Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

A Study of a New Class of Meromorphic Univalent Functions Through Subordination Applications

Hazha Zirar Hussain*

Department, of Mathematics College of Science, Salahaddin University, Erbil, IRAQ.

*Corresponding Author: Hazha Zirar Hussain

DOI: https://doi.org/10.31185/wjps.751
Received 10 January 2025; Accepted 09 March 2025; Available online 30 Jun 2025

ABSTRACT: In this research, I wanted to create a new beautiful class defined by subordination and the integral operator and the combination between them and I named it $\mathfrak{A}\mathfrak{L}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ so that I inferred results from the geometric properties of this nice class in terms of studying the radius of convexity and estimating the coefficient and other new results. I also presented the previous information related to the distinguished researchers who before me studied the subordination and the integral operator that I used in my beautiful research.

Keywords: Meromorphic Functions, Integral Operator, Coefficient Bounds, Radii of Starlikeness, Convolution.

1. INTRODUCTION

Suppose that c is a point inside the famous unit circle
$\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$,
Let $\mathfrak{L}_c$ be the class of all analytic normalized functions inside the perforated disk $\mathcal{U} \setminus \{c\} = \{z \in \mathbb{C}: c < |z| < 1\}$,such that each function is written as follows:

$$f(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty} a_n\ (z-c)^n . \qquad (1.1)$$

Here I will need a definition convolution or Hadamard product for two functions in the set that we defined above. This definition is actually found in many books on the geometry of analytical functions. I used the book found in the source [3] and the definition is as follows:
Let's assume that two functions belong to the set $\mathfrak{L}_c$

$$f_1(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty} a_{1,n}\ (z-c)^n ,$$

$$(1.2)$$

$$f_2(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty} a_{2,n}\ (z-c)^n,$$

then the convolution or Hadamard product for the two functions f_1 and f_2 is

$$(f_1 * f_2)(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty} a_{1,n}\ a_{2,n}\ (z-c)^n. \qquad (1.3)$$

I will also need a definition for subordination for two analytic functions and its present, of course, since it is one of the basic definitions in the subject of the geometry of analytical functions in most books in this field, for example, I used a book by the author found in the source [4].
The analytic function $f_1(z)$ is subordinate for the analytic function $f_2(z)$ if there is a Schwarz function $\omega(z)$ such that $z, \omega(z)$ they belong to $\mathcal{U}$ such that $f_1(z) = f_2(\omega(z))$, $z \in \mathcal{U}$ and is symbolized by $f_1(z) \prec f_2(z)$.
The set of meromorphic functions is one of the beautiful and extensive sets in terms of its study and applications by researchers in the past years as well as at the present time as an example [1], [2], [5], [7], [8], [9], [10].
In the year 2009, researcher Lashin developed the Jung Kim Srivastava operator, which is referred to it in the source number [6] for meromorphic functions.
Here we take this integral operator for meromorphic functions in the class $\mathfrak{L}_c$, which is written in the form (1.1) as follows:
If the function $\mathfrak{W}_{\sigma,\upsilon}: \mathfrak{L}_c \to \mathfrak{L}_c$is defined in a form such that for each function $f \in \mathfrak{L}_c$, then

*Corresponding author: hazha.hussain@su.edu.krd

$$\mathfrak{W}_{\sigma,\upsilon} = \mathfrak{W}_{\sigma,\upsilon}(z)f(z)$$

$$= \frac{\upsilon^{\sigma}}{\Gamma(\sigma)(z-c)^{\upsilon+1}}\int_0^{z-c} t^{\upsilon}\left(\log\left(\frac{z-c}{t}\right)^{\sigma-1} f(t)dt, \qquad (\sigma > 0, \qquad \upsilon > 0, z \in \mathcal{U}\backslash\{c\}\right)$$

$\mathfrak{W}_{\sigma,\upsilon}f(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n\,(z-c)^n$, $(\sigma > 0,\ \upsilon > 0)$, (1.4)

We notice that from the previous equation no. it is possible to deduce the following important result or observation:

$z\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)' = \upsilon\,\mathfrak{W}_{\sigma-1,\upsilon}\,f(z) - (\upsilon+1)\,\mathfrak{W}_{\sigma,\upsilon}f(z),\ (\sigma > 0,\ \upsilon > 0, z \in \mathcal{U}\backslash\{c\})$

Finally, I will define the new class, which I will call $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ by the integral operator referred to above for the functions defined in the presented class $\mathfrak{L}_c$ and defined in the form (1.1) as follows:

