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Mathematical Model for the Control of Unemployment in Nigeria Incorporating Vocational Education and Apprenticeship

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ABSTRACT: ***Background:*** *Youth unemployment presents a significant threat to Nigeria's socioeconomic stability, resulting in widespread issues such as poverty, insecurity, banditry, inequality, and robbery.* ***Objective:*** *In this paper, mathematical modelling for the control of unemployment in Nigeria incorporating vocational education and apprenticeship was formulated. The model equilibrium points were obtained and the local stability analysis was carried out.* ***Methods:*** *The basic reproduction number R_e was computed using the Next Generation Matrix technique and used to determine whether unemployment will persist or be eradicated in Nigeria. Sensitivity analysis was carried out to determine the most sensitive parameters.* ***Results:*** *Numerical simulations were presented in a graphical form to show the effects of sensitive parameters on the reproduction number. The stability analysis shows that unemployment will be eradicated in Nigeria if $R_e \leq 1$. It was revealed from the sensitivity analysis that the theoretical education graduate employment rate, γ is the most sensitive parameter with -0.9883 and the rate of transition of becoming highly skilled due to vocational education, β is the least sensitive parameter with -0.0409.* ***Conclusions:*** *It was shown from the sensitivity analysis and the graphical presentation that the theoretical education graduate employment rate, γ is the most sensitive negative parameter which will decrease the reproduction number. This implies that when employment is created for the graduates from the higher institutions the unemployment rate will reduce drastically from the society. It is recommended that the government and the private sector owners should create more job for the graduates. Optimal control analysis is recommended for future research to determine the minimal cost for the increase graduate employment rate and motivation of vocational education.*

KEYWORDS: *Unemployment; Vocational education; Apprenticeship; Stability analysis; Sensitivity analysis*

INTRODUCTION

The world currently grapples with numerous challenges including; illegal migration, corruption, climate change, pollution, discrimination, insecurity, social vices, and pandemics. Among these, unemployment stands out as one of the most pressing issues globally [1]. This concern is particularly alarming as it directly or indirectly exacerbates other significant problems such as poverty, banditry, inequality, and theft. Unemployment poses a significant threat not only to developing nations but also to developed countries. The National Bureau of Statistics (NBS) in Nigeria defines unemployment as a condition in which individuals have been without work for the previous seven days, are actively seeking employment within the last four weeks, and are available to work [2]. The rate of unemployment is calculated as the proportion of unemployed individuals to the total labour force, and this ratio typically increases during economic recessions. The World Employment and Social Outlook report from the International Labour Organization (ILO) estimates that the 'jobs gap'—which represents the number of individuals who want to work but are without jobs—will reach 402 million in 2024 [3]. Unemployment can lead to various economic, social, and political challenges for a country [4].

In addition to the growth of industries and public sectors, youth in various countries, including Nigeria, are confronting a significant jobs crisis. As of 2023, Nigeria's overall unemployment rate is reported at 5.0%, while the youth unemployment rate stands at 8.6%. Estimates from the International Labour Organization (ILO) indicate that global youth unemployment was at 12.6% in 2013, equating to approximately 73 million young people without jobs. A primary factor contributing to this high unemployment rate is the educational framework in many countries. While there are a limited number of technological, medical, and management institutions that equip students with essential skills, numerous other institutions produce graduates lacking practical skills, thereby exacerbating unemployment. Many individuals are academically qualified and eager to work, yet they find themselves without job offers or the means to start their own businesses due to a lack of skills. Consequently, it is crucial to foster skill development among youth, enabling them to demonstrate their potential in industries, public sectors, or through entrepreneurship. To address this issue, various governments, including Nigeria's, have initiated skill development programmes aimed at empowering their youth [5]. Technical Vocational Education and Training (TVET) is designed to prepare people with the necessary knowledge and vocational skills to fit into specific occupations. TVET can be described as that segment of education which provides recipients with fundamental knowledge and practical skills essential for entering the workforce as employees or self-employed individuals. It is defined as an educational approach that prepares persons for profitable employment as semi-skilled employees, experts, or sub-professionals [6].

Sirghi *et al.* [1] contributed to the discourse by presenting a mathematical model with four subgroups: the employed individuals, the unemployed individuals, the jobs in the marketplace, and the newly created jobs. In [7] a model was designed that considered the effects of limited jobs on unemployment and the creation of new jobs became constant. Inspired by the work of [8], Galindro and Torres [9] developed a model of unemployment and used optimal control to discover real strategies for reducing unemployment in Portugal. In [10] three-dimensional mathematical model was developed and analyzed in order to study the labour market, with a focus on unemployment/employment. The rate of change of new positions created by the public and private sectors depends on the historical levels of unemployment, taking into account the distributed time delay. They carried out the numerical simulations varying different parameters of the model. In [11] a model was formulated for the decrease of unemployment that emphasizes the importance of considering migration when designing plans to tackle unemployment. The model comprises five variables: the employed individuals, the of unemployed individuals, the jobs available, the migrants, and the new vacancies created through government intervention. Their analysis, which included both qualitative and quantitative methods, revealed that incorporating migration into government policy significantly reduced the early level of unemployment. Conversely, neglecting migration in policy considerations led to a marked increase in the unemployment rate. In a similar way, [12] developed an unemployment model focusing on poorer countries, inspired by the work of [8]. This model included three variables: the employees, the unemployed individuals, and the available vacancies. Their findings indicated that the rate of unemployment decreased and got to almost 7% when the basic reproductive ratio R_0 was around 15. Misra *et al.*, [13] developed a nonlinear mathematical model to examine the impact of the informal sector on unemployment dynamics in developing economies. Their model categorizes the workforce into three classes: unemployed, employed, and self-employed, with a separate variable for informal sector vacancies. Analysis using dynamical systems theory revealed a threshold quantity, the reproduction number, which determines the job creation rate needed to stabilize the system. Variations in this number led to changes in system behavior, including transcritical and saddle-node bifurcations. The study also proposed an optimal control model to determine effective government strategies for promoting employment and self-employment in the informal sector. Numerical validation supported the analytical findings, suggesting that unemployment can be reduce effectively if informal sector can promote self-employment. Eric and Calvin [14] developed a mathematical model to examine the influence of financial crises on unemployment. Their nonlinear ordinary differential equation system focuses on the dynamics of unemployment and employment rates. The study established the existence of a unique positive equilibrium and proved its global stability under certain conditions using a Lyapunov function. The authors proposed and compared two control strategies to improve employment rates at minimal cost. Their findings suggest that government assistance in helping unemployed individuals start their own businesses, leading to new job creation, is a more effective approach than supporting self-employed individuals in creating new vacancies. Numerical simulations validated the theoretical results, providing insights for policymakers to address unemployment challenges during financial crises. El Yahyaoui and Amine [15] developed a mathematical model to examine the impact of skills development on cyclical unemployment, categorizing the workforce into fundamentally unemployed, employed, and cyclically unemployed individuals. The model, represented by a system of nonlinear differential equations, assumes that government agendas can help reintegrate persons who have been out of work for a period. The analysis revealed that the model has a globally asymptotically stable equilibrium under certain conditions. Sensitivity analysis and numerical illustrations demonstrated that skills development among the cyclically unemployed can effectively reduce unemployment. The study emphasizes the importance of government initiatives and measures focusing on combating unemployment during economic crises by promoting skills development and professional improvement.

