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Approximation of unbounded functions of Szasz Durrmeyer involving Boas-Buck polynomials on simplex

Abeer Mahdi Salih

Ministry of Education Al-Rusafa Three

Abstract:

In this paper, we construct the Szasz-Durrmeyer operators defined by Boas-Buck polynomials and prove some approximation properties of these operators and then establish the convergence of these operators with the help of Ditzian- totik modulus of smoothness on simplex.

المستخلص

في هذا البحث نقوم بانشاء مؤثرات ساز -ديرماير بواسطة متعددات بوس بوك ونثبت بعض تقارب هذه المؤثراتبواسطة معامل التاثير ديتزيان توتك على فضاء simplex

Introduction and preliminaries.

Szasz operators are defined by Szasz in 1950[1], [6] , [9] which is defined as:

$$s_n(f,x) = e^{-sx} \sum_{r=0}^{\infty} \frac{(sx)^v}{v!} f\left(\frac{v}{s}\right)$$

Where $s \in N, x \geq 0$ *and* $f \in c[o,\infty)$. *the Durrmeyer operator modifications of the Bernstien Szasz and Baskakov operators.[6]*

The Bernstien- Durrmeyer operators are introduced in [9]

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt \; \ldots . (1,1)$$

Where $f \in l_p[0,1], p \geq 1, and p_{n,k}(x) = C_k^n x^k (1-x)^{n-k}$ ....(1,2)

In [9] Mazhar and Totik introduced the following Szasz Durrmeyer type modification of S_t

$$L_t(f,x) = \int_0^{\infty} f(u) H_t(x,u) du , x \geq 0, t > 0. \quad \ldots \quad \ldots (1,3)$$

Where $H_t(x,u) = t \sum_{k=0}^{\infty} \pi_{tk}(x) \pi_{t,k}(u)$ *and* $\pi_{t,k}(u) = e^{-tx} \frac{(tx)^k}{k!}, k = 0,1,2, \ldots.$

Introduced the linear positive operators which involve the Boas-Buck polynomials as follows

$$B_S(f,x) = \frac{1}{A(1)B(sx(H(1))}\sum_{v=0}^{\infty} p_v\,(sx)f\left(\frac{v}{s}\right), x \geq 0, s \in N \qquad \ldots\ldots(1,4)$$

And then Mursaleen [7] studied cholodowsky type generalization of Szasz operators which include

Boas-Buck type polynomials , the Boas-Buck polynomials have generating functions of the form,

$$A(u)B\big(xH(u)\big) = \sum_{v=0}^{\infty} p_v(x)u^v.$$

Where A(u),B(u) and H(u) are analytic functions
A(u)$=\sum_{r=0}^{\infty} a_r\, u^r$ $\ldots\ldots(1,5)$

B(u)$= \sum_{r=0}^{\infty} b_r\, u^r, H(u) = \sum_{r=0}^{\infty} h_r\, u^r$

With the conditions $a_0 t_0, b_r \neq 0, (r \geq 0)$ and $h_1 \neq 0$[8] we assume

1. A(1)$\neq 0, H'(1) = 1, P_r(x) \geq 0, v = 0,1,2,\ldots$
2. B:$R \to (0,\infty)$
3. The power series (4),(5) converges in the disk $|u| < r, (R > 1)$

In(3) by using $A(u) = 1, H(u) = u, and\ B(u) = e^u$

In view of the generating functions(4) we obtain $p_v(s,x) = \frac{(sx)^v}{v!}$ And the operators (3)reduce to the szaszoperators(1)in the current paper, we aim to construct a generalization of Szasz Mirakjan Durrmeyer operators [6]by utilizing the Boas-Buch- type polynomials then we show that the Boas-Buch- type polynomials include the appel polynomials.

