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Some results of T-quasi fuzzy prime submodules

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ABSTRACT

In this work, the topic of T-quasi fuzzy prime submodule was studied, which is one type of partial fuzzy model. This work was organized into three sections, each of which contains a study that starts with an overview of the subject, followed by an explanation of the fundamental definitions, and finally, an explanation of the topic of □-quasi fuzzy prime submodule, bolstered by examples and observations. The final section finds the relationship between this concept T-quasi fuzzy prime submodule with some other concepts such as the concept multiplication fuzzy module and the concept of maximal fuzzy ideal. This section has also been enhanced with many non-examples, observations, and properties that prove that relationship.

Introduction

In [1], we introduce the concept of the fuzzy set, and in [2], Zuhedi introduced the concept of fuzzy modules. In this work, we study some results of the ⍰-⍰-quasi-prime submodule (T-quasi fuzzy prime submodules) E is a ⍰-module additionally a proper ⍰-submodule of □ be ⍰-⍰-quasi-prime submodule, wherever a_{j} , $b_{\mathfrak{s}}$is a ⍰-singleton of $\mathfrak{R}$, e_{r} ⊆ E to the extent that $a_{t}b_{\mathfrak{s}}e_{r}$ ⊆ □ then either$a_{j}^{2}e_{r}$ ⊆□ or $b_{\mathfrak{s}}^{2}e_{r}$⊆□. In particular, a fuzzy ideal D of $\mathfrak{R}$ is a □-⍰-quasi -prime ideal the short (□-quasi-fuzzy-prime ideal) of R should D a ⍰-⍰-quasi- prime the short (□-quasi-fuzzy-prime of $\mathfrak{R}$). Some sources, including [3] [4] [5] [6], were used to obtain a portion of the important properties, helping [7] [8] [9] [10] [11] in finding connections and proofs for those properties.

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Among the most important results that were reached in this research include E being a □ -⍰-quasi-prime module if and only if E is a □-⍰-quasi-prime submodule. This feature helped make the connection between the ⍰-submodules the short (fuzzy submodule) in the ordinary and fuzzy states. Another important property is that E is a multiplication ⍰-module the short (fuzzy module). Additionally, □ is a proper ⍰-submodule of E. Then, the following sentences are comparable.

1. □ be a □-⍰-quasi -prime submodule.
2. [□:_R E] be a □-⍰- prime ideal.
3. □ is a □-⍰- prime submodule.

In addition to some of the important results and examples found in this research.

□-□-quasi - prime submodule

Throughout this part, we define the □ -⍰-quasi prime submodule, which is an extension of the quasi - ⍰-prime submodule. We also provide some observations and examples of this type of submodule. We also provide descriptions and fundamental features.

Definition (2.1): [12]

We suppose E is a ⍰-module and a proper ⍰-submodule of □ being a □-⍰-quasi-prime submodule. If wherever a_t, b_s is a ⍰-singleton of $\mathfrak{R}$, $e_r \subseteq$ E to the extent that $a_j b_s e_r \subseteq$□ that either $a_j^2 e_r \subseteq$□ or $b_s^2 e_r \subseteq$□. Specifically, an ⍰- ideal D of $\mathfrak{R}$ is a □-⍰-quasi-prime ideal of $\mathfrak{R}$ should D be a □-⍰-quasi -prime of $\mathfrak{R}$.

Proposition (2.2):

E is a □-⍰-quasi- prime module if and only if E is □-quasi-prime submodule.

proof:

Define: E: U⟶ [0,1] as follows

$E(e) = \begin{cases} 1 & if \quad e \in \mathcal{U} \\ 0 & otherwise \end{cases}$

Define:$\mathcal{V}$:□ ⟶ [0,1] as follows

$\mathcal{V}\ (e)=\begin{cases} 1 & if \quad e \in □ \\ 0 & if \quad otherwise \end{cases}$

We define $(0_1)\ (\mathcal{Y}) = \begin{cases} 1 & if \quad y = 0 \\ 0 & otherwise \end{cases}$

Clearly, $a_j b_s e_r \subseteq □ \Rightarrow a_j^2 e_r \subseteq □$

$(a\ b\ e)_\eta \subseteq □ \qquad \eta = \min\{j, s, r\}$. [1] [2]

a b $e \in □_\eta$. $(a^2 e)_\eta \Rightarrow a^2 e \in □_\eta \leq □_j \Rightarrow a^2 e \in □_j$.