Several researchers have studied the mathematical model of unemployment considering different parameters in recent times; [14], [16]–[20]. Inspired by the work of [18], we formulated the mathematical model of unemployment in Nigeria, incorporating vocational education and apprenticeship, following the recommendations of [19]. In the present work, the highly skilled and apprentice classes are also incorporated to control unemployment in the country. The present study intends to: obtain the equilibrium states of the model, compute the basic reproduction number, carry out the local and global stability of the unemployment-free equilibrium, conduct a sensitivity analysis, and present the graphical simulation of the basic reproduction number with the sensitive parameters. The organization of the paper is as follows: Section 2 presents the model formulation, equilibrium states of the model, basic reproduction number R_e and stability analysis of the equilibrium states. Section 3 presents variables and parameters estimation, sensitivity analysis, graphical simulation, and discussion of results. The conclusion is in section 4.

MATERIALS AND METHODS

Model Formulation

The model employs a compartmental structure to represent the flow of individuals between different workforce states: Vocational class $V(t)$, Theoretical class $T(t)$, Unemployed class $U(t)$, Highly-skilled class $Hs(t)$, Apprenticed class $A(t)$, and the Employed class $E(t)$.

The Vocational class is made up of individuals enrolled in vocational education programs aimed at equipping them with skills for self-employment upon graduation (transitioning to the Highly-Skilled class, $Hs(t)$). The Theoretical class comprise individuals pursuing traditional academic education or those leaving vocational programs after enrollment as a result of poor policy implementation by government and private sectors. The employed class includes persons who have been employed or self-employed after graduation from vocational education or apprenticeship. We assumed that the individuals disengage from all compartments $(V(t), T(t), Hs(t), A(t)$, and $E(t))$ due to factors like retirement, career changes, or migration at the rate μ. Also, individuals in the employed class $E(t)$ are assumed to remain employed unless actively seeking new opportunities (greener pastures). We also assumed that unemployed individuals graduating from the theoretical class either withdrawal by migration or move to apprenticed class to learn high-demand skills that will prepare them for future employment during their waiting period.

$$\begin{cases} \dfrac{dV}{dt} = \phi\Lambda - (\alpha + \beta + \mu)V, \\ \dfrac{dT}{dt} = (1-\phi)\Lambda + \alpha V - \psi\theta(1-\rho)TU - (\gamma + \mu)T, \\ \dfrac{dU}{dt} = \psi\theta(1-\rho)TU - \omega\tau(1-\sigma)U - \mu U, \\ \dfrac{dHs}{dt} = \beta V - (a + \eta + \mu)Hs, \\ \dfrac{dA}{dt} = \omega\tau(1-\sigma)U - (b + \eta + \mu)A, \\ \dfrac{dE}{dt} = \gamma T + (a+\eta)Hs + (b+\eta)A - \mu E. \end{cases} \tag{1}$$

Figure 1 shows the schematic diagram of the model, while Table 1 the definition of the variables and parameters.

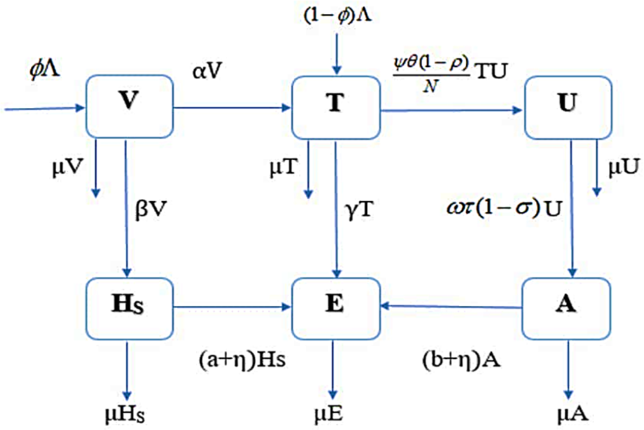

Figure 1. The schematic diagram of the model

Table 1. Definition of variables and parameters of the model

Symbol	Description
$V(t)$	Number of individuals in vocational education programs at time t
$T(t)$	Number of individuals in theoretical education at time t
$U(t)$	Number of unemployed individuals actively seeking employment at time t
$Hs(t)$	Number of highly-skilled individuals graduated from $V(t)$ at time t
$A(t)$	Number of apprenticed at time t
$E(t)$	Number of individuals with gainful employment at time t
Λ	Total number of recruitment
ϕ	Rate of motivation for vocational education programs
$1-\phi$	Proportion of individuals recruited into theoretical education
μ	Rate of withdrawal
α	the Rate of loss of interest in vocational education
γ	Theoretical education graduate employment rate
β	Rate of transition of becoming highly-skilled due to vocational education
a	Highly-skilled vocational graduate employment rate
b	Rate of transition of becoming employed due to apprenticeship
ψ	Per capital probability of becoming unemployed
θ	the Rate of influence of individuals in $U(t)$ on $T(t)$ class
ρ	Vocational program successful implementation rate
δ	the Tendency of becoming unemployed defined as $\delta=\psi\theta(1-\rho)$
ω	Per capital probability of becoming employed
τ	the Rate of influence of individuals in $A(t)$ on $U(t)$ class
σ	Apprenticeship program successful implementation rate
ε	Tendency of becoming employed defined as $\varepsilon=\omega\tau(1-\sigma)$
η	Rate of government and private investment

The model integrates three control strategies to analyze their impact on unemployment transmission dynamics; Vocational Education $U(t)$ compartment represents the intervention of vocational education programs. The model will track successful implementation of vocational education on theoretical $T(t)$ and the movement of individuals from vocational $V(t)$ into highly-skilled $Hs(t)$ or the employed class $E(t)$ upon graduation. The model also incorporates the Apprenticeship Program $A(t)$

as a crucial intervention to bridge the gap between unemployment and employment. An investment parameter η is introduced to represent the level of government or private sector investment in both vocational education and apprenticeship programs.

