P- ISSN 1991-8941 E-ISSN 2706-6703 *Journal of University of Anbar for Pure Science (JUAPS)* Open Access
2025,(19), (01):299 – 303

Some properties of $\mathcal{Z}$-small prime modules

Aya A. Salman, Alaa A. Elewi *

Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq;

ARTICLE INFO
Received:11 / 08 /2024
Accepted: 04/ 09 /2024
Available online:21/ 06 /2025

DOI: 10.37652/juaps.2024.152157.1293

Keywords:
Small submodule, prime submodule, Z-small submodule, prime modules and small prime modules.

ABSTRACT

Let E be a commutative ring with 1 ,and H be a unital(left) E-module. This paper introduces new properties of Z-small prime modules. Where an E-module H is called a Z-small prime module if and only if $ann\, H = ann\, K$, for every non-zero submodule K of H such that $K \ll_Z H$. A submodule K of an E-module H is defined as Z-small (briefly $K \ll_Z H$) if $K + B = H$ with $B \supseteq Z_2(E)$ and B is a submodule of H, then $B = H$. Among these properties, the following is established: if H is finitely generated, faithful multiplication E-module, then H is a small Z-small prime E-module if and only if E itself is a Z-small prime ring. Furthermore, an E-module H is proven as a Z-small prime if and only if the E-module $\frac{E}{ann\, H}$ is cogenerated by every non-trivial Z-small submodule of H.

1. Introduction

In this paper, all rings are assumed to have an identity, and all modules are unitary left E-modules. An E-module H is called a small-prime module if $\mathrm{ann}_E H = \mathrm{ann}_E K$ for every non-zero Z-small submodule K of H [1]. In [2], Aya A. Salman and Alaa A. Elewi generalized this concept by introducing the concept of a Z-small prime module, where an E-module H is said to be a Z-small prime module if $ann\, H = ann\, K$ for every non-zero Z-small submodule K of H. A submodule K of an E-module H is called a Z-small submodule in H (denoted by $K \ll_Z H$) if $K + L = H$, with $L \supseteq Z_2(H)$, "where Z_2(H) is the submodule of H containing Z(E) such that $\frac{Z_2(E)}{Z(E)}$ is the singular submodule of $\frac{E}{Z(E)}$ and called the second singular submodule of H"[3]. Thus, $L = H$ [3].

Every Z-small prime module is clearly a small prime module, but not conversely in general, as shown in [2]. This paper introduces and investigates new properties of these types of modules.

*Corresponding author at : Department of Mathematic, College of Science, University of Baghdad, Baghdad, Iraq
ORCID:https:// https://orcid.org/ 0000-0002-1729-6210 ,
Tel: +964 7703995905
Email: alaa.elewi@sc.uobaghdad.edu.iq

2. New properties of $\mathcal{Z}$-small prime modules

This section discusses the new properties of Z-small prime modules.

Before presenting the propositions, the following concepts are first introduced.
Let H be an E-module. If H has a maximal submodule containing $Z_2(H)$, $Rad_Z(H) = \cap\{K < H;\ K \text{ is a maximal submodule containing } Z_2(H)\}$ is first defined. If H has no maximal submodule containing $Z_2(H)$, then $Rad_Z(H) = H$. Equivalently, $Rad_Z(H) = \sum \{K \leq H;\ K \leq_Z H\}$ [4].

Proposition 2.1. [2] An E-module H is a Z-small prime if and only if $ann\, H = ann <h>$ for every non-zero element $h \in H$, such that $<h> \ll_Z H$.

Proposition 2.2. Let H be an E-module such that $Rad_Z(H)$ is a proper direct summand of H. If $Rad_Z(H)$ is a Z-small prime E-module and $ann\, H = ann\, Rad_Z(H)$, then H is also a Z-small prime E-module.

Proof. Let $0 \neq x \in H$ such that $\langle x \rangle \ll_Z H$, then $x \in Rad_Z H$ [3]. Thus, $\langle x \rangle \ll_Z H$ by [4]. Based on Proposition 2.1, $ann\, Rad_Z(H) = ann\langle x \rangle$. Therefore, by assumption, $ann\, H = ann\, \langle x \rangle$. The result is obtained through Proposition (2.1)

Proposition 2.3. [2] A non-trivial direct summand of a Z-small prime E-module is also a Z-small prime E-module.

Proposition 2.4. Let $H = H_1 \bigoplus H_2$ be an E-module such that $ann\, H_1 + ann\, H_2 = E$. Therefore, H is a Z-small E-module if and only if H_1 and H_2 are Z-small prime E-modules.