We consider the Szasz Durrmeyer including Boas-Buch- type polynomials as follows.

$$D_s(f,x) = \frac{1}{A(T)B(sxH(1))} = \sum_{V=0}^{\infty} p_v(sx)\int_0^{\infty} fu\, H_t(x,u)du. \qquad \ldots(1,6).$$

Lemma 2.1for all $x \in S$, we write

$D_S(1,X) = 1$

$$D_S(e_1,x) = \frac{B'\big(sxH(1)\big)}{(s-2)B\big(sxH(1)\big)}x + \frac{A(1)+A'(1)}{A(1)}, S > 2$$

$$D_S(e_2,x) = \frac{B''\big(sxH(1)\big)}{B\big(sxH(1)\big)}x^2 + \frac{A(1)+2A'(1)+A(1)H''(1)B''\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)} + \frac{A(1)+A'(1)+A''(1)}{A(1)}$$

$$D_S(e_3,x)=\frac{B'''\big(sxH(1)\big)}{B\big(sxH(1)\big)}x^3+\frac{A(1)+A'(1)+A(1)H''(1)B''\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)}$$

$$+\frac{A(1)+A'(1)+A(1)H''(1)+A''(1)+A'(1)H''(1)B'\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)}+$$

$$\frac{A(1)+A'(1)+A''(1)+A'''(1)}{A(1)}$$

$$D_S(e_4,x)=\frac{B''''\big(sxH(1)\big)}{B\big(sxH(1)\big)}x^4+\frac{A(1)+A'(1)+A(1)H''(1)B'''\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)}x^3$$

$$+\frac{B''\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)}[A(1)+A'(1)+A(1)H''(1)+A''(1)+A'(1)H''(1)+A(1)+H'''(1)$$

$$+A(1)H''(1)^2]x^2+\frac{B'\big(sxH(1)\big)}{A(1)B\big(sxH(1)\big)}$$

$$[A(1)+A'(1)+A''(1)+A'''(1)+A'(1)H''(1)+A'(1)H''(1)+A''(1)H''(1)$$

$$+A'(1)H''(1)+A(1)H''''(1)]x\frac{A(1)+A'(1)+A''(1)+A'''(1)+A''''(1)}{A(1)}$$

Where $e_m=t^m for\ m=1,2,3,4$

$$\sum_{v=0}^{\infty}p_v(sx)=1,\int_0^v u\frac{tu}{(1+t)^4}dt=\frac{1}{s-1}$$

By using the generating functions of the Boas-Buch type polynomials by taking (4) we get

$\sum_{v=0}^{\infty}p_v(sx)=A(1)B\big(sxH(1)\big)$ …(2,1)

$\sum_{v=0}^{\infty}vp_v(sx)=A'(1)B\big(sxH(1)\big)+A(1)B'\big(sxH(1)\big)$ …(2,2)

$$\sum_{v=0}^{\infty}v^2p_v(sx)=\big(A'(1)+A''(1)\big)B\big(sxH(1)\big)+\big(A(1)+A'(1)+A(1)H''(1)\big)B'\big(sxH(1)\big)$$

$+A(1)B''\big(sxH(1)\big)$ …(2,3)

$\sum_{v=0}^{\infty}v^3p_v(sx)=\big(A'(1)+A''(1)+A'''(1)\big)B\big(sxH(1)\big)+\big(A(1)+A'(1)+A(1)H''(1)+A''(1)+A'(1)H''(1)+A(1)H'''(1)\big)B'\big(sxH(1)\big)$ …(2,4)

$\big(A(1)+A(1)H''(1)\big)B''\big(sxH(1)\big)+A(1)B'''\big(sxH(1)\big)$

$$\sum_{v=0}^{\infty} v^4 p_v(sx) = \left(A^{\prime}(1) + A^{\prime\prime}(1) + A^{\prime\prime\prime}(1) + A^{\prime\prime\prime\prime}(1)\right)B\left(sxH(1)\right)$$
$$+ \left(A(1) + A^{\prime}(1) + A^{\prime}(1)H^{\prime\prime}(1) + A^{\prime\prime}(1) + A^{\prime}(1)H^{\prime\prime}(1) + A(1) + A(1)H^{\prime\prime\prime\prime}(1)\right)B^{\prime}\left(sxH(1)\right)$$

$$\left(A(1) + A^{\prime}(1) + A(1)H^{\prime\prime}(1) + A^{\prime\prime}(1) + A^{\prime}(1)H^{\prime\prime}(1) + A(1)H^{\prime\prime\prime}(1) + A(1) + A^{\prime\prime}(1)^2\right)B^{\prime\prime}\left(sxH(1)\right) + \left(A(1) + A^{\prime}(1) + A(1)H^{\prime\prime}(1)\right)B^{\prime\prime\prime}\left(sxH(1)\right)$$

$$+A(1)B^{\prime\prime\prime\prime}\left(sxH(1)\right)$$
….(2,5)