Equilibrium States of the Model

At equilibrium, the right-hand side (RHS) of the model (1) equals zero, which gives,

$$\phi\Lambda - k_1 V^* = 0, \tag{2}$$

$$(1-\phi)\Lambda + \alpha V^* - \delta T^* U^* - k_2 T^* = 0, \tag{3}$$

$$\delta T^* U^* - k_3 U^* = 0, \tag{4}$$

$$\beta V^* - k_4 Hs^* = 0, \tag{5}$$

$$\varepsilon U^* - k_5 A^* = 0, \tag{6}$$

$$\gamma T^* + (a+\eta) Hs^* + (b+\eta) A^* - \mu E^* = 0, \tag{7}$$

where,

$$\begin{aligned} &\delta = \psi\theta(1-\rho); \varepsilon = \omega\tau(1-\sigma), \\ &k_1 = \alpha+\beta+\mu; k_2 = \gamma+\mu; k_3 = \varepsilon+\mu; k_4 = a+\eta+\mu; k_5 = b+\eta+\mu\ . \end{aligned} \tag{8}$$

From (4),

$$(\delta T^* - k_3) U^* = 0, \tag{9}$$

$$U^* = 0, \tag{10}$$

or

$$\delta T^* - k_3 = 0\ . \tag{11}$$

Equation (10) above is the necessary condition for the Unemployment-Free Equilibrium (UEF) state, while equation (11) is for the Unemployment Endemic Equilibrium (UFE) state.

Unemployment-Free Equilibrium State

In this subsection, we consider the situation where there is no unemployment. Equation (10) has shown that the Unemployment-Free Equilibrium state exists.

Let the Unemployment-Free Equilibrium, E_0 be given by

$$E_0 = \left(V^0,\ \ T^0,\ \ U^0,\ \ Hs^0,\ \ A^0,\ \ E^0\right) = \left(V^*,\ \ T^*,\ \ U^*,\ \ Hs^*,\ \ A^*,\ \ E^*\right)\ . \tag{12}$$

Thus, substituting (10) and (12) into (2), (3), (5), (6) and (7) gives,

$$\phi\Lambda - k_1 V^\circ = 0, \tag{13}$$

$$(1-\phi)\Lambda + \alpha V^o - k_2 T^o = 0, \tag{14}$$

$$\beta V^\circ - k_4 Hs^\circ = 0 \tag{15}$$

$$k_5 A^\circ = 0, \tag{16}$$

$$\gamma T^\circ + (a+\eta) Hs^\circ + (b+\eta) A^\circ - \mu E^\circ = 0\ . \tag{17}$$

From (13),

$$V^o = \frac{\phi\Lambda}{K_1}\ . \tag{18}$$

From (16)

$$A^\circ = 0\ . \tag{19}$$

Substituting (18) into (14) gives

$$T^0 = \frac{K_1\left(1-\phi\right)\Lambda + \alpha\phi\Lambda}{k_1 k_2}\ . \tag{20}$$

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Substituting (18) into (15) gives

$$H_s^0 = \frac{\phi\beta\Lambda}{k_1 k_4} \ . \tag{21}$$

Substituting (19), (20) and (21) into (17) gives

$$E^0 = \frac{k_1 k_4 \gamma \left(1-\phi\right)\Lambda + k_4 \alpha\phi\gamma\Lambda + k_2 \phi\beta\Lambda \left(\alpha+\eta\right)}{k_1 k_2 k_4} \ . \tag{22}$$

Equations (18), (19), (20), (21) and (22) are the Unemployment-Free Equilibrium point E_0;

$$E_o = \begin{pmatrix} V^o \\ T^o \\ U^o \\ Hs^o \\ A^o \\ E^o \end{pmatrix} = \begin{pmatrix} \frac{\phi\Lambda}{k_1} \\ \frac{k_1(1-\phi)\Lambda+\alpha\phi\Lambda}{k_1 k_2} \\ 0 \\ \frac{\beta\phi\Lambda}{k_1 k_4} \\ \frac{(k_1(1-\phi)\Lambda+\alpha\phi\Lambda)k_4\gamma+k_2(\alpha+\eta)\beta\phi\Lambda}{k_1 k_2 k_4 \mu} \end{pmatrix} . \tag{23}$$

Unemployment Endemic Equilibrium State

In this sub-section, we are looking at the situation where the unemployment exists. In equation (11), we saw that there can be such a situation.

The endemic equilibrium state of the model is denoted by

$$E_1 = \left(V^1, T^1, U^1, Hs^1, A^1, E^1\right) = \left(V^*, T^*, U^*, Hs^*, A^*, E^*\right) \ . \tag{24}$$

Thus, from (11),

$$T^1 = \frac{k_3}{\delta} \ . \tag{25}$$

Substituting (24) and (25) into (2) to (7) gives

$$\phi\Lambda - k_1 V^1 = 0, \tag{26}$$

$$(1-\phi)\Lambda + \alpha V^1 - \delta T^1 U^1 - k_2 T^1 = 0, \tag{27}$$

$$\beta V^1 - k_4 Hs^1 = 0, \tag{28}$$

$$\varepsilon U^1 - k_5 A^1 = 0, \tag{29}$$

$$\gamma T^1 + (a+\eta) Hs^1 + (b+\eta) A^1 - \mu E^1 = 0 \ . \tag{30}$$

From (24)

$$V^1 = \frac{\phi\Lambda}{k_1} \ . \tag{31}$$

Substituting (31) into (25) gives

$$U^1 = \frac{k_1(1-\phi)\Lambda\delta + \alpha\phi\Lambda - k_1 k_2 k_3}{k_1 k_3 \delta} \ . \tag{32}$$

Substituting (31) into (27) gives

$$H_s^1 = \frac{\beta\phi\Lambda}{k_1 k_4} \ . \tag{33}$$

Substituting (32) into (29) gives

$$A^1 = \varepsilon \left(\frac{k_1(1-\phi)\Lambda\delta + \alpha\phi\Lambda - k_1k_2k_3}{k_1k_3k_5\delta} \right) . \tag{34}$$

Substituting (25), (33) and (34) into (30) gives

$$E^1 = \frac{k_1k_3^2k_4k_5\gamma + (a+\eta)k_3k_5\beta\phi\Lambda\delta + (b+\eta)\varepsilon k_4\left(k_1(1-\phi)\Lambda\delta + \alpha\phi\Lambda - k_1k_2k_3\right)}{k_1k_3k_4k_5\delta\mu}. \tag{35}$$

Equations (25), (31), (32), (33), (34) and (35) are the Unemployment Endemic Equilibrium point E_1

$$E_1 = \begin{pmatrix} V^1 \\ T^1 \\ U^1 \\ Hs^1 \\ A^1 \\ E^1 \end{pmatrix} = \begin{pmatrix} \frac{\phi\Lambda}{k_1} \\ \frac{k_3}{\delta} \\ \frac{k_1(1-\phi)\Lambda\delta+\alpha\phi\Lambda-k_1k_2k_3}{k_1k_3\delta} \\ \frac{\beta\phi\Lambda}{k_1k_4} \\ \varepsilon\left(\frac{k_1(1-\phi)\Lambda\delta+\alpha\phi\Lambda-k_1k_2k_3}{k_1k_3k_5\delta}\right) \\ \frac{k_1k_3^2k_4k_5\gamma+(a+\eta)k_3k_5\beta\phi\Lambda\delta+(b+\eta)\varepsilon k_4(k_1(1-\phi)\Lambda\delta+\alpha\phi\Lambda-k_1k_2k_3)}{k_1k_3k_4k_5\delta\mu} \end{pmatrix} . \tag{36}$$

Basic Reproduction Number (R_e)

The basic reproduction number of the average amount of secondary infections generated by an infectious person in a susceptible individual during the infectious period. It is computed using the next-generation matrix, where the infected classes are considered. In this model the unemployment class is considered as the infected class. The spectral radius of the FV^{-1} matrix is the basic reproduction number.