Proof. Let $0 \neq K \ll_Z H$.

$ann\, H_1 + ann\, H_2 = E$; therefore, $K = K_1 \bigoplus K_2$, where $K_i \subseteq H_i$ for $i = 1{,}2$ [7]. Based on [4], $K \ll_Z H$ if and only if $K_i \ll_Z H_i$ for $i = 1{,}2$. Thus, $ann(K) = ann(K_1 \oplus K_2) = ann(K_1) \cap ann(K_2)$ based on [5]. H_1 and H_2 are Z-small prime modules; therefore, $ann\, K = ann\, H_1 \cap ann\, H_2$. Meanwhile, H_1 and H_2 are Z-small prime E-modules; therefore, $ann\, K = ann\, (H_1 \bigoplus H_2)$. That is, H is a Z-small prime module, contrary to that by Proposition 2.3.

The following concepts and propositions are necessary to derive the next propositions.

Recall that an E-module H is defined as distributive if, for all K, L, $W \subseteq H$, $K \cap (W + L) = (K \cap W) + (K \cap L)$ [6]. Equivalently, $K + (W \cap L) = (K + W) \cap (K + L)$ [6].

Recall that $K \subseteq H$ is called fully-invariant if an endomorphism exists from H to H, $f(K) \subseteq K$ by[8].

An E-module H is defined as duo if every submodule of H is fully-invariant [9].

Proposition 2.5. [2] Every non-zero submodule of Z-small prime module is also a Z-small prime submodule.

Proposition 2.6. Let H_1, H_2 be E-modules, $H = H_1 \bigoplus H_2$ such that H is a duo-module. Therefore, H is Z-small prime E-module if and only if H_1 and H_2 are Z-small prime E-modules.

Proof. Let $0 \neq K \ll_Z H$. K is fully invariant; therefore, $K = (K \cap H_1) \bigoplus (K \cap H_2)$ based on [9].

$K \cap H_1$, $K \cap H_2$ are non-zero proper Z-small submodules of H_1 and H_2, respectively. H_1 and H_2 are Z-small prime modules; therefore, based on Proposition 2.5, $K = (K \cap H_1) \bigoplus (K \cap H_2)$ is a Z-small prime E-module. Conversely, it directly follows Proposition 2.3.

Proposition 2.7. Let $H = H_1 \bigoplus H_2$ be a distributive E-module such that $H_1, H_2 \subseteq H$. Thus, H is a Z-small prime if and only if H_1 and H_2 are Z-small prime E-modules for each for every non-zero Z-small submodule K of H, provided $K \cap H_i \neq H_i$ for $i = 1{,}2$.

Proof. (⇒) cleared by previous proposition.

(⇐) Let K be a non-zero Z-small submodule of H. H is distributive E-module; thus. $K = K \cap H_1 \bigoplus K \cap H_2$. Similarly, the result is obtained based on the proof of Proposition 2.6.

Proposition 2.8. Let K be a submodule of an E-module H. If $\frac{H}{K}$ is a Z-small prime, then $[K{:}H] = [K{:}T]$ for every Z-small submodules T of H such that $K \subseteq T$.

Proof. Let $0 \neq T \ll_Z H$ and $K \subseteq T$. Thus, based on [3], $\frac{T}{K} \ll_Z \frac{H}{K}$. However, $\frac{H}{K}$ is a Z-small prime; therefore, $ann\, \frac{H}{K} = ann\, \frac{T}{K}$. Hence, $[K{:}H] = [K{:}T]$.

Corollary 2.9. If K is a Z-small prime submodule of an E-module H, then $[K{:}H] = [K{:}T]$ for all Z-small submodules T of H such that $K \subseteq T$.

3. Other properties of Z-small prime modules

This section investigates the other properties of this type of module.

Remark 3.1. If H is an E-module such that $\frac{E}{ann\, H}$ is an integral domain and H is a torsion-free $\frac{E}{ann\, H}$-module, then H is a prime by [4]. Therefore, under these conditions, H is a Z-small prime E-module, while the reverse does not generally hold. However, this condition cannot be guaranteed based on this example.

The opposing concept that satisfies certain conditions is shown in the following proposition.

Proposition 3.2. Consider H as a Z-small prime E-module, where every non-trivial cyclic submodule is Z-small. Then, $\frac{E}{ann\,H}$ is an integral domain, and H is a torsion-free $\frac{E}{ann\,H}$-module.

