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Generalization to The AMK Integral Transform and Its Applications

Mohanad Faisal Salman Algburi*, Qin Zhao

School of Mathematics and Statistics, Wuhan University of Technology, Wuhan, China.
Email:Muhannadalwurdy@gmail.com

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ABSTRACT

Since the proposal of Laplace to integral transformation in 1780, many integral transformations have been proposed. Some are based on Laplace, and others are totally independent. Integral transformations play an essential role in solving versatile problems related to various fields due to their phenomenal ability to transform complex problems from one domain (their original domain) to another, where they can be solved and manipulated much more smoothly and easily. The results could be left in their alternative domain or transformed back via the inverse of that integral transformation into their original domain.

This article proposes a novel type of integral transformation. Defined within the interval [0, π/2] and utilizing trigonometric function kernels f(ρ) and g(ρ), this new transformation is explored through its basic theorems, properties, and derivatives. The properties of the proposed integral transformation have been successfully and efficiently employed to find exact solutions for specific types of ordinary linear differential equations of the first and second order, expressed in terms of (sin(N)).

Introduction

The ability of integral transformations to convert differential or integral equations, or their systems, into simpler algebraic equations for easier solutions has motivated mathematicians to develop, generalize, and propose new integral transformations capable of addressing specific or more general problems in scientific, medical, and engineering fields [1]. Many proposed integral transformations are modifications or derivatives of the Laplace integral transformation. Some of the most notable recent integral transforms include Sumudu [2], Aboodh [3], ElZaki [4], Complex AL-Tememe [5], AL Zughair [6], SEE [7], AMK [8], ARA [9], and many others [10].

Although the generalization of trigonometric functions has been discussed in some works such as [12, 13], the generalization proposed by H. Jafari [11] for the Laplace transform inspired the integral transformation presented in this work.

*Corresponding author at: School of Mathematics and Statistics, Wuhan University of Technology, Wuhan, China
ORCID:https:// https://orcid.org/0009-0006-5630-5235 ,
Tel: +964 7700015172
Email: Muhannadalwurdy@gmail.com

A general formula or generalization for the AMK [8] integral transform within the interval $\left[0,\frac{\pi}{2}\right]$ has been proposed and discussed, along with its properties and basic theorems. Finally, the generalized transformation has been applied to solve actual problems using its properties.

General Concepts

Definition (1), [1]

If the function $F(\rho)$ represents the convergent integral transformation to the function η that lies in the interval (a,b), then $F(\rho)$ can be defined as:

$$I\{\eta(N)\} = F(\rho) = \int_a^b k(\rho,N)\eta(N)dN,\ \ a,b \in \mathbb{R}$$

, and k is the kernel of the transformation.

Definition (2), [8]

For the trigonometric function $\eta(\sin(N))$ where $N \in \left[0,\frac{\pi}{2}\right]$, the AMK convergent integral transformation is defined as:

$$AMK\,\{\eta(\sin(N)\} = \int_{N=0}^{\frac{\pi}{2}} \cos(N)\,(\sin(N))^{\rho}\,\eta(\sin(N))dN = F(\rho)$$

and ρ is constant.

ZhM* Integral Transformation

This work proposes a generalization to the AMK integral transformation, which will be called the ZhM* integral transformation.

Definition (3)

The ZhM* integral transform technique for the function $\eta(\sin(N))$ where $N \in \left[0, \frac{\pi}{2}\right]$ is defined as:

$$ZhM^*\{\eta(\sin(N))\} = f(\rho)\int_{N=0}^{\frac{\pi}{2}} \cos(N)\,(\sin(N))^{g(\rho)}\,\eta(\sin(N)\,dN$$
$$= F\big(f(\rho), g(\rho)\big) = F(\rho)\,.$$

Where $f(\rho)$ and $g(\rho)$ are two functions of parameter ρ with $f(\rho), g(\rho) \neq 0$, and the above integral is convergent in the closed interval $\left[0, \frac{\pi}{2}\right]$.

Property (1): (Linear Property)

$$ZhM^*\{\alpha\, h(\sin(N)) \pm \beta\, k(\sin(N))\} = \alpha\, ZhM^*\{h(\sin(N))\} \pm \beta ZhM^*\{K(\sin(N))\}.$$

Where α and β are constants.