Now let the K-functional of simplex are

$$K = \left\{f; x \in S, \|f(x)\|_{p,s} \leq ae^{bx}, a, b\ positive\ and\ finite\right\}$$

Suppose that

$$\lim_{s\to\infty} \frac{B^{\prime}(s)}{B(s)} = 1, \lim_{s\to\infty} \frac{B^{\prime\prime}(s)}{B(s)} = 1, \lim_{s\to\infty} \frac{B^{\prime\prime\prime}(s)}{B(s)} = 1, \lim_{s\to\infty} \frac{B^{\prime\prime\prime\prime}(s)}{B(s)} = 1. \quad \ldots(2,5)$$

Theorem 2:2

Let $f \in S \cap K$ and equations of (2,5)be satisfied then $\lim_{s\to\infty} D(f, x) = f(x)$ and the operators D converges uniform

Proof

From lemma 2.1 and (2,5) we get

$f\ m = 0, 1,2, \ldots$

Then the new operator D converge uniformly , we complete the proof by applying universal Korovkin property with respect to positive linear operators

Lemma 2.3

For all $x \in S$,we have

$$D_3(t - x; x) = \left(3\frac{B^{\prime}(3xH(1))}{B(2xH(1))} - 1\right)x + \frac{A(1)+A^{\prime}(1)}{A(1)}, s > 3 \quad \ldots(2,6)$$

$$D_4((t - x)^2; x) = \left(\frac{16B^{\prime\prime}\left(4xH(1)\right)}{2B\left(2xH(1)\right)} - \frac{8B^{\prime}\left(4xH(1)\right)}{2B\left(4xH(1)\right)} + 1\right)x^2 +$$
$$\left(\left(16A(1) + 8A^{\prime}(1) + 4A(1)H^{\prime\prime}(1)\right)B^{\prime}\left(4xH(1)\right) - \frac{2A(1) + 2A^{\prime}(1)}{2A(1)}\right)x +$$
$$\frac{2A(1)+4A^{\prime}(1)+A^{\prime\prime}(1)}{2A(1)}, s > 4 \quad \ldots(2,7)$$

$$D_6((t-x)^4;x)=\left(\frac{6^4B''''\big(6xH(1)\big)}{4!\,B\big(6xH(1)\big)}-\frac{4*6^3B'\big(6xH(1)\big)}{4!\,B\big(6xH(1)\big)}+\frac{6^3B''\big(6xH(1)\big)}{4!\,B\big(sxH(1)\big)}\right.$$

$$-\frac{24B'\big(6xH(1)\big)}{4B\big(6xH(1)\big)}+1)x^4+\left[\frac{6^3\left(16A(1)+4A'(1)+6A(1)B'''\big(6xH(1)\big)\right)}{4!\,A(1)B\big(6xH(1)\big)}\right.$$

$$-\frac{4*6^2\big(9A(1)+3A'(1)+3A(1)H''(1)\big)B''\big(6xH(1)\big)}{4!\,A(1)B\big(6xH(1)\big)}$$

$$\left.+6^2\frac{\big(4A(1)+2A'(1)+A(1)H''(1)\big)B'\big(6xH(1)\big)}{12A(1)B\big(6xH(1)\big)}-\frac{4A(1)+4A'(1)}{4A(1)}\right]x^3+$$

$$\left[\frac{6B''\big(6xH(1)\big)}{4!\,A(1)B\big(6xH(1)\big)}\{72A(1)+48A'(1)+48A(1)H''(1)+6A''(1)\right.$$

$$+12A'(1)H''(1)+4A(1)H'''(1)+3A(1)\big(H''(1)\big)^2\}$$

$$-\frac{24\big(18A(1)+18A'(1)+9A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1)\big)B'\big(6xH(1)\big)}{4!\,A(1)B\big(6xH(1)\big)}$$

$$\left.+\frac{12A(1)+24A'(1)+6A''(1)}{12A(1)}\right)x^2$$

$$+\left[\frac{6B'\big(6xH(1)\big)}{4!\,A(1)B\big(6xH(1)\big)}\{96A(1)+144A'(1)+48A(1)H''(1)+48A''(1)+4\,A'''(1)\right.$$

$$+72A(1)H''(1)+48A'(1)H''(1)+16A(1)H'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)$$
$$+A(1)H''''(1)\}$$

$$\left.-\frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{4!\,A(1)}\right)x$$

$$+\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A''''(1)}{4!A(1)},s>6 \qquad \ldots(2,7)$$

