz_{xy}) = -y\frac{\partial}{\partial y}\big(z(1, y)\big) + yq\frac{\partial F(q(p), y)}{\partial y}.$$

Proof: Let us apply P_q transform to function xyz_{xy}

$$P_q(xyz_{xy}) = \int_1^{\infty} x^{-(q(p)+1)}\, xyz_{xy}dx$$

$= y\int_1^{\infty} x^{-q(p)}\, z_{xy}dx$

$$= y\frac{\partial}{\partial y}\left(\int_1^{\infty} x^{-q(p)}\, z_x dx\right) = y\frac{\partial}{\partial y}\big(-z(1,y) + qF(q(p),y)\big)$$

$$= -y\frac{\partial}{\partial y}\big(z(1,y)\big) + yq\frac{\partial F(q(p),y)}{\partial y}.$$

2.2 Applying P_q transform on partial differential Euler equation

Definition 3: The Euler equations (or couchy Euler equations) of first and second order are given by :

$$axz_x + byz_y + cz = h(x,y)$$

$$ax^2z_{xx} + bxyz_{xy} + cy^2z_{yy} + dxz_x + eyz_y + fz = h(x,y)$$

Where a, b, c, d, e, f are constants.

Theorem 9: The solution of the problem

$$axz_x + byz_y + cz = h(x,y),\ \ z(1,y) = f(y)$$

is $P_q^{-1}\big(\frac{y^{-\frac{aq(p)+c}{b}}}{b}\int \quad y^{\frac{aq(p)+c-b}{b}}\big(P_q f(x,y) + az(1,y)\big)dy\big)$.

Proof: Let's apply P_q transform

$$P_q(axz_x + byz_y + cz) = P_q h(x,y)$$

$$aP_q(xz_x) + bP_q(yz_y) + cP_q z = P_q h(x,y)$$

$$a(-z(1,y) + \ q(p)F(q(p),y)) + by\frac{\partial F(q(p),y)}{\partial y} + cF(q(p),y) = P_q h(x,y)$$

$$by\frac{\partial F(q(p),y)}{\partial y} + (aq(p) + c)F(q(p),y) = P_q h(x,y) + az(1,y)$$

$$\frac{\partial F(q(p),y)}{\partial y} + \frac{aq(p) + c}{by}F(q(p),y) = \frac{P_q h(x,y) + az(1,y)}{by}$$

The above equation is linear from first order.

$\lambda = e^{\int \ \frac{aq(p)+c}{by}dy} = y^{\frac{aq(p)+c}{b}}$

$$\frac{\partial}{\partial y}\left(y^{\frac{aq(p)+c}{b}}F(q(p),y)\right) = y^{\frac{aq(p)+c}{b}}\left(\frac{P_q h(x,y) + az(1,y)}{by}\right)$$

$$\frac{\partial}{\partial y}\left(y^{\frac{aq(p)+c}{b}}F(q(p),y)\right) = \frac{y^{\frac{aq(p)+c-b}{b}}}{b}\big(P_q h(x,y) + az(1,y)\big)$$

$$y^{\frac{aq(p)+c}{b}}F(q(p),y) = \frac{1}{b}\int y^{\frac{aq(p)+c-b}{b}}\left(P_q h(x,y) + az(1,y)\right)dy$$

$$F(q(p),y) = \frac{y^{-\frac{aq(p)+c}{b}}}{b}\int y^{\frac{aq(p)+c-b}{b}}\left(P_q h(x,y) + az(1,y)\right)dy$$

$$z(x,y) = P_q^{-1}\left(F(q(p),y)\right)$$

$$= P_q^{-1}\left(\frac{y^{-\frac{aq(p)+c}{b}}}{b}\int y^{\frac{aq(p)+c-b}{b}}\left(P_q h(x,y) + az(1,y)\right)dy\right)$$

Theorem 10: The solution of

$$ax^2 z_{xx} + bxyz_{xy} + cxz_x + dyz_y + ez = h(x,y),$$

$$z(1,y) = f_1(y), z_x(1,y) = f_2(y)$$

is

$$z(x,y) = P_q^{-1}\left(y^{-\frac{aq^2-aq+cq+e}{bq+d}}\int \frac{y^{\frac{aq^2+(c-a-b)q+e}{bq+d}}}{bq+d}\left(az_x(1,y) + a(q-1)z(1,y) + by\frac{\partial z(1,y)}{\partial y} + cz(1,y) + P_q h(x,y)\right)dy\right)$$

Proof: Let's apply P_q transform

$$P_q\left(ax^2 z_{xx} + bxyz_{xy} + cxz_x + dyz_y + ez\right) = P_q h(x,y)$$

$$aP_q(x^2 z_{xx}) + bP_q(xyz_{xy}) + cP_q(xz_x) + dP_q(yz_y) + eP_q z = P_q h(x,y)$$

$$a(-z_x(1,y) + (1-q)z(1,y) + q(q-1)F(q(p),y)) + b\left(-y\frac{\partial}{\partial y}\left(z(1,y)\right) + yq\frac{\partial F(q(p),y)}{\partial y}\right) + c(-z(1,y) + q(p)F(q(p),y)) + dy\frac{\partial F(q(p),y)}{\partial y} + eF(q(p),y) = P_q h(x,y)$$

$$(dy + byq)\frac{\partial F(q(p),y)}{\partial y} + (aq(q-1) + cq + e)F(q(p),y) = P_q h(x,y) + az_x(1,y) + a(q-1)z(1,y) + by\frac{\partial z(1,y)}{\partial y} + cz(1,y)$$

$$\frac{\partial F(q(p),y)}{\partial y}+\frac{aq(q-1)+cq+e}{dy+byq}F(q(p),y)=\frac{P_q h(x,y)+az_x(1,y)+a(q-1)z(1,y)+by\frac{\partial z(1,y)}{\partial y}+cz(1,y)}{dy+byq}$$