Property (2): (Shifting Property)

If $F(f(\rho), g(\rho))$ is defined as $\text{ZhM}^*\{\eta(\sin(N))\} = F(f(\rho), g(\rho))$ and a is a constant, then:

$$ZhM^*\{(\sin(N))^a\,\eta(\sin(N))\} = F(f(\rho), g(\rho) + a).$$

Proof: By the definition,

$$ZhM^*\{(\sin(N))^a\,\eta(\sin(N))\}$$
$$= f(\rho)\int_{N=0}^{\frac{\pi}{2}} \cos(N)\,(\sin(N))^{g(\rho)}(\sin(N))^a\,\eta(\sin(N)\,dN\,,$$
$$= f(\rho)\int_{N=0}^{\frac{\pi}{2}} \cos(N)\,(\sin(N))^{g(\rho)+a}\,\eta(\sin(N))dN,$$
$$= F(f(\rho), g(\rho) + a)\,.$$

Definition (4): Suppose that $\eta(\sin(N))$ is a function and $hM^*\{\eta(\sin(N))\} = F(f(\rho), g(\rho))$, then $\eta(\sin(N))$ is called the inverse of F and defined as: $(ZhM^*)^{-1}\{F\big(f(\rho), g(s)\big)\} = \eta(\sin(N))\,.$

Theorem (1)

Assume that $F(f(\rho), g(\rho)) = ZhM^*\{\eta(\sin(N))\}$ and a is any constant, then:

$$(ZhM^*)^{-1}\{F(f(\rho), g(\rho) + a)\} = (\sin N)^a (ZhM^*)^{-1}\{F(f(\rho), g(\rho)\}\,.$$

ZhM* Transform Technique of Frequently Encountered Functions

Theorem (2)

(1) If $\eta(\sin(N)) = k$, where k is any constant, then

$$ZhM^*\{k\} = \frac{kf(\rho)}{g(\rho)+1}\,,\ g(\rho) > -1.$$

Proof: By the definition,

$$ZhM^*\{k\} = f(\rho)\int_{N=0}^{\frac{\pi}{2}} \cos(N)(\sin(N))^{g(\rho)}\,k\,dN,$$
$$= kf(\rho)\int_{0}^{\frac{\pi}{2}} \cos(N)(\sin(N))^{g(\rho)}\,dN\,,$$
$$= kf(\rho)\left[\frac{(\sin(N))^{g(\rho)+1}}{g(\rho)+1}\right|_0^{\frac{\pi}{2}} = \frac{kf(\rho)}{g(\rho)+1}\,.$$

(2) If $\eta(\sin(N)) = (\sin(N))^n$, then $ZhM^*\{(\sin(N))^n\} = \frac{f(\rho)}{g(\rho)+(n+1)}$, and $g(\rho) > -(n+1)$.

Proof: By the definition,

$$ZhM^* = \{(\sin(N))^n\}$$
$$= f(\rho)\int_{0}^{\frac{\pi}{2}} \cos(N)(\sin(N))^{g(\rho)}(\sin(N))^n\,dN,$$
$$= f(\rho)\int_{0}^{\frac{\pi}{2}} \cos(N)(\sin(N))^{g(\rho)+n}\,dN,$$
$$ZhM^*\{(\sin(N))^n\} = f(\rho)\left[\frac{(\sin(N))^{g(\rho)+(n+1)}}{g(\rho)+(n+1)}\right|_0^{\frac{\pi}{2}},$$
$$= \frac{f(\rho)}{g(\rho)+(n+1)}\,,\ g(\rho) > -(n+1)\,.$$

(3) If $\eta(\sin(N)) = \cos(a\,ln(sin(N)))$, where a is any constant, then

$$ZhM^*\{\cos(a\ln(\sin(N)))\} = \frac{f(\rho)\cdot(g(\rho)+1)}{(g(\rho)+1)^2 + a^2}\,.$$

Proof: By the definition,

$$ZhM^*\{\cos(a\ln(\sin(N)))\}$$
$$= f(\rho)\int_{N=0}^{\frac{\pi}{2}} \cos(N)\,(\sin(N))^{g(\rho)}\cos(a\,ln(sin(N)))\,dN\,.$$

Since,

$$\cos(a\ln(\sin(N))) = \frac{1}{2}\left[e^{ai\ln(\sin(N))} + e^{-ai\ln(\sin(N))}\right],$$
$$i \in \mathbb{C}, = \frac{1}{2}\left[(\sin(N))^{ai} + (\sin(N))^{-ai}\right].$$

After simple computations, we get:

$$ZhM^*\{\cos(a\ln(\sin(N)))\} = \frac{f(\rho)\cdot(g(\rho)+1)}{(g(\rho)+1)^2+a^2}\,.$$

Similarly,

(4) If $\eta(\sin(N)) = \sin(a\ln(\sin(N)))$, where a is any constant, then

$$ZhM^*\{\sin(a\ln(\sin(N)))\} = \frac{af(\rho)}{(g(\rho)+1)^2+a^2}\,.$$

With $\sin(a\ln(\sin(N))) = \frac{1}{2i}\left[(\sin(N))^{ai} - (\sin(N))^{-ai}\right]$. where $\in \mathbb{C}$.