Or $b_s^2 e_r \subseteq □ \Rightarrow (b^2 e)_\eta \subseteq □ \Rightarrow b^2 e \in □_\eta \leq □_j \quad \eta = \{j, s, r\}$

$b^2 e \in □_j$. $□_j$ is a ⍰ -quasi prime submodule.

$(\Leftarrow)\ a^2 b^2 e \in □$. that either $a^2 e \in □$

$(a^2 b^2 e)_j \subseteq □ \Rightarrow (a^2 b^2 e \in □)_j$. [8] [4] [10]

$(a^2 b^2 e)_j \subseteq □. \Leftrightarrow □(a^2 b^2 e) \geq j$. Then, $(a^2 b^2 e)_j \subseteq □$.let $j = \{j, s, r\}$

$a_j b_s e_r \subseteq □$

Remarks and Examples (2.3):

1. Each quasi-⍰- prime submodule is a ⍰-⍰-quasi -prime submodule.

Proof:

We assume □ be a quasi-⍰- prime submodule of a fuzzy module E. Additionally, we let $a_j b_s e_r \subseteq □$, where a_j, b_s are ⍰- singletons of $\mathfrak{R}$, as well $e_r \subseteq E$. Given that □ is quasi-⍰-prime, then either $a_j e_r \subseteq □$ or $b_s e_r \subseteq □$. Thus, the $a_j^2 e_r \subseteq □$ or $b_s^2 e_r \subseteq □$. Therefore, □ is a ⍰-⍰-quasi- prime submodule.

2. The inverse of (1) is generally not true, for instance:

E :$\mathcal{U} \longrightarrow$ [0,1] as follows

E$(e)=\begin{cases} 1 & if \ \ e \in \mathcal{U} \\ 0 & otherwise \end{cases}$

$E_j = \mathcal{U} = z_8$

□: → [0,1] as follows

$□(e)=\begin{cases} t & if \quad e \in □ = (4) \\ 0 & otherwise \end{cases}$

$□_j$=□

Now: $□ = □_j = (4)$ such that 2.2.1=4∈ □
1=2∉ □ , □ is not a prime submodule. Therefore, □ *is a* quasi -⍰-prime submodule by (2.2).

3. Every fuzzy prime submodule is a ⍰ -⍰-quasi-prime submodule.

Proof:

Since every ⍰-prime submodule is quasi- ⍰-prime submodule by [4] and by (1) complete the proof.

4. The converse of (3) is false, in general, in the example:

let $\mathcal{U}$=z⊕z, □=2□⊕(0)

Define $E:\mathcal{U} \rightarrow$ [0,1] as follows

E (e)
$= \begin{cases} 1 & if \quad e \in \mathcal{U} \\ 0 & otherwise \end{cases}$

Define: □:$\mathcal{U} \rightarrow$ [0,1] as follows

$□(e) = \begin{cases} t & if \quad e \in □ \\ 0 & otherwise \end{cases}$

Clearly, $□_j = □$ additionally $E_j = \mathcal{U}$

But □ is not a prime submodule because 2(3,0) = (6,0)∈ □ as well (3,0)∉ □ *and* 2∉ [2z ⊕(0): z⊕z = 0 □ is not ⍰-prime [4].

5. All ⍰-⍰- prime submodules are ⍰-⍰-quasi- prime submodules.

Proof:

We let E be a ⍰-module and let □ be ⍰ -⍰-prime submodule and suppose $a_j b_s e_r \subseteq □$, where a_j, b_s are fuzzy singletons *if* $\mathfrak{R}$, and $e_r \subseteq E$, □ *is a* T-⍰- prime submodule. Then, either $a_j e_r \subseteq □$ *or* $b_s^2 \subseteq [□:E]$. Thus, $a_t^2 e_r \subseteq$ A or $b_s^2 e_r \subseteq □$. Therefore, □ is a ⍰-⍰-quasi-prime submodule.