Definition 1.1. For every function f in the given class $\mathfrak{L}_c$ that is written in the form (1.1) and satisfies the following condition:

$\zeta\frac{\mathfrak{W}_{\sigma,\upsilon}(f*f_1)(z)}{\mathfrak{W}_{\sigma,\upsilon}(f*f_2)(z)} \prec \zeta - \frac{(C-D)(z-c)}{1+D(z-c)},\ \ ((z-c)\in\mathcal{U}),$ (1.5)

It is an element of the new wonderful class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ such that $-1 \le D \le C \le 1, \zeta > 0, \sigma > 0, \upsilon > 0$, $\mathfrak{W}_{\sigma,\upsilon}(f*f_2)(z) \neq 0$.

Now that the new class we have defined is ready, it is time to study it from the perspective of its special geometric properties, such as determining the value of the coefficients of its elements, or rather the functions that belong to it, as well as determining the exact radius within the unit disc $\mathcal{U}$, which determines the function's belonging to the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$. I also try to study and determine under what condition a function in this beautiful set can acquire the convexity and starlikeness, and I also try to study other geometric properties for my new class.

2. SOME GEOMETRIC PROPERTIES OF THE NEW CLASS STRUCTURE $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$

We begin our study of the geometric properties of the class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ by determining the value of the coefficients of the functions that belong to it and determining the necessary and sufficient condition for determining the value of their coefficients, on the basis which they belong to the class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, which we present with the proof in the following theorem.

Theorem 2.1. Let us take an element from the class $\mathfrak{L}_c$ let it f , which formulated as (1.1), then f will be an element of the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, which we define in Definition 1.1, if the following inequality is satisfied:

$\sum_{n=1}^{\infty}\left(\frac{n+\upsilon+1}{\upsilon}\right)^{\sigma}(\zeta a_{1n}(1+D) - a_{2n}(\zeta(1+D)+C-D))a_n \ \le\ C-D.$ (2.1)

This condition reaches its limit at the elements of the class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, which is formulated as:

f(z) = $\frac{1}{z-c} + \left(\frac{n+\upsilon+1}{\upsilon}\right)^{\sigma}\frac{C-D}{(\zeta a_{1n}(1+D) - a_{2n}(\zeta(1+D)+C-D))}a_n\,(z-c)^n.$ (2.2)

Proof. Let's start the proof by taking an element f from the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$. My real goal is to prove the validity of the previous inequality no. (2.1).

So according to my definition of the class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, it satisfies the following condition which is equivalent to the following inequality:

$$\zeta\frac{\mathfrak{W}_{\sigma,\upsilon}(f*f_1)(z)}{\mathfrak{W}_{\sigma,\upsilon}(f*f_2)(z)} = \zeta - \frac{(C-D)(z-c)}{1+D(z-c)}.$$

Which is equivalent to the following inequality

$$\left|\frac{\zeta\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n\,(a_{1n}-a_{2n})(z-c)^{n+1}}{(C-D) - \sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n\,(\zeta D a_{1n} + \{(C-D) - \zeta D\}a_{2n})(z-c)^{n+1}}\right|$$

$$\le \frac{\zeta\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n\,(a_{1n}-a_{2n})|z-c|^{n+1}}{(C-D) - \sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n\,(\zeta D a_{1n} + \{(C-D) - \zeta D\}a_{2n})|z-c|^{n+1}}$$

$$\le 1.$$

If the value of $(z-c)$ reaches the limits, i.e the peak, then the required inequality is achieved

$\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D) - a_{2n}(\zeta(1+D)+C-D))\ a_n \le\ C-D.$

In other words, $f \in \mathfrak{A}\mathfrak{L}_c(C,\ D,\ \zeta,\sigma,\upsilon)$.
Thus, we have completed the proof of the theorem.
Naturally, from the previous theorem, we directly obtain the following important result:

Corollary 2.2. If we take an element or rather a function f from the new class $\mathfrak{A}\mathfrak{L}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, which is written in the form (1.1), then its coefficients are satisfied:

$$a_n \leq \frac{C-D}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))}, \quad (n \geq 1), \qquad (2.3)$$

As we previously noted from the previous theorem, the result reaches its peak for the function given by the equation (2.2).