The above equation is linear from first order.

$$\lambda=e^{\int\frac{aq(q-1)+cq+e}{dy+byq}dy}=y^{\frac{aq^2-aq+cq+e}{bq+d}}$$

$$\frac{\partial}{\partial y}\left(y^{\frac{aq^2-aq+cq+e}{bq+d}}F(q(p),y)\right)=y^{\frac{aq^2-aq+cq+e-bq-d}{bq+d}}\left(\frac{P_q h(x,y)+az_x(1,y)+a(q-1)z(1,y)+by\frac{\partial z(1,y)}{\partial y}+cz(1,y)}{d+bq}\right)$$

$$y^{\frac{aq^2-aq+cq+e}{bq+d}}F(q(p),y)=\frac{1}{bq+d}\int y^{\frac{aq^2-aq+cq+e-bq-d}{bq+d}}\left(P_q h(x,y)+az_x(1,y)+a(q-1)z(1,y)+by\frac{\partial z(1,y)}{\partial y}+cz(1,y)\right)dy$$

$$z(x,y)=P_q^{-1}\big(F(q(p),y)\big)=P_q^{-1}\left(\frac{y^{-\frac{aq^2-aq+cq+e}{bq+d}}}{bq+d}\int y^{\frac{aq^2-aq+cq+e-bq-d}{bq+d}}\left(P_q h(x,y)+az_x(1,y)+a(q-1)z(1,y)+by\frac{\partial z(1,y)}{\partial y}+cz(1,y)\right)dy\right)$$

3. Examples

Example 1: Let us consider the following problem.

$xz_x+yz_y-5z=0,\ z(1,y)=y^4$

Solution: Let us apply P_q transform to the given equation.

$$a=1,b=1,c=-5,h(x,y)=0$$

$$z(x,y)=P_q^{-1}\left(y^{5-q(p)}\int y^{q(p)-5-1}\,y^4dy\right)$$

$$= P_q^{-1}\left(y^{5-q(p)}\int y^{q(p)-2}\,dy\right)$$
$$= P_q^{-1}\left(y^{5-q(p)}\frac{y^{q(p)-1}}{q(p)-1}\right)$$
$$= P_q^{-1}\left(\frac{y^4}{q(p)-1}\right) = xy^4$$

Example 2: Let us consider the following problem.

$xz_x - 4yz_y + 2z = xy,\ z(1,y) = 0$

Solution: Let us apply P_q transform to the given equation.

$$a = 1, b = -4, c = 2, h(x,y) = xy$$
$$\frac{y^{-\frac{aq(p)+c}{b}}}{b}\int y^{\frac{aq(p)+c-b}{b}}\left(P_q h(x,y) + az(1,y)\right)dy$$
$$z(x,y) = P_q^{-1}\left(\frac{y^{\frac{2+q(p)}{4}}}{-4}\int y^{\frac{q(p)+6}{-4}}\frac{y}{q(p)-1}\,dy\right)$$
$$= \frac{1}{-4}P_q^{-1}\left(y^{\frac{2+q(p)}{4}}\int \frac{y^{\frac{q(p)+2}{-4}}}{q(p)-1}\,dy\right)$$
$$= \frac{1}{-4}P_q^{-1}\left(-y^{\frac{2+q(p)}{4}}\frac{y^{\frac{q(p)-2}{-4}}}{q(p)-1}\frac{4}{q(p)-2}\right)$$
$$= P_q^{-1}\left(\frac{y}{(q(p)-1)(q(p)-2)}\right) = yP_q^{-1}\left(\frac{1}{q(p)-2} - \frac{1}{q(p)-1}\right)$$
$$= y(x^2 - x)$$

Example 3: Let's solve the problem following

$$x^2 z_{xx} - xyz_{xy} - xz_x = 0\ , z(1,y) = y^2 + y, z_x(1,y) = 4y^2$$

Solution: Let us apply P_q transform to the given equation.

$$a = 1, b = -1, c = -1, d = e = 0$$
$$F(q(p),y) = y^{-\frac{q^2-q-q}{-q}}\int \frac{y^{-\frac{q^2-q}{-q}}}{-q}(4y^2 + (q-1)(y^2+y) - y(2y+1) - y^2 - y)dy$$
$$F(q(p),y) = y^{q-2}\int \frac{y^{1-q}}{-q}(qy^2 + (q-3)y)dy$$

$$= y^{q-2} \int \left(-y^{3-q} + \frac{3-q}{q} y^{2-q}\right) dy$$

$$= y^{q-2} \left(\frac{y^{4-q}}{q-4} + \frac{3-q}{q} \frac{y^{3-q}}{3-q}\right)$$

$$= \frac{y^2}{q-4} + \frac{y}{q}$$

$z(x, y) = P_q^{-1}\left(\frac{y^2}{q-4} + \frac{y}{q}\right) = x^4y^2 + y.$

Conclusion

In this study, we applied a general polynomial transform to find a solution to Euler's Partial Differential Equations. In addition, for Partial differential equations of second order, we observed that if the initial conditions are z(1,y) and z_x(1,y), and there is no z_yy in the general form of the Partial differential equation, the exact solution is easily obtained. But if there is z_yy in the Partial differential equation, it will be difficult to calculate the transformation of the exact solution.

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