Proof:

$D_3(t-x;x)=D_3(e_1;x)-xD_3(1;x),s>3$

$$=\left(\left(3\frac{B'\big(3xH(1)\big)}{B\big(2xH(1)\big)}-1\right)x+\frac{A(1)+A'(1)}{A(1)}\right)$$

$=D_3(e_1;x)-x(1)$

$=D_3(e_1;x)-xD_3(1;x),s>3$

$D_4((t-x)^2;x) = D_4(e_1;x) - x^2 D_4(1;x), s \succ 4$

$$D_4((t-x)^2;x) = \frac{8B''(4xH(1))x^2}{B(2xH(1))} - \frac{4B'(4xH(1))}{B(4xH(1))}x^2 + x^2 +$$

$$\left(8A(1) + 4A'(1) + 2A(1)H''(1)\right)B'(4xH(1)) - A'(1)x + \frac{A(1) + 2A'(1) + 2A''(1)x}{A(1)}$$

$$= D_4(e_1;x) - x^2 D_4(1;x)$$

And then from (2,7) and $s \succ 6$

$$D_6((t-x)^4;x) = D_6(e_4;x) - 4xD_6(e_3;x) + 6x^2 D_6(e_2;x) - 4x^3 D_6(e_1;x) + x^4 D_6(1;x)$$

Lemma 2.4

For all $x \in S$,we have

$$\lim_{s\to\infty} s\, D_s(t-x;x) = x\mu_1(x) + \frac{A(1) + A'(1)}{A(1)}$$

$$\lim_{s\to\infty} s\, D_s((t-x)^2;x) = x^2\mu_2(x) + x\left(2 + H''(1)\right)$$

$$\lim_{s\to\infty} s^2\, D_s((t-x)^4;x) = x^4\mu_3(x) + x^3\mu_4(x) + x^2\left(12 + 12H''(1)\right) + 3\left(H''(1)\right)^2$$

Where

$$\mu_1(x) = \lim_{s\to\infty} s\left(\frac{sB'\left(sxH(1) - (s-2)B(sxH(1))\right)}{(s-2)B(sxH(1))}\right)$$

$$\mu_2(x) = \lim_{s\to\infty} s\left(\frac{s^2B''\left(sxH(1) - 2s(s-3)B'(sxH(1))\right) + (s-2)(s-3)B(sxH(1))}{(s-2)(s-3)B(sxH(1))}\right)$$

$$\mu_3(x)$$
$$= \lim_{s\to\infty} s^2\left(\frac{s^4B''''\left(sxH(1) - 4s^3(s-5)B''(sxH(1))\right) + 6s^2(s-4)(s-5)B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))}\right.$$
$$\left. - \frac{4s(s-3)(s-4)(s-5)B'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} + 1\right)$$

$$\mu_4(x) = \lim_{s\to\infty} s^2\left(\frac{s^3\left(16A(1) + 4A'(1) + 6A(1)H''(1)\right)B''''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))}\right.$$
$$- \frac{4s^2(s-5)\left(9A(1) + 3A'(1) + 3A(1)H''(1)\right)B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}$$

$$+\frac{6s(s-4)(s-5)\big(4A(1)+2A^{\prime}(1)+3A(1)H^{\prime\prime}(1)\big)B^{\prime}\big(sxH(1)\big)}{(s-2)(s-3)(s-4)(s-5)A(1)B\big(sxH(1)\big)}$$

$$-\frac{(s-3)(s-4)(s-5)\big(4A(1)+4A^{\prime}(1)\big)B\big(sxH(1)\big)}{(s-2)(s-3)(s-4)(s-5)A(1)B\big(sxH(1)\big)}$$

Lemma2.5

Let $\lim_{s\to\infty}\frac{B^{\prime}(s)}{B(s)}=1, \lim_{s\to\infty}\frac{B^{\prime\prime}(s)}{B(s)}=1$

Be satisfied for all $f\in S$, then $\|D_s(x)\|\le C$ where C is positive constant.