Let F be the rate of appearance of new unemployment in the population and let V be the rate of transfer of unemployment from compartment U to another. Following the approach of [21] we define F and V as follows

$$F = \frac{\partial F_i}{\partial U_i}(E_o), \text{ and } V = \frac{\partial V_i}{\partial U_i}(E_o) . \tag{37}$$

From the model system, we shall use only the third equation of (1) for the derivation.

Recall from (4) that:

$$\frac{dU^*}{dt} = \delta T^* U^* - k_3 U^* .$$

It can be deduced that $F_i = [\delta T^* U^*]$ and $V_i = [k_3 U]$

$$F = \frac{\partial F_i}{\partial U_i}(U_o) = \delta T^*(U_o) = \delta T^o = \delta\left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2}\right), \tag{38}$$

$$V = \frac{\partial V_i}{\partial U_i}(U_o) = k_3,$$

$$V^{-1} = \frac{1}{k_3} . \tag{39}$$

Multiplying (38) by (39), we get

$$FV^{-1} = \delta\left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2k_3}\right), \tag{40}$$

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$$|FV^{-1} - \lambda I| = 0, \tag{41}$$

$$\lambda = \delta \left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2k_3} \right) . \tag{42}$$

The basic reproduction number is given as

$$R_e = \delta \left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2k_3} \right). \tag{43}$$

Equation (43) above is the effective reproduction number for the model system.

Stability Analysis of the Equilibrium States

The local and global stability analyses of the model (1) will be carried out. The Jacobian matrix technique is used for the local stability, and the Lyapunov function is used for the global stability.

Local Stability of Unemployment-Free Equilibrium Point

Theorem 1: The unemployment-free equilibrium point E_0 is said to be locally asymptotically stable if all the eigenvalues of the Jacobian matrix at UFE are all negative and $R_e < 1$.
Proof: We use the Jacobian matrix technique to analyze the stability of the model.
The Jacobian of model system (1) to (6) at unemployment-free equilibrium is given by

$$J = \begin{pmatrix} -k_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & -\delta U - k_2 & -\delta T^0 & 0 & 0 & 0 \\ 0 & \delta U & \delta T^0 - k_3 & 0 & 0 & 0 \\ \beta & 0 & 0 & -k_4 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & -k_5 & 0 \\ 0 & \gamma & 0 & a+\eta & b+\eta & -\mu \end{pmatrix}, \tag{44}$$

$$J[E_0] = \begin{pmatrix} -k_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & -k_2 & -\delta k_6 & 0 & 0 & 0 \\ 0 & 0 & \delta k_6 - k_3 & 0 & 0 & 0 \\ \beta & 0 & 0 & -k_4 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & -k_5 & 0 \\ 0 & \gamma & 0 & a+\eta & b+\eta & -\mu \end{pmatrix}, \tag{45}$$

where $k_6 = T^o = \frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2}$

Performing row operation on (3.47) the characteristic equation $\mid J(E_0) - \lambda I \mid = 0$ gives,

$$(-k_1 - \lambda_1)(-k_2 - \lambda_2)(\delta k_6 - k_3 - \lambda_3)(-k_4 - \lambda_4)(-k_5 - \lambda_5)(-\mu - \lambda_6) = 0, \tag{46}$$

$$\lambda_1 = -k_1, \lambda_2 = -k_2, \lambda_3 = \delta k_6 - k_3, \lambda_4 = -k_4, \lambda_5 = -k_5, \lambda_6 = -\mu, \tag{47}$$

for the unemployment-free equilibrium state of the model system to be stable $\lambda_i < 0; i = 1,2,3,4,5,6$.But from (47) it is revealed that all the $\lambda_i < 0$ except λ_3.

Therefore, to proof that the model is stable there is need to show that $\lambda_3 < 0$, which will give the second condition on Theorem 1,i.e. $(R_e < 0)$

$$\Rightarrow \delta k_6 - k_3 < 0,$$

but $k_6 = \frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2}$

$$\begin{cases} \delta \left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2} \right) - k_3 < 0, \\ \delta \left(\frac{k_1(1-\phi)\Lambda + \alpha\phi\Lambda}{k_1k_2} \right) < k_3 \ . \end{cases} \tag{48}$$

Divide both sides by k_3,

$$\delta\left(\frac{k_1(1-\phi)\Lambda+\alpha\phi\Lambda}{k_1k_2k_3}\right)<1\ . \tag{49}$$

Substituting (43) into (49) gives,

$$R_e<1. \tag{50}$$

Obviously, $\lambda_1,\lambda_2,\lambda_4,\lambda_5<0$ and if the inequality (49) is true, the unemployment-free equilibrium state is asymptotically stable.

Global Stability of Unemployment-Free Equilibrium

Theorem 2: If $R_e\leq 1$, the UFE is globally asymptotically stable (GAS).
Proof
Define the Lyapunov function

$$F\left(V,T,U,H_s,A,E\right)=k_3U\ . \tag{51}$$

Differentiating (46) with respect to t gives

$$\frac{dF}{dt}=k_3\frac{dU}{dt}, \tag{52}$$

$$\frac{dF}{dt}=k_3\left(\delta T-k_3\right)U, \tag{53}$$

since at UFET$\leq T^0$

$$\frac{dF}{dt}\leq k_3\left(\delta T^0-k_3\right)U,$$

$$\frac{dF}{dt}\leq k_3\left[\frac{\delta\left(K_1\left(1-\phi\right)\Lambda+\alpha\phi\Lambda\right)-k_1k_2k_3}{k_1k_2}\right]U\ . \tag{54}$$

Divide(54) through by k_3^2 and simplify gives

$$\frac{dF}{dt}\leq k_3^2\left[\mathrm{R_e}-1\right]U. \tag{55}$$

Hence, from (55) $\mathrm{R_e}\leq 0$ implies that $\frac{dF}{dt}\leq 0$, we therefore conclude that $F(V,T,U,H_s,A,E)$ is negative definite and this proves that the model is Globally Asymptotically Stable.

RESULTS AND DISCUSSION

In this section, we present the estimated variables and parameters used in the model in Table 2 and the estimations can be seen in Appendix A. The sensitivity analysis was carried out and the sensitivity indices are presented in Table 3 and Figure 2. The basic reproduction number and sensitive parameters are presented graphically in Figure 3 to 7.

Variables and Parameters Estimation

The values of variables and parameters were estimated using the technique of [22] and [23] based on the available data from the National Bureau of Statistics (NBS), National Board for Technical Education (NBTE), National Directorate of Employment (NDE), Federal Ministry of Labor and Employment (FMLE), International Labor Organization (ILO), and other reliable related sources. The details of the estimations are in Appendix A, and the values are presented in Table 2.