6. The converse of (5) is not true in general

   let $\mathcal{U} = z_{16}$, $□ = (8)$

   Define: : $\mathcal{U} \to [0,1]$ as follows

$$\mathrm{E}(e) = \begin{cases} 1 & if \quad e \in \mathcal{U} \\ 0 & otherwise \end{cases}$$

Define: □: □ → [0,1] as follows

$$□(e) = \begin{cases} \mathrm{t} & if \quad e \in □ \\ 0 & otherwise \end{cases}$$

$$□_{j} = □ \quad , \quad \mathrm{E}_{j} = \mathcal{U}$$

$□_{\mathrm{t}}$=(8) $is\ a$ ⍰-quasi prime submodule since 16=2.4.2∈ $□_{j}$ either 2.4=8∈ (8) or 2.2∉ (8) $is\ a$ ⍰-quasi prime submodule ⇒ (8) $is\ a$ ⍰-⍰-quasi-prime submodule by Proposition (2.2). However, (8) $is\ a$ be⍰ - prime submodule
4∈ (8).(4) ∉ (8) is not a ⍰-prime submodule. is not ⍰-⍰- prime submodule.

7. Every maximal ⍰-submodule of a ⍰-module E $is\ a$ ⍰-⍰-quasi-prime submodule.

   Proof:

   Given that every maximal ⍰-submodule is ⍰-⍰-prime submodule ((ch. 2(3.)) and (5))), the proof is complete.

8. In general. the inverse of (7) is not true

   Define: E: $\mathcal{U} \to [0,1]$ as follows

$$\mathrm{E}(e) = \begin{cases} 1 & if \quad e \in \mathcal{U} \\ 0 & otherwise \end{cases}$$

   Define: □ : $\mathcal{U} \to [0,1]$ .as follows

$$□\ (e) = \begin{cases} \mathrm{t} & if \quad e \in \mathcal{U} \\ 0 & otherwise \end{cases}$$

$□_{j}$=□ = (0) is not a maximal submodule ⟹ □ $is\ not\ a$ maximal ⍰-submodule by [4]. However, $□_{j}$=(0) isa ⍰-quasi prime submodule because $a.b$ 0 ∈ $□_{j}$ ⇒ 0∈ $□_{j}$ $a.0 \in □_{j}$ $or\ b.0 \in □_{j}$ (0) $is\ a$ ⍰-⍰-quasi submodule by proposition (2.2).

Note: ⍰-⍰-quasi-prime submodule ↛ ⍰-⍰- prime submodule.

Proposition (2.4):

A proper ⍰-submodule □ of E $is\ a$ ⍰-⍰-quasi -prime submodule if and only if $[□{:_{\mathfrak{R}}}(e_r)]$ is a ⍰-⍰-prime ideal. For all $e_r \subseteq E$, $e_r \nsubseteq □$.

Proof:

(⇒)We suppose that □ is⍰ -⍰-quasi-prime submodule and let $a_j b_s \subseteq [□{:_{\mathfrak{R}}}(e_r)]$, where $e_r \subseteq$ E, $e_r \nsubseteq □$. Additionally a_j, b_s are ⍰-singletons of $\mathfrak{R}$,. Thus, $a_j b_s(e_r) \subseteq □ \Rightarrow a_j b_s n_{\mathrm{t}} e_r \subseteq □$. However, E $is\ a$ ⍰-⍰-quasi-prime submodule that either $a_{\mathrm{t}}^2 e_r \subseteq □$ or $b_s^2 n_{\mathrm{t}}^2 e_r \subseteq □$ (since it is ⍰-⍰-quasi-prime). Thus, either $a_j^2 \subseteq [□{:_{\mathfrak{R}}}(e_r)]$ r $b_s^2 \subseteq [□{:_{\mathfrak{R}}}(e_r)]$. thus $[□{:_{\mathfrak{R}}}(e_r)]$ is ⍰-⍰-prime ideal.

(⇐) If $[□{:_{\mathfrak{R}}}(e_r)]$ $is\ a$ ⍰-⍰-prime ideal where $e_r \subseteq$ E and $e_r \nsubseteq □$. To prove that □ is ⍰-⍰-quasi -prime:

We let $a_j b_s e_r \subseteq □$, where a_j, b_s are be a ⍰-singleton of $\mathfrak{R}$, and $e_r \subseteq$ E. But $a_j b_s \subseteq [□{:_{\mathfrak{R}}}(e_r)]$ is ⍰ -⍰-prime ideal, then either $a_j^2 \subseteq [□{:_{\mathfrak{R}}}(e_r)]$ or $b_s^2 \subseteq [□{:_{\mathfrak{R}}}(e_r)]$. Thus, either $a_j^2 \chi_s \subseteq □$ or $b_s^2 e_r \subseteq □$, so □ $is\ a$ ⍰-⍰-quasi -prime

Proposition (2.5):

We let $\mathcal{H}$ and W are two ⍰-submodules of a⍰-module E. If for every $e_r \subseteq W$, $[\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$ which is ⍰-fuzzy-prime ideal, then $[\mathcal{H}{:_{\mathfrak{R}}} W]$ $is\ a$ ⍰-⍰- prime ideal.