Proof:

By using lemma (2, 1) we have

$D_s(x)=D_s(1,x)+D_s(e_2,x)$

Then

$$\|D_s(x)\|=sup\left\{\left|\frac{s^2B^{\prime\prime}\big(sxH(1)\big)}{(s-2)(s-3)B\big(sxH(1)\big)}\right|\frac{x^2}{\psi}+\left|\frac{s\big(4A(1)+2A^{\prime}(1)+A(1)H^{\prime\prime}(1)\big)B^{\prime}\big(sxH(1)\big)}{(s-2)(s-3)B\big(sxH(1)\big)}\right|\frac{x}{\psi}\right.$$

$$\left.+\left|\frac{2A(1)+2A^{\prime}(1)+A^{\prime\prime}(1)}{(s-2)(s-3)A(1)}+1\right|\frac{1}{\psi}\right\}, s>3$$

Since $sup\frac{1}{\psi}=1, sup\frac{x}{\psi}=\frac{1}{2}, sup\frac{x^2}{\psi}=1$ we have

$$\|D_s(x)\|\le sup\left\{\left|\frac{s^2B^{\prime\prime}\big(sxH(1)\big)}{(s-2)(s-3)B\big(sxH(1)\big)}\right|\frac{x^2}{\psi}\right.$$
$$+\left|\frac{s\big(4A(1)+2A^{\prime}(1)+A(1)H^{\prime\prime}(1)\big)B^{\prime}\big(sxH(1)\big)}{(s-2)(s-3)B\big(sxH(1)\big)}\right|\frac{x}{\psi}$$

By using (10), there exist a positive constant R

$\|D_s(x)\|\le R$

Theorem 3.1 for all $f\in S$ then $\lim_{s\to\infty}\|D_sf(x)-f(x)\|=0$

Proof:

Froom Korovkin theorem for m=0, 1, 2

$$\lim_{s\to\infty}\|D_s(t^m,x)-x^m\|=0 \qquad \ldots(3.1)$$

Then from lemma 2.1 we have

$\lim_{s\to\infty}\|D_s(1,x)-1\|=0$

$$\lim_{s\to\infty}\|D_S(e_1,x)-e_1(x)\| = sup\left\{\left|\frac{sB'(sxH(1))}{(s-2)B(sxH(1))}-1\right|\frac{x}{\psi}+\left|\frac{A(1)+A'(1)}{(s-2)A(1)}\right|\frac{1}{\psi}\right.$$

$$\leq \frac{1}{2}\left|\frac{sB'(sxH(1))}{(s-2)B(sxH(1))}-1\right|+\left|\frac{A(1)+A'(1)}{(s-2)A(1)}\right| \qquad \text{... (3.2)}$$

$\lim_{s\to\infty}\|D_S(e_1,x)-e_1(x)\| = 0, for\ s>2$
...(3.3)

In the same way

$$\lim_{s\to\infty}\|D_S(e_2,x)-e_2(x)\|$$

$$= sup\left\{\left|\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-1\right|\frac{x^2}{\psi}\right.$$

$$+\left|\frac{s(4A(1)+2A'(1)+A(1)H''(1))B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}\right|\frac{x}{\psi}$$

$$\left.+\left|\frac{2A(1)+2A'(1)+A''(1)}{(s-2)(s-3)A(1)}+1\right|\frac{1}{\psi}\right\}$$

$$\leq sup\left\{\left|\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-1\right|+\left|\frac{s(4A(1)+2A'(1)+A(1)H''(1))B'(sxH(1))}{2(s-2)(s-3)A(1)B(sxH(1))}\right|\right.$$

$$\left.+\left|\frac{2A(1)+2A'(1)+A''(1)}{(s-2)(s-3)A(1)}+1\right|\right\}$$

For $s>3, we\ have$

$$\lim_{s\to\infty}\|D_S(e_2,x)-e_2(x)\| = 0$$

Thus we get
$\lim_{s\to\infty}\|D_s(t^m,x)-x^m\| = 0$

From theorem we get $\lim_{s\to\infty}\|D_sf(x)-f(x)\| = 0$

Lemma3.2

If $f\in S$, then $\|D_sf(x)-f(x)\|_\psi \leq 2\left(2+\mathcal{G}_0(s)+\sqrt{\mathcal{G}_1(s)}\right)\mathcal{W}\left(f,\sqrt{\mathcal{G}_0(s)}\right)$ ...(3.4)