Theorem 2.3. If we take an element or in fact a function from the new class $\mathfrak{A}\mathfrak{L}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ which is written in the form (1.1), then it satisfies the following two inequalities:

$$\frac{1}{r} - \frac{C-D}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}r$$

$$\leq |f(z)| \leq \frac{1}{\mathrm{r}} + \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}\mathrm{r}, \quad (2.4)$$

and

$$\frac{1}{r^2} - \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}$$

$$\leq |f'(z)| \leq \frac{1}{r^2} + \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}, \quad (2.5)$$

Proof. Based on Theorem (2.1) which is essential for deducing most of the properties, we simply have the following condition satisfied:

$$\sum_{n=1}^{\infty} a_n \leq \frac{C-D}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}. \quad (2.6)$$

Since we are working inside the unit disc $\mathcal{U}$, we obtain the following inequality:

$$|f(z)| \leq \frac{1}{|z-c|} + \sum_{n=1}^{\infty} a_n\, |z-c\,|^n,$$

$$\leq \frac{1}{r} + r\sum_{n=1}^{k} a_n\ ,$$

$$\leq \frac{1}{\mathrm{r}} + \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}\mathrm{r},$$

and

$$|f(z)| \geq \frac{1}{|z-c|} - \sum_{n=1}^{\infty} a_n\, |z-c\,|^n,$$

$$\geq \frac{1}{r} - r\sum_{n=1}^{k} a_n$$

$$\geq \frac{1}{\mathrm{r}} - \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}\mathrm{r},$$

Again, based on the previous theorem, the following inequality is correct:

$$\sum_{n=1}^{\infty} na_n \leq \frac{C-D}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}.$$

From this we note that

$$|f'(z)| \leq \frac{1}{|z-c|^2} + \sum_{n=1}^{k} na_n\ |z-c\ |^{n-1},$$

$$\leq \frac{1}{r^2} + \sum_{n=1}^{k} na_n\ ,$$

$$\leq \frac{1}{r^2} + \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} (\zeta a_{11}(1+D) - a_{21}(\zeta(1+D)+C-D))},$$

also

$$|f'(z)| \geq \frac{1}{|z-c|^2} - \sum_{n=1}^{\infty} na_n\ |z-c\ |^{n-1}$$

$$\geq \frac{1}{r^2} - \sum_{n=1}^{k} na_n$$

$$\geq \frac{1}{r^2} - \frac{\mathrm{C}-\mathrm{D}}{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} (\zeta a_{11}(1+D) - a_{21}(\zeta(1+D)+C-D))},$$

Thus, we have completed the proof of the interesting theorem

We will continue this research in the same manner by studying another geometric properties of the class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ and discussing the meromorphic starlikeness of order η. For this set, we give the condition for the radius on which the function has this property, and we always remember that we are working inside the famous unit disc $\mathcal{U}$, and we discuss this in the following theorem.

Theorem 2.4. If we take an element or function f in the class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, that we have defined by (1.1), then the radius of the disk ρ , which we take for the function f to be a meromorphically starlike of order η, is determined or measured by the following inequality:

$$\rho = inf_{n\geq 1} \left\{ \frac{(1-\eta)(\zeta a_{1n}(1+D) - a_{2n}(\zeta(1+D)+C-D))}{(n+2-\eta)(C-D)} \right\}^{\frac{1}{n+1}}. \quad (2.7)$$

The function that achieves the peak is the function whose formula is given in (1.1).

Proof. According to the definition of the definition of the meromorphic starlike function of order η, our goal is to prove the following inequality

$$\left| \frac{(z-c)\left(\mathfrak{W}_{\sigma,\upsilon} f(z)\right)'}{\mathfrak{W}_{\sigma,\upsilon} f(z)} + 1 \right| \leq 1+\eta. \quad (2.8)$$

Let's take the right side of the previous inequality as follows:

$$\left| \frac{(z-c)\left(\mathfrak{W}_{\sigma,\upsilon} f(z)\right)'}{\mathfrak{W}_{\sigma,\upsilon} f(z)} + 1 \right| = \left| \frac{\sum_{n=1}^{\infty}(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n(z-c)^n}{1/(z-c) + \sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n(z-c)^n} \right|,$$

$$= \left| \frac{\sum_{n=1}^{\infty}(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n(z-c)^{n+1}}{1+\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n(z-c)^{n+1}} \right|, \quad (2.9)$$

$$\leq \frac{\sum_{n=1}^{\infty}(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n|z-c|^{n+1}}{1-\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n|z-c|^{n+1}}.$$