Table 2. Summary of variables and parameters estimation

Variables/ Parameters	Values per year	Source
N	95,246,975	V_1
$E(0)$	72,006,713	V_2
$A(0)$	757,213	V_3
$U(0)$	3,028,854	V_4
Λ	19,525,630	V_5
$Hs(0)$	468,615	V_6
$V(0)$	3,436,511	V_7
$T(0)$	15,549,069	V_8
ϕ	0.20	Assumed
α	0.12	Assumed
β	0.40	Assumed
μ	0.0071	P_1
ψ	0.040	P_2
θ	0.025	P_2
ρ	0.14	P_2
δ	0.00086	P_2
ω	0.60	P_3
τ	0.61	P_3
σ	0.21	P_3
ε	0.28914	P_3
γ	0.60	P_4
a	0.90	P_5
b	0.85	P_6
η	0.12	Assumed

Sensitivity Analysis

In this sub-section, a sensitivity analysis is conducted to show that each parameter in the basic reproduction number considered is sensitive to unemployment. By way of definition, the sensitivity index measures the relative change in a variable concerning the relative change in the parameters involved.

The normalized forward sensitivity index of a variable to a parameter is used to carry out the sensitivity analysis following the approaches of [24] and [25].

The normalized forward sensitivity indices with respect to a parameter value, P is defined as

$$S_P^{R_0} = \frac{\partial R_0}{\partial P} \times \frac{P}{R_0}, \tag{56}$$

where,

$$P = \{\alpha, \beta, \gamma, \phi, \rho\}. \tag{57}$$

The sensitivity indices are given in Table 3, where the parameters positive and negative values.

Table 3. Sensitivity indices of the effective reproduction number and the parameters

Parameters	Sensitivity indices
α	0.0416
β	-0.0409
γ	-0.9883
ϕ	-0.1827
ρ	-0.1628

The local and global stability analyses reveal that the unemployment-free equilibrium were stable if $R_e < 1$ and $R_e \leq 1$ respectively. Ayoade *et al.* [18] carried out only local stability analysis and the condition of stability was based on some parameters less than zero and not on the implementation success ratio that may be equivalent to basic reproduction number R_e.

The sensitivity analysis shows that the rate of loss of interest in vocational education, α is the only positive value, which implies that the basic reproduction number will increase and the unemployment will increase in the population. This is a strong indication that government should encourage vocational education by funding and awareness. Several people believe that it is those who are not brilliant that are in vocational schools. It is also revealed that the theoretical education graduate employment rate, γ is the most sensitive negative parameter which will decrease the reproduction number. The only means by which people are gainfully employed is when they graduate from tertiary institutions. This implies that when the graduates from the higher institutions are employed the unemployment will reduce drastically from the society. Other parameters that contribute to the decrease in unemployment are rate of motivation for vocational education programs ϕ, vocational program successful implementation rate ρ and the rate of transition of becoming highly-skilled due to vocational education, β. The sensitivity analysis has helped to identify the parameters that are responsible for increase and decrease of unemployment compared to the work of [18] where there is no such analysis.

It is beneficial to determine the impact of the parameters on the basic reproduction number R_e which increases or decreases it the most for the sake of effective implementation of the proper control strategies needed to curtail the increase of unemployment. Figure 2 is the presentation the sensitivity indices in a chart for clear understanding.

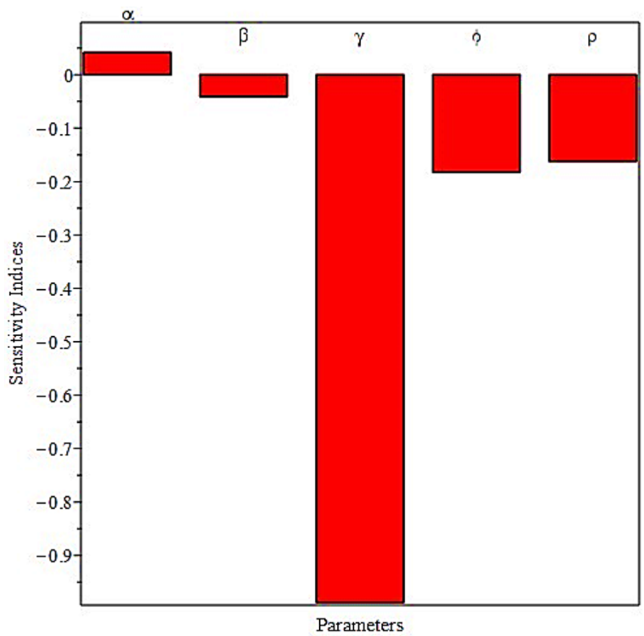

Figure 2. Sensitivity indices chart

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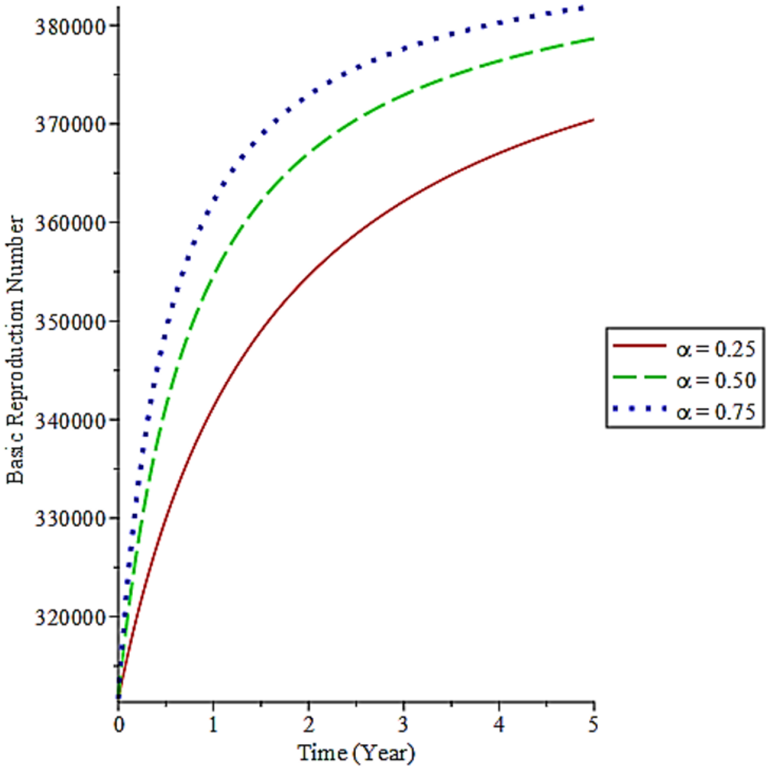

Figure 3. Effect of rate of loss of interest in vocational education on basic reproduction number