(5) If $\eta(\sin(N)) = \cosh(a\ln(\sin(N)))$, where a is any constant, then $ZhM^*\{\cosh(a\ln(\sin(N)))\} = \frac{f(\rho)(g(\rho)+1)}{(g(\rho)+1)^2-a^2}$.

With,

$\cosh(a\ln(\sin(N))) = \frac{1}{2}[(\sin(N))^a + (\sin(N))^{-a}]$.

(6) If $\eta(\sin(N)) = \sinh(a\ln(\sin(N))$, where a is any constant then $ZhM^*\{\sinh(\mathrm{a}\ln(\sin(N)))\} = \frac{-f(\rho)\,a}{(g(\rho)+1)^2-a^2}$.

With

$$\sinh(a\ln(\sin(N))) = \frac{1}{2}[(\sin(N))^a - (\sin(N))^{-a}]\,.$$

(7) If $\eta(\sin(N)) = [\ln(\sin(N))]^n, n \in \mathbb{N}$ then

$$ZhM^*\{[\ln(\sin(N))]^n\} = \frac{f(\rho)n!\,(-1)^n}{(1+g(\rho))^{n+1}}\,,\ g(\rho) > -1\,.$$

Theorem (3)

If the $\eta(\sin(N))$ is defined as $N > 0$ and its derivatives $\frac{d\eta(\sin(N))}{dN}, \frac{d^2\eta(\sin(N))}{dN^2}, \cdots, \frac{d^n(\sin(N))}{dN^n}$ and thy are exist, then:

$$\begin{aligned} ZhM^*\{(\sin(N))^n\,\eta^{(n)}(\sin(N))\} &= f(\rho)[\eta^{(n-1)}(1) \\ &- (n+g(\rho))\eta^{(n-2)}(1) \\ &+ (-1)^{n-1}(n+g(\rho))((n-1) \\ &+ g(\rho))\eta^{(n-3)}(1) + \cdots \\ &+ (g(\rho)+n)((n-1)+g(\rho)) + \cdots \\ &+ (g(\rho)+2)\eta(1) \\ &+ (-1)^n[(g(\rho)+n)!\,F(f(\rho),g(\rho))]\,. \end{aligned}$$

Where $n \in \mathbb{N}$.

Notes:

- If n=1 then:

  $ZhM^*\{(\sin(N))\,\eta'(\sin(N))\} = f(\rho)\eta(1) - (g(\rho)+1)F(f(\rho),g(\rho))$.
- If n=2 then:

  $ZhM^*\{(\sin(N))^2\,\eta''(\sin(N))\} = f(\rho)\eta'(1) - -f(\rho)(2+g(\rho))\eta(1) + (g(\rho)+1)(g(\rho)+2)F(f(\rho),g(\rho))$.

Applications

This section will demonstrate the proposed ZhM* integral transformation's capability in solving first- and second-order linear ordinary differential equations of a trigonometric nature.

Application (1)

Solving first-order linear ordinary differential equation using ZhM^*integral transformation.

$$(\sin(N)\,\eta'(\sin(N)) - 4\eta(\sin(N)) = \sinh(2\ln(sin(N))\,,$$

with $\eta(1) = -3$.

Solution: By applying ZhM^* technique, we have

$$ZhM^*\{\sin(N)\,\eta'(\sin(N))\} - 4\,ZhM^*\,\{\eta(\sin(N))\} = ZhM^*\{\sinh(2\ln(\sin((N))\}$$

$$f(\rho)[\eta(1)] - (g(\rho)+1)F(f(\rho),g(\rho)) - 4F(f(\rho),g(\rho)) = \frac{-2f(\rho)}{(g(\rho)+1)^2-4}\,,$$

$$-3f(\rho) - (g(\rho)+1)F - 4F = \frac{-2f(\rho)}{(g(\rho)+1)^2-4}\,,$$

$$-3f(\rho) - (g(\rho)+5)F = \frac{-2f(\rho)}{(g(\rho)+1)^2-4}\,,$$

$$-(g(\rho)+5)F(f(\rho),g(\rho)) = \frac{-2f(\rho)}{(g(\rho)+1)^2-4} + 3f(\rho)\,,$$

$$F(f(\rho),g(\rho)) = \frac{2f(\rho)}{(g(\rho)+5)[(g(\rho)+1)^2-4]} - \frac{3f(\rho)}{g(\rho)+5}\,,$$

$$= \frac{2f(\rho) - 3f(\rho)[(g(\rho)+1)^2-4]}{(g(\rho)+5)[(g(\rho)+1)^2+4]}\,,$$

$$= \frac{2f(\rho) + [-3g^2(\rho) - 6g(\rho) - 3 + 12]f(\rho)}{(g(\rho)+5)[(g(\rho)+1)^2+4]}\,,$$

$$= \frac{[-3g^2(\rho) - 6g(\rho) + 11]f(\rho)}{(g(\rho)+5)[(g(\rho)+1)^2+4]}\,.$$