It is less than or equal to $\eta - 1$

$$\sum_{n=1}^{\infty}(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n|z-c|^{n+1} \leq (1-\eta)\left(1-\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n|z-c|^{n+1}\right).$$

From which we get

$$\frac{\sum_{n=1}^{\infty}(n+2-\eta)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} a_n|z-c|^{n+1}}{(1-\eta)} \leq 1. \quad (2.10)$$

Now by applying the basic Theorem 2.1 related to the coefficients of the function in the class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, we arrive at the following:

$$\frac{\sum_{n=1}^{\infty}(n+2-\eta)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} |z-c|^{n+1}}{(1-\eta)} \leq \frac{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma} (\zeta a_{1n}(1+D) - a_{2n}(\zeta(1+D)+C-D))}{C-D}, (n \geq 1).$$

From this we get the radius that determines whether the function has this property

$$|z-c|<\left\{\frac{(1-\eta)(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))}{(n+2-\eta)(C-D)}\right\}^{\frac{1}{n+1}}.$$

Thus, we have finished the proof of the theorem.

Theorem 2.5. If we take a function $f(z)$ in our new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$, which is written as (1.1), then it is meromorphically convex of order

$$\eta=inf_{n\geq 1}\left\{\left\{\frac{(1-\eta)(\zeta a_{11}(1+D)-a_{21}(\zeta(1+D)+C-D))}{n(n+2-\eta)(C-D)}\right\}\right\}^{\frac{1}{n+1}}. \quad (2.11)$$

where η is the radius of the open disk $|z-c|<\eta$ in which the function has this property.
The result is sharp for the function f(z) given by (2.2).

Proof. Our goal is to prove that the function f in the class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ has the property of meromorphically convex of order η . In order to prove this, we must prove that it satisfies the following condition:

$$\left|\frac{(z-c)\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)''}{\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)'}+2\right|\leq 1-\eta. \quad (2.12)$$

Now we take the left side of the previous equation

$$\left|\frac{(z-c)\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)''+2\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)'}{\left(\mathfrak{W}_{\sigma,\upsilon}f(z)\right)'}\right|=\left|\frac{\sum_{n=1}^{\infty}n(n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}a_n(z-c)^{n+1}}{-1+\sum_{n=1}^{\infty}n\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}a_n(z-c)^{n+1}}\right|,$$

This quantity is less than or equal to

$$\leq\frac{\sum_{n=1}^{\infty n}n(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}a_n|z-c|^{n+1}}{1-\sum_{n=1}^{\infty}n\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}a_n|z-c|^{n+1}},$$

In order to satisfy condition (2.12), it must be equal to $1-\eta$, thus

$$\sum_{n=1}^{\infty}n(n+1)\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}a_n|z-c|^{n+1}\leq(1-\eta)\left(1-\sum_{n=1}^{\infty}n\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}a_n|z-c|^{n+1}\right),$$

Therefore, we simplify it and get

$$\frac{\sum_{n=1}^{\infty}n(n+2-\eta)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}a_n|z-w|^{n+1}}{(1-\eta)}\leq 1. \quad (2.13)$$

Once again we use the result (2.2) to get

$$\frac{\sum_{n=1}^{\infty}n(n+2-\eta)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}|z-c|^{n+1}}{(1-\eta)}\leq\frac{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))}{C-D},(n\geq 1),$$

From it we get the required

$$|z-c|<\left\{\frac{(1-\eta)(\zeta b_1(1+D)-c_1(\zeta(1+D)+C-D))}{n(n+2-\eta)(C-D)}\right\}^{\frac{1}{n+1}}.$$

And thus we have completed the proof of the theorem.
Now I want to prove that the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ that we have known has an important and basic property, which is the property of closure under linear combination through the proof of the following theorem.

Theorem 2.6. If we take two functions $g_1(z),\ g_2(z)$ n the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ defined by the form
$g_j(z)=\frac{1}{z-c}+\sum_{n=1}^{\infty}b_{n,j}\ (z-c)^n\ ,\ (j=1,\ 2),$
Then the function $k(z)$ defined by the form

$$k(z)=t\,f_1(z)+(1-t)f_2(z),\qquad (0\leq t\leq 1)$$

Belongs to the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$.