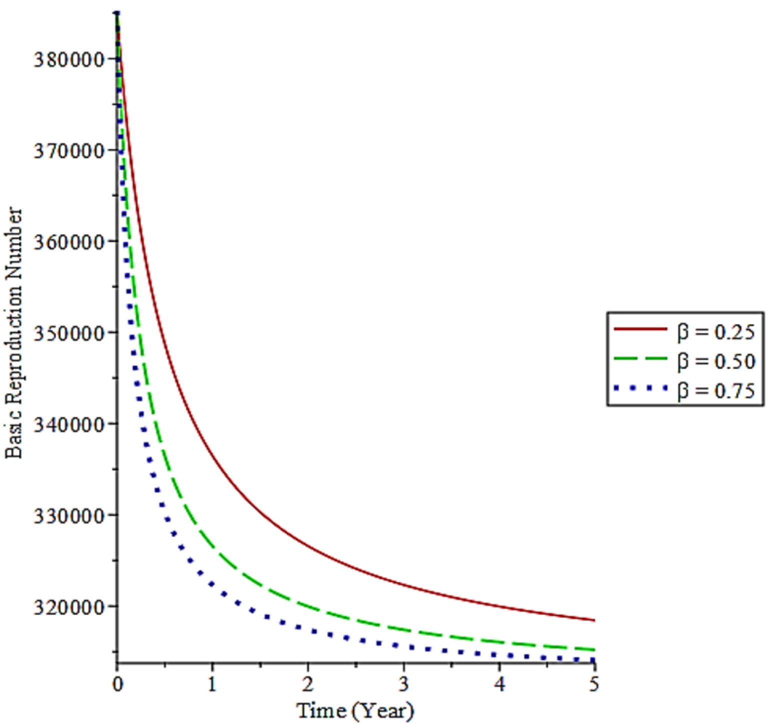

Figure 4. Effect of rate of transition of becoming high-skilled due to vocational education on basic reproduction number

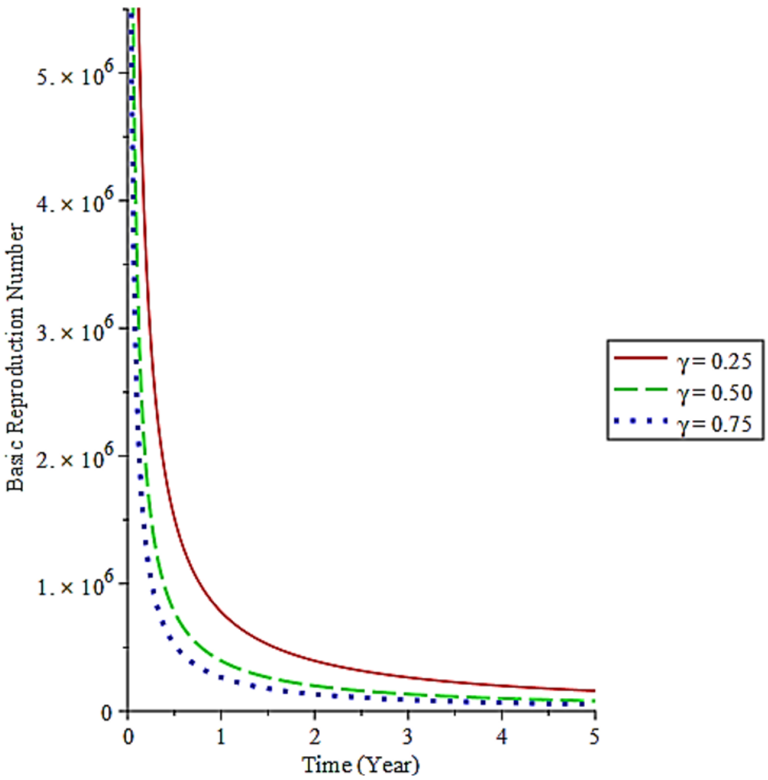

Figure 5. Effect of theoretical education graduate employment rate on basic reproduction number

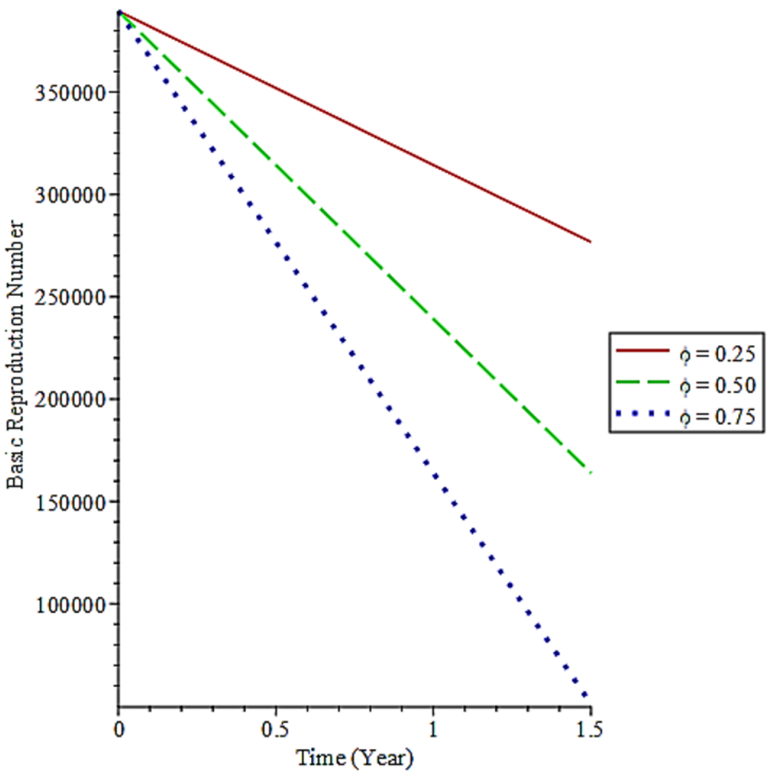

Figure 6. Effect of rate of motivation for vocational education on basic reproduction number

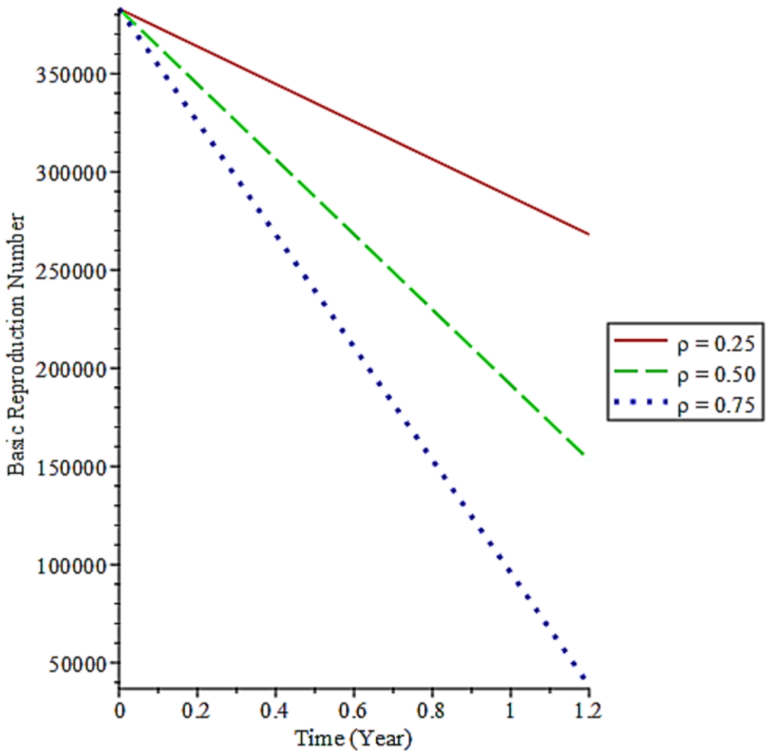

Figure 7. Effect of vocational program successful implementation on the basic reproduction number