Where

$$\mathcal{G}_0(s) = \left(\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-\frac{2sB'(sxH(1))}{(s-2)B(sxH(1))}+1\right)$$
$$+\frac{1}{2}\left(\frac{s(4A(1)+2A'(1)+A(1)H''(1))B'(sxH(1))}{(s-2)(s-3)B(sxH(1))}-\frac{2A(1)+2A'(1)}{(s-2)A(1)}\right)$$

$+\frac{2A(1)+2A'(1)+A''(1)}{(s-2)(s-3)A(1)}, s \geq 4$
…(3.5)

$$\mathcal{G}_1(s) = \left(\frac{s^4 B''''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} - \frac{4s^3 B'''(sxH(1))}{(s-2)(s-3)(s-4)B(sxH(1))}\right.$$
$$\left.+\frac{6s^2 B''(sxH(1))}{(s-2)(s-3)B(sxH(1))} - \frac{4s^2 B'(sxH(1))}{(s-2)B(sxH(1))} + 1\right)$$
$$+\frac{3\sqrt{3}}{16}\left(\frac{s^3(16A(1)+4A'(1)+6A(1)H''(1))B''''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))}\right.$$
$$-\frac{4s^2(9A(1)+3A'(1)+3A(1)H''(1))B''(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}$$
$$\left.+\frac{6s(4A(1)+2A'(1)+3A(1)H''(1))B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))} - \frac{4A(1)+2A'(1)}{(s-2)A(1)}\right)$$
$$+\frac{1}{4}\left(\frac{s^2 B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{72A(1)+48A'(1)+48A(1)H''(1)\right.$$
$$+6A''(1)+12A'(1)H''(1)+4A(1)H'''(1)+3A(1)\left(H''(1)\right)^2\}$$
$$-\frac{4s(18A(1)+18A'(1)+9A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1))B'(6xH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}$$
$$\left.+\frac{12A(1)+24A'(1)+6A''(1)}{12A(1)}\right)$$
$$+\frac{3\sqrt{3}}{16}\left(\frac{sB'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{96A(1)+144A'(1)+48A''(1)\right.$$
$$+4A'''(1)$$
$$+72A(1)H''(1)+48A'(1)H''(1)+16A(1)H'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)$$
$$+A(1)H''''(1)\}$$
$$\left.-\frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{4!\,A(1)}\right)x$$
$$+\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A''''(1)}{4!A(1)}, s \succ 6 \qquad \ldots(3.6)$$

Proof

From lemma 4.1 we get

$$\|\mathcal{D}(f;x)-f(x)\| \leq sup\left\{\int \left|2((1-x)^2)\big(1+(\mathcal{D}(t-x)^2;x)\big) + \mathcal{D}\left(1+(t-x)^2\left(\frac{t-x}{\propto}\right);x\right)\mathcal{W}(f,\propto)\right| dx\right\}$$

By Cauchy-Schwarz inequality in $\mathcal{D}\left(1+(t-x)^2\left(\frac{t-x}{\propto}\right);x\right)$ *and lemma* 2.3 we get

$$\|\mathcal{D}(f;x)-f(x)\| \leq Csup\left\{\int \left|2((1+x^2))\left(1+\mathcal{G}_0(s)(1+x^2)+\sqrt{(1+x^2)} + \sqrt{\mathcal{G}_1(s)}\sqrt{((1+x^2))^3}\right)\mathcal{W}(f,\propto)\right| dx\right\}$$

$$\leq Csup\left\{\int \left|\sqrt{((1+x^2))^5}\left(2+\mathcal{G}_0(s)++\sqrt{\mathcal{G}_1(s)}\right)\mathcal{W}(f,\propto)\right| dx\right\}$$

$$\leq Csup\left\{\int \left|\left(2+\mathcal{G}_0(s)++\sqrt{\mathcal{G}_1(s)}\right)\mathcal{W}(f,\mathcal{G}_0(s))\right| dx\right\} \blacksquare$$

Conclusion:

In the present investigation , we have defined the approximation of operators by modifying Szasz- Durrmeyer operators through the incorporation of the Boas- Buck polynomial, the initial finding indicate that the operators retain several desirable approximation of their counter parts while exhibiting unique convergence behavior attributable to the nature of the Boas- Buck basis and we have proven a convergence a theorem for this operators , we have obtained quantitative estimates for the convergence using K-functional these estimates provide bounds on how quickly the new operators approximation .

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