Proof. In order for the function $k(z)$ given in the theorem to belong to the new class $\mathfrak{AL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ it must satisfy the following condition

$$\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta b_{1n}(1+D)-b_{2n}(\zeta(1+D)+C-D))\ (t\,b_{n,1}+(1-t)b_{n,2}),$$

$$= t\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta b_{1n}(1+D)-b_{2n}(\zeta(1+D)+C-D))\ b_{n,1}$$
$$+(1-t)\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta b_{1n}(1+D)-b_{2n}(\zeta(1+D)+C-D))\ \ b_{n,2}\le(C-D).$$

Thus, the function $k(z)$ belongs to the class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$.
Which is required by the theorem.
Thus, we have completed the proof of the theorem.
In the context of the closure of the new class under linear combinations, we prove the following theorem as well.

Theorem 2.7. If we take the functions $g_j(z)$ in the new class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$ of that, defined by the form
$g_j(z) = \frac{1}{z-c} + \sum_{n=1}^{\infty} b_{n,j}\ (z-c)^n\ ,\ (j\in\{1,\ 2,\dots t) and\ 0< k_j<1$
such that

$$\sum_{j=1}^{t} k_j = 1.$$

Then the function H defined by
$H(z)=\sum_{j=1}^{t} k_j\ g_j(z),$
is also belongs to the new class $\mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$.

Proof. For every $(j\in\{1,\ 2,\dots t)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))\ b_{n,j}\le\ C-\ D.$$

Since

$$H(z)=\sum_{j=1}^{t}k_j\ g_j(z)=\sum_{j=1}^{t}k_j\left(\frac{1}{z-c}+\sum_{n=1}^{\infty}b_{n,j}\ (z-c)^n\right)=\ \frac{1}{z-c}+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{t}k_j\ b_{n,j}\right)(z-c)^n\ .$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))}{(C-D)}\left(\sum_{j=1}^{t}k_j\ a_{n,j}\right)$$
$$=\sum_{j=1}^{t}k_j\left(\sum_{n=1}^{\infty}\frac{\left(\frac{\upsilon}{n+\upsilon+1}\right)^{\sigma}(\zeta a_{1n}(1+D)-a_{2n}(\zeta(1+D)+C-D))}{(C-D)}a_{n,j}\right)$$
$$\le\ \sum_{j=1}^{t}k_j\ \ =1.$$

Hence $H(z)\in\ \mathfrak{UL}_c(C,\ D,\ \zeta,\sigma,\upsilon)$.
Hence the proof is complete.

(1)

REFERENCES

[1] MB. Ahmad. "Class of meromorphic functions partially shared values with their differences or shifts". arXive:2108.07562vl[math.CV]. 17 aug 2021. https://doi.org/10.48550/arXiv.2108.07562

[2] L. Cotirla and A. Totoi, "On classes of meromorphic functions defined by subordination and convolution". Symmetry. Vol. 15, no.1763, 2023. https://doi.org/10.3390/sym15091763

[3] PL. Duren." Univalent Functions". Springer-Verlag New York Berlin Heidelberg Tokyo. 2001.

[4] JA. Jenkins." Univalent functions and conformal mapping". Springer-Verlag, Berlin Gottingen Heidelberg. 1958. https://doi.org/10.1007/978-3-662-34447-7

[5] MH. Lafta. "Some geometric properties for a certain class of meromorphic univalent functions by differential operator". Iraqi Journal of Science. Vol. 62, no.8, pp.2667-2675, 2021. https://doi.org/10.24996/ijs.2021.62.8.19

[6] AY. Lashin. "On certain subclasses of meromorphic functions associated with certain integral operators", CMA., vol. 59, pp524-531, 2010. https://doi.org/10.1016/j.camwa.2009.06.015

[7] X. Liu. "A class of meromorphic functions with Yang's conjecture concerning periodicity". Mediterr. J. Math., vol. 21, no. 82, 2024. https://doi.org/ 10.1007/s00009-024-02627-y

[8] SGA Shah, S. Noor, M. Darus and WU. Haq and S. Hussain S. "On meromorphic functions defined by a new class of Liu-Srivastava integral operator". IJAA, vol. 18, no.6, pp.1056-1065, 2020. https://doi.org/10.28924/2291-8639

[9] V. Srinivas , S. Lalitha and KV. Sitavani . "On a certain class of mromorphic functions"., IJAEM.vol. 4, no. 8, pp. 784-786, 2022.

[10] B. Venkateswarlu, PT. Reddy, NI. Rajkumar and S. SreelakshmiS. "On a subclass of meromorphic functions with positive coefficients defined by rapid operator". PJM. Vol. 41, no. 3, pp. 553-568, 2022. https://doi.org/10.22199/issn.0717-6279-3933