Figure 2 gives the pictorial representation of the sensitivity analysis, it shows that the theoretical education graduate employment rate, γ is the most sensitive parameter with almost 99% decrease in unemployment. The least sensitive parameter, rate of transition of becoming highly-skilled due to vocational education, β will decrease the unemployment by 4%. This is due to the reality on ground that even the government policies do not favour technical education. Figure 3 shows the effect of different rates of loss of interest in vocational education α on basic reproduction number R_e, it is observed that as α increases with time the reproduction number increases. It increases from 0 to 380,000 at $\alpha = 0.25, R_e = 370,000, \alpha = 0.50,\ R_e = 378,000$ and $\alpha = 0.75, R_e = 382,000$. It is an indication that if hatred for vocational education is not checked the rate of unemployment will increase. Figure 4, shows the effect of different rates of transition of becoming highly-skilled due to vocational education β on the basic reproduction number R_e, it is revealed that as β increases with time R_e decreases. The decrease is from 385,000 to 0 at $\beta = 0.25, R_e = 318,000, \beta = 0.50, R_e = 304,000$ and $\beta = 0.75,\ R_e = 300,000$. It is an indication that if hatred for vocational education is not checked the rate of unemployment will increase. Figure 4, shows the effect of different rates of transition of becoming highly-skilled due to vocational education β on the basic reproduction number R_e, it is revealed that as β increases with time R_e decreases. The decrease is from 385, 000 to 0 at $\beta = 0.25, R_e = 318,000,\ \beta = 0.50,\ R_e = 304,000$ and $\beta = 0.75,\ R_e = 300,000$. Figure 5 shows the effect of theoretical education graduate employment rate, γ on the basic reproduction number R_e, it is revealed that as γ increases with time the R $_e$ decreases. It decreases from 28,000,000 to 0 in a short time, at $\gamma = 0.25, R_e = 150,000, \gamma = 0.50, R_e = 100,000$ and $\gamma = 0.75, R_e = 50,000$. Figure 6 shows the effect of different rates of motivation for vocational education programs ϕ on the basic reproduction number R_e,it is revealed that as ϕ increases with time the R_e decreases. It decreases from 390,000 to 0 at $\phi = 0.25, R_e = 280,000, \phi = 0.50, R_e = 170,000$ and $\phi = 0.75, R_e = 50,000$. Figure 7 shows the effect of the vocational program successful implementation rate ρ on the basic reproduction number R_eIt is revealed that as ρ increases with time the R_e decreases. It decreases from 380,000 to 0 at $\rho = 0.25, R_e = 270,000,\ \rho = 0.50,\ R_e = 155,000$ and $\rho = 0.75,\ R_e = 40,000$.

It is observed that the successful implementation of vocational program and the motivation for vocational education programs are key factors in decreasing unemployment. It takes a short time for ρ and ϕ to reduce unemployment compared to γ and β. It takes γ and β five years to reduce unemployment to the barest minimum while ρ and ϕ takes one year, two months, and one year six months.

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CONCLUSION

The mathematical modelling of the dynamics of unemployment incorporating vocational education and apprenticeship as controls using non-linear ordinary differential equations. The equilibrium points were obtained and the unemployment-free equilibrium was analyzed locally and globally for stability using Jacobian matrix technique and Lyapunouv function. The basic reproduction number R_e,was computed and unemployment-free equilibrium was stable when $R_e < 1$. The sensitivity analysis with some parameters of the model was carried out with respect to the R_e. The effect of the sensitive parameters on the effective reproduction number R_e was presented graphically. It was shown from the sensitivity analysis that the theoretical education graduate employment rate, γ is the most sensitive negative parameter which will decrease the reproduction number. It was observed from Figure 5 that the impact of theoretical education on the graduate employment rate, γ on the basic reproduction number was very high; it was revealed that the decrease was drastic in a short time. This implies that when employment is created for graduates from higher institutions, the unemployment rate will reduce drastically in society. We recommend that the government and the private sector owners create more jobs for the graduates, and vocational education should be prioritized by motivating the students in vocational schools with scholarships and increasing enrollment. Also, manufacturing industries should be established to absorb the graduates from vocational schools. Consequently, this work can be extended to include other factors such as industrialization as a means to reduce unemployment.

SUPPLEMENTARY MATERIAL

The following supporting information can be downloaded at: https://mjs.uomustansiriyah.edu.iq/index.php/MJS/article/view/1655/818.

AUTHOR CONTRIBUTIONS

Samuel Abu Somma: Methodology, Formal analysis, Writing–original draft. Rokeeb O. Omotinuwe: Software, and Validation. Nurat Olamide Abdurrahman: Visualization, Writing–review, and editing.

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DATA AVAILABILITY STATEMENT

All data generated or analyzed during this study are included in this published article (and its supplementary material file).

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CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

REFERENCES

[1] N. Sirghi, M. Neamtu, and D. Deac, “A dynamic model for unemployment control with distributed delay,” in *Business administration; Mathematical methods in finance and business administration; proceedings of the 8th WSEAS international conference on business administration (ICBA ’14), proceedings of the 2n*, vol. 2014, 2014, pp. 42–48. [Online]. Available: `https://scienceon.kisti.re.kr/srch/selectPORSrchArticle.do?cn=NPAP11348289`.

[2] National Bureau of Statistics (NBS), “Annual Nigerian labour force survey report,” National Bureau of Statistics (NBS), Nigeria, 2023. [Online]. Available: `https://www.nigerianstat.gov.ng/pdfuploads/Annual_Nigerian_Labour_Force_Survey_Report.pdf` (visited on 05/27/2025).

[3] International Labour Organization, "World employment and social outlook: Trends 2024," ILO; Geneva, 2024, p. 113. doi: `https://doi.org/10.54394/HQAE1085`. (visited on 05/27/2025).

[4] A. K. Singh, P. K. Singh, and A. K. Misra, "Combating unemployment through skill development," *Nonlinear Analysis: Modelling and Control*, vol. 25, no. 6, pp. 919–937, 2020. doi: `10.15388/namc.2020.25.20598`.