Proof. Aiming to prove that $\frac{E}{ann\,H}$ is an integral domain, let $\bar{t}, \bar{s} \in \frac{E}{ann\,H}$. Conversely, $ann < sh >= ann(H) = ann < th >$. Therefore, either $t \in ann\,H$, or $s \in ann\,H$. That is, either $\bar{t} = 0$ or $\bar{s} = 0$; hence, $\frac{E}{ann\,H}$ is an integral domain. Suppose that $eh = 0$ such that $h \neq 0$, Therefore, $\bar{e} = 0$, where $\bar{e} \in \frac{E}{ann\,H}$. Hence, $eh = 0$, indicating that $e \in ann\,\langle h \rangle = ann\,H$ based on Proposition 2.1. $\bar{e} = 0$; therefore, H is a torsion-free $\frac{E}{ann\,H}$-module.

The following corollaries are directly obtained using Proposition 3.2 and [4].

Corollary 3.3. Let H be an E-module, where every non-trivial cyclic submodule is Z-small. Then, H is a Z-small prime E-module if and only if $\frac{E}{ann\,H}$ is an integral domain and H is a torsion-free $\frac{E}{ann\,H}$-module.

Corollary 3.4. Let H be a faithful E-module such that every non-trivial cyclic submodule is Z-small. Thus, H is a Z-small prime if and only if E is an integral domain and H is torsion-free.

Corollary 3.5. Let H be a hollow E-module. Therefore, H is a Z-small prime E-module if and only if $\frac{E}{ann\,H}$ is an integral domain and H is a torsion-free $\frac{E}{ann\,H}$-module.

The next result is another characterization of Z-small prime E-module H in terms of the E-module $\frac{E}{ann\,H}$.

Definition 3.6.[12] An E-module H is finitely cogenerated if, for every set $\{A_i : i \in I\}$ of submodules A_i of H with $\bigcap_{i\in I} A_i = 0$, a finitely subset $\{A_i : i \in J\}$, $J \subseteq I$ is finite with $\bigcap_{i\in J} A_i = 0$.

Theorem 3.7. An E-module H is a Z-small prime if and only if the E-module $\frac{E}{ann\,H}$ is cogenerated by every non-trivial Z-small submodule of H.

Proof. (⇒) Suppose that H is a Z-small prime E-module. Let $0 \neq K \ll_Z H$ and $0 \neq k \in K$. Therefore, $\langle k \rangle \ll_Z H$ based on [3]. However, H is a Z-small prime; hence, $ann\,H = ann\,\langle k \rangle$ is based on Proposition 2.1. Based on the first isomorphism theorem, ${}^{E}/_{annH} = {}^{E}/_{ann(k)} \cong E_n =< n >$ is a submodule of K. Thus, a monomorphism from $\frac{E}{ann\,H}$ into K exists, where $\frac{E}{ann\,H}$ is cogenerated by K.

(⇐) Aiming to prove H is a Z-small prime, let $0 \neq K \ll_Z H$. Then, $\frac{E}{ann\,H}$ is cogenerated by K. Thus, there exists a monomorphism,say $f: E/\mathrm{ann(H)} \to K$, exists for some index set I. Let $e \in ann\,K$. Thus, $eK = 0$. Now, $f(\bar{1}) \in K$; hence, $(f(\bar{e}))(i) \in K_i$, $\forall\, i \in I$. However, $f(\bar{e}) = ef(\bar{1})$. Therefore $(f(\bar{e}))(i) = (ef(\bar{1}))(i) \in K_i$, $\forall i \in I$. Hence, $(f(\bar{1}))(i) = 0, \forall i \in I$, yielding $f(\bar{e}) = 0$. However, f is a monomorphism; therefore $\bar{e} = 0$. That is, $e + ann\,H = ann\,H$, and $e \in ann\,H$. Therefore, $ann\,H = ann\,K$; that is, H is a Z-small prime module.

Definition 3.8. [10] An E-module H is regarded as a multiplication module if, for every submodule K of H, an ideal I of E exists such that K = IH.

Theorem 3.9. Let H be finitely-generated, faithful multiplication E-module. Then, H is a Z-small prime E-module if and only if E is a Z-small prime ring.