By applying partial fraction:

$$(ZhM^{*})^{-1}\{F(f(\rho),g(\rho))\} = (ZhM^{*})^{-1}\left\{\frac{Af(\rho)}{g(\rho)+5} + \frac{[B\,g(\rho)+C]f(\rho)}{(g(\rho)+1)^2-4}\right\}$$

After simple computations, we get:

$$A=\frac{-17}{6}, \qquad B=\frac{-1}{6} \text{ and } C=\frac{1}{2}.$$

Then,

$$\eta(\sin(N))=\frac{-17}{6}\sin^4(N)-\frac{1}{6}\cosh(2\ln(\sin(N)))-\frac{1}{3}\sinh(2\,ln(sin(N))).$$

Application (2)

Solving second-order linear ordinary differential equation using ZhM^{*}integral transformation.

$$-(\sin(N))\eta'(\sin(N))+(\sin(N))^2\eta''(\sin(N)) + \eta(\sin(N))=\ln(\sin(N)).$$

With $\eta(1)=-1$ and $\eta'(1)=2$. (Assuming that $(\rho)=1$.

Solution: By applying ZhM* technique, we have:

$$ZhM^{*}\{(\sin(N))^2\eta''(\sin(N))\} - ZhM^{*}\{\sin(N)\,\eta'(\sin(N))\} + ZhM^{*}\{\eta(\sin(N)\} = ZhM^{*}\{\ln(\sin(N))\},$$

$$\eta'(1)-(g(\rho)+2)\eta(1)+(g(\rho)+2)(g(\rho)+1)\,F - \eta(1)+(g(\rho)+1)F+F = \frac{-1}{(g(\rho)+1)^2}.$$

where $F=F(1,g(\rho))$.

After simple computations, we obtain:

$$F=\frac{-g^3(\rho)-7g^2(\rho)-11g(\rho)-6}{(g(\rho)+1)^2(g(\rho)+2)^2}.$$

By taking the inverse of the ZhM* technique and partial fraction, we get the exact solution:

$$\eta(\sin(N))=2+\ln(\sin(N))-3\sin(N) + 4\sin(N)\ln(sin(N)).$$

Conclusions

This work proposed a general formula for the AMK integral transformation. The most notable feature of the AMK transformation is its trigonometric kernel $\sin(N)$ and $\cos(N)$, which distinguishes it from other integral transformations. This work introduced a generalization of the AMK integral transformation with significant capability in solving first and second-order ordinary differential equations that contain trigonometric functions, which are typically difficult to solve. However, the proposed ZhM* demonstrated an excellent capability in solving such equations through its theorems and properties, providing exact solutions.

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تعميم على التحويل التكاملي AMK وتطبيقاته

مهند فيصل سلمان الجبوري*، تشين تشاو

كلية الرياضيات والإحصاء، جامعة ووهان للتكنولوجيا، ووهان، الصين.
Email: Muhannadalwurdy@gmail.com

الخلاصة:

منذ اقتراح لابلاس للتحويل التكاملي في عام 1780، تم اقتراح العديد من التحويلات التكاملية. بعضها يعتمد على لابلاس، والبعض الآخر مستقل تمامًا. تلعب التحويلات التكاملية دورًا أساسيًا في حل المشكلات المتنوعة المتعلقة بمجالات مختلفة نظرًا لقدرتها الهائلة على تحويل المشكلات المعقدة من مجال (مجالها الأصلي) إلى آخر، حيث يمكن حلها والتلاعب بها بسلاسة وسهولة أكبر. يمكن ترك النتائج في مجالها البديل أو تحويلها مرة أخرى عبر معكوس هذا التحويل التكاملي إلى مجالها الأصلي.

تقترح هذه المقالة نوعًا جديدًا من التحويلات التكاملية. يتم تعريف هذا التحويل الجديد ضمن الفترة [0، π/2] واستخدام نوى الدوال المثلثية f(ρ) وg(ρ)، ويتم استكشافه من خلال نظرياته الأساسية وخصائصه ومشتقاته. تم استخدام خصائص التحويل التكاملي المقترح بنجاح وكفاءة لإيجاد حلول دقيقة لأنواع معينة من المعادلات التفاضلية الخطية العادية من الدرجة الأولى والثانية، معبرًا عنها بشكل ($\sin(N)$) .

الكلمات المفتاحية: التحويلات التكاملية؛ تحويل *AMK*؛ المعادلات التفاضلية العادية.