Proof:

We let p_j, g_i in $\mathfrak{R}$ such that $p_j g_i \subseteq [\mathcal{H}{:_{\mathfrak{R}}} W]$, so $p_j g_i e_r \subseteq \mathcal{H}$, for every $e_r \subseteq W$, $p_j g_i \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_r)$ ... (*).

But $[\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$ is ⍰-⍰- prime ideal. Thus, either $p_j^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$ or $g_i^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$. Therefore, for any $e_r \subseteq w$ either $p_j^2 e_r \subseteq \mathcal{H}$ or $g_i^2 e_r \subseteq \mathcal{H}$, we suppose that $p_j^2 \nsubseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$ and $g_i^2 \nsubseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_r)]$, so $e_1 e_2$ exists in V such that $p_j^2 e_1^2 \nsubseteq W$ and $g_i^2 e_2^2 \nsubseteq W$. Thus, $p_j^2 \nsubseteq [\mathcal{H}{:_R}(e)]$ and $g_i^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_2)]$ by (*) $p_i g_j \subseteq [\mathcal{H}{:}(e_1)]$ which is ⍰ -⍰-prime ideal, so $g_i^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_1)]$. Thus, $g_i^2 e_1 \subseteq \mathcal{H}$, like that $p_j g_i \subseteq [\mathcal{H}{:}(e_2)]$ implies that $g_i^2 e_2 \subseteq \mathcal{H}$.

By contrast, by (*), $p_j g_i \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_1 + e_2)]$, so either $p_j^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_1 + e_2)]$ or $g_i^2 \subseteq [\mathcal{H}{:_{\mathfrak{R}}}(e_1 + e_2)]$. Thus, either $p_j^2(e_1 + e_2) \subseteq \mathcal{H}$ or $g_i^2(e_1 + e_2) \subseteq \mathcal{H}$. This result implies that either $p_i^2 e_1 + p_i^2 e_2 = h_1 \subseteq \mathcal{H}$ or $g_i^2(e_1) + g_i^2(e_2) = h_2 \subseteq \mathcal{H}$. Therefore, either $p_j^2 e_1 = h_1 - p_j^2 e_2 \subseteq \mathcal{H}$ or $g_i^2 e_2 = h_2 - g_i^2 e_1 \subseteq \mathcal{H}$, which is contradiction. Thus, either $p_j^2 \subseteq [\mathcal{H}{:_{\mathcal{R}}} W]$ or $g_i^2 \subseteq [\mathcal{H}{:_{\mathcal{R}}} W]$. Hence $[\mathcal{H}{:}\ \mathrm{W}]$, is ⍰-⍰- prime ideal.

Proposition (2.6):

We suppose E is a ⍰-module of $\mathfrak{R}$ and $\mathcal{D}$ is a proper ⍰-submodule of e. Subsequently, $\mathcal{D}$ is a ⍰-⍰-quasi-prime submodule of E *if and only if* $[\mathcal{D}:_{\mathfrak{R}} P]$ *is a* ⍰-⍰- prime ideal of $\mathfrak{R}$ every ⍰-submodule L *of* E, where$[\mathcal{D}:_{\mathfrak{R}} L]= \{r_j : r_j$be a ⍰- singleton of $\mathfrak{R}$, $r_j L \subseteq \mathcal{D}\}$.

Proof:

We let $\mathcal{D}$ be a ⍰-⍰-quasi-prime submodule of E. Thus, by proposition (2.4), $[\mathcal{D}:(e_r)]$ *is* ⍰-⍰- prime ideal *of* $\mathfrak{R}$, for every $e_r \subseteq$ E, $e_r \not\subseteq J$. □ gain by proposition (2.4), so $[\mathcal{D}:_{\mathfrak{R}}(e_r)]$ *is* ⍰-⍰- prime ideal for each $e_r \subseteq L$. Thus, by Lemma (2.5), $[\mathcal{D}:_{\mathfrak{R}} L]$ *is a* ⍰-⍰-prime ideal *of* $\mathfrak{R}$.