Proof. Let I be a non-trivial Z-small ideal of E. Then, based on [3], IH is a Z-small of H. If $IH = 0$, then $I \subseteq ann\, H = 0$, which is a contradiction. Hence, $0 \neq IH \ll_Z H$. However, H is a Z-small prime; thus, $ann\, H = ann\, IH$. However, $ann\, H = 0$; thus, $ann\, IH = 0$.But $ann\, I \subseteq ann\, IH$. $ann\, I = 0$; hence, E is a Z-small prime ring, proving the "on if" part. Aiming to prove the "If" part: Let $0 \neq K \ll_Z H$, then $[K:H] \ll_Z E$ [3]. However, H is multiplication, which implies that $K = [K:H]H$ based on [12]; hence, $[K:H] \neq 0$. E is a Z-small prime ring; thus, $ann\, [K:H] = 0$. Conversely, H is a faithful module. Therefore, $ann\, [K:H]H = ann\, [K:H]$; thus, $ann\, K = 0$.

Corollary 3.10. Let H be a cyclic faithful E-module. Therefore, H is a Z-small prime E-module if and only if E is a Z-small prime ring.

Proposition 3.11. Let H be a multiplication E-module. If H is a Z-small prime S-module, then H is a Z-small prime E-module (where $S = End_E(H)$).

Proof. Let $0 \neq K \ll_Z H$, claiming the existence of $e \in ann_E H$; thus, $eH \neq 0$. Define $f: H \to H$ by $f(h) = eh, \forall\ h \in H$. f is a well-defined E-homomorphism and $f \neq 0$ because $f(K) = eK = 0$ implies $f \in ann_S K = ann_S H$ (H is a Z-small prime S-module). Thus, $f(H) = 0$; that is, $f = 0$, which is a contradiction. Therefore, $ann\, K = ann\, H$ and H is a Z-small prime E-module.

Recall that an E-module is known as a scalar module if $\forall\ f \in End_E(H)$, $f \neq 0$, $\exists\ e \in E$, $e \neq 0$ such that $f(h) = eh, \ \forall h \in H$ [11].

Proposition 3.12. Let H be a finitely-generated multiplication E-module. If H is a Z-small prime E-module, then H is a Z-small prime S-module.

Proof. H is a finitely-generated multiplication E-module that yields H, which is a scalar E-module by [11]. Let K be a non-trivial Z-small S-submodule of H. Thus, K is a non-trivial Z-small E-submodule of H. Claim that $\exists\ f \in S$, $f \in ann_S K$ and $f \notin ann_S H$ such that $f(x) = ex, \forall x \in H$. Therefore, $eK = 0$ and $e \in ann\, K = ann\, H$. Hence, $Eh = 0$. Therefore, $f(h) = 0$, which is a contradiction; thus, $ann_S H = ann_S K$.

Corollary 3.13. Let H be a finitely-generated multiplication E-module. Therefore, H is a Z-small prime E-module if and only if H is a Z-small prime S-module.

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بعض خصائص المقاسات الاولية الصغيرة من النمط −Z

اية عامر سلمان، الاء عباس عليوي

قسم علوم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق
email: alaa.elewi@sc.uobaghdad.edu.iq

الخلاصة:

لتكن E حلقة ابدالية ذات عنصر محايد و H مقاس احادي (ايسر) على الحلقة E. في هذا البحث سنعطي خصائص جديدة للمقاسات الأولية الصغيرة من النمط −Z أعطيت حيث ان المقاس H على الحلقة E يقال له انه مقاس أولي صغير من النمط −Z اذا وفقط اذا $ann\, H = ann\, K$ لكل مقاس جزئي K غير صفري من H حيث أن $K \ll_Z H$ حيث يقال للمقاس الجزئي K من المقاس H على الحلقة E انه صغير من النمط Z (بصورة مختصرة $K \ll_Z H$) اذا $H = K + B$ بحيث $B \supseteq Z_2(E)$ وحيث B هو مقاس جزئي من المقاس H من بين هذه الخصائص: إذا كان H مقاس مخلص منتهي التولد جدائي على الحلقة E فان H هو مقاس أولي صغير من النمط −Z على الحلقة R اذا وفقط اذا كانت الحلقة E اولية صغيرة من النمط Z. أيضا برهنا أن المقاس H هو مقاس أولي صغير من النمط Z اذا وفقط اذا المقاس $\frac{E}{ann\, E}$ على الحلقة E هو متجانس التولد لكل مقاس جزئي صغير من النمط Z غير تافه من المقاس H.

الكلمات المفتاحية:المقاسات الجزئية الصغيرة، المقاس الجزئي الاولي، المقاس الجزئي الصغير من النمط −Z، المقاس الاولي والمقاس الجزئي الاولي الصغ.