[5] A. K. Misra, A. K. Singh, and P. K. Singh, "Modeling the role of skill development to control unemployment," *Differential Equations and Dynamical Systems*, vol. 30, no. 2, pp. 397–409, 2017. doi: `10.1007/s12591-017-0405-3`.

[6] R. Audu, S. Maaji, A. Idris, S. Jido, and I. Shehu, "Entrepreneurship education for alleviating youth unemployment in Nigeria: A case study of technical vocational education and training," pp. 86–99, 2019. [Online]. Available: `https://www.scribd.com/document/821611145/Entrepreneurship-education-for-alleviating-youth-unemployment-in-Nigeria-1`.

[7] S. Al-Sheikh, R. Al-Maalwi, and H. A. Ashi, "A mathematical model of unemployment with the effect of limited jobs," *Comptes Rendus. Mathématique*, vol. 359, no. 3, pp. 283–290, 2021. doi: `10.5802/crmath.164`.

[8] S. B. Munoli and S. Gani, "Optimal control analysis of a mathematical model for unemployment," *Optimal Control Applications and Methods*, vol. 37, no. 4, pp. 798–806, 2015. doi: `10.1002/oca.2195`.

[9] A. Galindro and D. F. M. Torres, "A simple mathematical model for unemployment: A case study in Portugal with optimal control," *Statistics, Optimization & Information Computing*, vol. 6, no. 1, 2018. doi: `10.19139/soic.v6i1.470`.

[10] L. F. Vesa, "Dynamics of a simple mathematical model for unemployment with time delay," *Annals of West University of Timisoara - Mathematics and Computer Science*, vol. 58, no. 2, pp. 85–95, 2022. doi: `10.2478/awutm-2022-0019`.

[11] L. Harding and M. Neamţu, "A dynamic model of unemployment with migration and delayed policy intervention," *Computational Economics*, vol. 51, no. 3, pp. 427–462, 2016. doi: `10.1007/s10614-016-9610-3`.

[12] R. M. Al-Maalwi, H. A. Ashi, and S. Al-sheikh, "Unemployment model," *Applied Mathematical Sciences*, vol. 12, no. 21, pp. 989–1006, 2018. doi: `10.12988/ams.2018.87102`.

[13] A. Misra, M. Kumari, and M. Sajid, "Role of informal sector to combat unemployment in developing economy: A modeling study," *Heliyon*, vol. 10, no. 13, p. e33378, 2024. doi: `10.1016/j.heliyon.2024.e33378`.

[14] E. R. Njike-Tchaptchet and C. Tadmon, "Mathematical modeling of the unemployment problem in a context of financial crisis," *Mathematics and Computers in Simulation*, vol. 211, pp. 241–262, Sep. 2023. doi: `10.1016/j.matcom.2023.04.014`.

[15] M. El Yahyaoui and S. Amine, "Mathematical modeling of unemployment dynamics with skills development and cyclical effects," *Partial Differential Equations in Applied Mathematics*, vol. 11, p. 100 800, Sep. 2024. doi: `10.1016/j.padiff.2024.100800`.

[16] R. Al-Maalwi, S. Al-Sheikh, H. Ashi, and S. Asiri, "Mathematical modeling and parameter estimation of unemployment with the impact of training programs," *Mathematics and Computers in Simulation*, vol. 182, pp. 705–720, Apr. 2021. doi: `10.1016/j.matcom.2020.11.018`.

[17] E. Y. Mohamed, "Stability and control analysis of an unemployment model incorporating integration programs effect," *Advanced Mathematical Models & Applications*, vol. 9, no. 2, pp. 267–285, 2024. doi: `10.62476/amma9267`.

[18] A. Ayoade, O. Odetunde, and B. Falodun, "Modeling and analysis of the impact of vocational education on the unemployment rate in Nigeria," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 15, no. 1, pp. 550–564, 2020. [Online]. Available: `https://www.pvamu.edu/aam/wp-content/uploads/sites/182/2020/06/32-R1316_AAM_Ayodade_AA_080919_Posted_060420.pdf`.

[19] H. A. Ashi, R. M. Al-Maalwi, and S. Al-Sheikh, "Study of the unemployment problem by mathematical modeling: Predictions and controls," *The Journal of Mathematical Sociology*, vol. 46, no. 3, pp. 301–313, 2021. doi: `10.1080/0022250x.2021.1931173`.

[20] E. Kaslik, M. Neamţu, and L. F. Vesa, "Global stability analysis of an unemployment model with distributed delay," *Mathematics and Computers in Simulation*, vol. 185, pp. 535–546, Jul. 2021. doi: `10.1016/j.matcom.2021.01.010`.

[21] S. A. Somma, R. T. Balogun, F. Y. Eguda, N. O. Abdurrahman, P. W. Adama, and E. M. Yisa, "Stability analysis of the mathematical modelling of transmission and control of rabies incorporating vaccination class," *Dutse Journal of Pure and Applied Sciences*, vol. 8, no. 1a, pp. 36–44, 2022. doi: `10.4314/dujopas.v8i1a.4`.

[22] S. A. Somma, N. I. Akinwande, P. Gana, O. D. Ogwumu, T. T. Ashezua, and F. Y. Eguda, "Stability and bifurcation analysis of a mathematical modeling of measles incorporating vitamin A supplement," *Sule Lamido University Journal of Science and Technology*, vol. 2, no. 1, pp. 1–18, 2021. [Online]. Available: `http://repository.futminna.edu.ng:4000/items/f0f57469-9705-45b6-9e07-bb087697ad5c`.

[23] N. I. Akinwande, S. A. Somma, R. O. Olayiwola, T. T. Ashezua, R. I. Gweryina, F. A. Oguntolu, O. N. Abdurahman, F. S. Kaduna, T. P. Adajime, F. A. Kuta, S. Abdulrahman, A. I. Enagi, G. A. Bolarin, M. D. Shehu, and A. Usman, "Modelling the impacts of media campaign and double dose vaccination in controlling COVID-19 in Nigeria," *Alexandria Engineering Journal*, vol. 80, pp. 167–190, Oct. 2023. doi: 10.1016/j.aej.2023.08.053.

[24] S. A. Somma and N. I. Akinwande, "Sensitivity analysis for the mathematical modelling of monkey pox virus incorporating quarantine and public enlightenment campaign," *FULafia Journal of Science and Technology*, vol. 6, no. 1, pp. 54–61, 2020. [Online]. Available: https://lafiascijournals.org.ng/index.php/fjst/article/view/187.

[25] N. I. Akinwande, T. T. Ashezua, R. I. Gweryina, S. A. Somma, F. A. Oguntolu, A. Usman, O. N. Abdurrahman, F. S. Kaduna, T. P. Adajime, F. A. Kuta, S. Abdulrahman, R. O. Olayiwola, A. I. Enagi, G. A. Bolarin, and M. D. Shehu, "Mathematical model of COVID-19 transmission dynamics incorporating booster vaccine program and environmental contamination," *Heliyon*, vol. 8, no. 11, p. e11513, 2022. doi: 10.1016/j.heliyon.2022.e11513.