Now, for the converse, let $a_j b_s e_r \subseteq \mathcal{D}$. a_j , b_s be a ⍰-singleton *of* $\mathfrak{R}$, $e_r \subseteq$ E i.e. $a_j b_s \subseteq [\mathcal{D}:_{\mathfrak{R}} (e_r)]$ and $e_r \subseteq L \subseteq$ E by supposition $[\mathcal{D}:_{\mathfrak{R}} (e_r)]$ *is* ⍰-⍰-prime ideal. Then, either $a_j^2 \subseteq [\mathcal{D}:_{\mathfrak{R}} (e_r)]$ or $b_s^2 \subseteq [\mathcal{D}:_{\mathfrak{R}} (e_r)]$ are mean $a_j^2 e_r \subseteq \mathcal{D}$ or $b_s^2 e_r \subseteq \mathcal{D}$. Thus, $\mathcal{D}$ *is a* ⍰-⍰-quasi-prime submodule.

Corollary (2.7):

We let E bea ⍰-module of $\mathfrak{R}$. Additionally, □ a proper ⍰-submodule of E . If □ *is a* ⍰-⍰-quasi -prime submodule of $\mathcal{U}$, then $[\square:_R$ E$]$ *is a* ⍰-⍰-prime ideal of $\mathfrak{R}$.

In general, the corollary's inverse is not true.

Example (2.8):

We consider the Z-module Q+Z, and a submodule □=z⊕z. Then, $[\square:_{z}$ E$]= [z\oplus z:_{\mathfrak{R}} Q\oplus z]= (0_j)$ which be a ⍰ ⍰--prime ideal of z. However, □ is not ⍰-⍰-quasi-prime submodule since $[\square:_{z} (\frac{1}{6}, 0)]$ =6z is not ⍰-⍰-prime ideal of z.

Definition (2.9): [4]

An ⍰-ideal I of a ring $\mathfrak{R}$ is known as a principal ⍰-ideal If $e_j \subseteq I$ exists to the extent that I= (e_j) each $m_s \subseteq I$, there is also a ⍰-singleton of $\mathcal{R}$ such that $m_s = a_i e_j$, where s,i,j $\subseteq [0,1]$, that is I= (e_j)=$\{m_s \subseteq I/m_s$=$a_i e_j$ for some ⍰-singleton a_i of $\mathcal{R}\}$.

Remark (2.10):

We suppose E is a ⍰-module over PID, $\mathfrak{R}$. □ *is* a ⍰-⍰-quasi prime submodule of E if $[\square:_{\mathfrak{R}}$ E$]$ is non-trivial ⍰-⍰- prime ideal of $\mathfrak{R}$.

The ensuing observation indicates that the intersection of ⍰-⍰-quasi- prime submodule may not be ⍰-⍰-quasi -prime submodule.

Remark (2.11):

Any two intersecting points ⍰-⍰-quasi-prime submodule of a ⍰- module need not be a ⍰-⍰-quasi-prime submodule. For example, $\square_1$= (2) = {0,2,4}, $\square_2$ = {0,3}, $\square_1 \cap \square_2 = (0_1)$*is* not a ⍰-⍰-quasi-prime submodule of z_6 since 2.3.1=6 $\subseteq(0_j)$, 2^2.1=4$\not\subseteq(0_j)$ *a* 3^2.1$\not\subseteq(0_j)$.

Relationships between ⍰-⍰-quasi-prime submodules with another quasi-⍰- submodule.

Proposition (3.1):

□ is a proper ⍰-submodule of E. we assume E is a multiplication □-module. Consequently, the subsequent claims are interchangeable.

1. □ is ⍰-⍰-quasi -prime submodule.
2. $[\square:_{\mathfrak{R}}$ E$]$ is ⍰- ⍰-prime ideal.
3. □ *is* ⍰-⍰- prime submodule.

Proof:

(1)⇒(2) We let □ *is* ⍰-submodule E. Then, we let $a_j b_s \subseteq[\square:_{\mathfrak{R}}$ E$]$ ⇒ $a_j b_s e_r \subseteq$ □ ∀ $e_r \subseteq$ E because □ *is a* ⍰-⍰- quasi- prime submodule of E , then either $a_j^2 \subseteq [\square:(e_r)]$ or $b_s^2 \subseteq [\square:(e_r)]$. Then, by Lemma (2.6), $[\square:_{\mathfrak{R}}$ E$]$ *is a* ⍰-⍰- prime ideal of $\mathfrak{R}$.

(2)⇒(3) $[\square:_{\mathfrak{R}}$ E$]$ *is* ⍰- ⍰-prime ideal, then □ *is a* ⍰-⍰-prime submodule.

(3)⇒(1) □ *is* ⍰ -⍰-prime submodule,. then □ is a ⍰-⍰-quasi-prime submodule by (remark and example (2.3) (5)).

Proposition (3.2):

J, H are two ⍰- submodules of E. If J *is a* ⍰-⍰-quasi-prime submodule *of* E and H is not contained in J. Then, H∩J is a ⍰-⍰-quasi- prime submodule of H.

proof:

Given that H$\not\subset J$, then $J \cap H$ is a proper ⍰-submodule of H

Now, we let $a_{ȷ}, b_s$ be □-singletons $of\ \Re$, and $b_s \subseteq H$ such that $\chi_t y_i b_s \subseteq H \cap J$, so $\chi_{ȷ} y_i b_s \subseteq H\ and\ \chi_{ȷ} y_i e_r \subseteq J$. However, J is a T-□-quasi- prime submodule of E. Thus, either $\chi_{ȷ}^2 b_s \subseteq J$ or $y_i^2 b_s \subseteq J$but $b_s \subseteq H\ as$ well $\chi_{ȷ}^2 b_s \subseteq J$ so that $\chi_{ȷ}^2 b_s \subseteq H \cap J$ or $y_i^2 b_s \subseteq H \cap J$.

Proposition (3.3):

1. We let E and E* be two fuzzy modules and let $\eta{:}E \rightarrow E^*$ be a epimorphism. If J is a □ -quasi-fuzzy prime submodule of E*, then $\eta^{-1}(J)$ is also a □-quasi fuzzy prime submodule of E.
2. If η:E $\rightarrow$ E*is an epimorphism such that $ker\ \eta \subseteq H$ where H is a quasi-fuzzy prime submodule of V. subsequently $\eta(H)$ be a □-quasi- fuzzy prime submodule of η^*.

Proof (1):

To prove $\eta^{-1}(J)is$ a □-□-quasi- prime submodule of E, we want to prove $[\eta^{-1}(J){:}_{\mathcal{R}} J]$ is a □-□- prime ideal. $\forall J \leq E^*$ such that$\eta^{-1}(J) \subsetneq H$

We let $\chi_{ȷ} y_i \subseteq [\eta^{-1}(J){:}_{\Re} H]$ and so $\chi_{ȷ} y_i\ H \subseteq \eta^{-1}(J)$. Thus, $\eta(\chi_t y_i H) \subseteq \eta\left(\eta^{-1}(J)\right)$, so $\chi_{ȷ} y_i(\eta(H) \subseteq J$. Therefore, $\chi_{ȷ} y_i \subseteq [J{:}\eta(H)]$, but J $is\ a$ □-□-quasi- prime submodule f E*. Either $\chi_{ȷ}^2 \subseteq [J{:}\eta(H)]$or $y_i^2 \subseteq [J{:}\eta(H)]$. Thus, either $\chi_{ȷ}^2\eta(H) \subseteq J$or $y_i^2\eta(H) \subseteq J$, i.e. either $\chi_{ȷ}^2 H \subseteq \eta^{-1}(J)$ or $y_i^2 H \subseteq \eta^{-1}(J)$. Thus, either $\chi_{ȷ}^2 \subseteq [\eta^{-1}(J){:}H]$ or $y_i^2 \subseteq [\eta^{-1}(J){:}H]$, i.e. $[\eta^{-1}(J){:}H]\ is$ □-□- prime ideal. Thus, $\eta^{-1}(J)\ is$ □-□-a quasi-prime submodule of E.

Proof (2):

To prove which $\eta(H)\ is\ a$ □-□-quasi-prime submodule of E*, we prove that $[\eta(H){:}_{\Re} J^*]\ \ is\ a$ □-□-prime ideal $of\ \Re$, for all $J^* \leq E^*$ and $J^* \nsubseteq \eta(H)$. Given that η is an epimorphism, thereafter $\eta^{-1}(J^*) = J^*$
let $\eta\ (J) = J^*$. Consequently, $\eta(J) \supsetneq \eta(H)$.

To prove which $[\eta(H){:}_{\Re}\eta(J)]\ is$ □- □-prime ideal $of\ \Re$. We let $\chi_{ȷ}, y_i$ be □-singletons $of\ \Re$, such that $\chi_{ȷ} y_i \subseteq [\eta(H){:}_{\Re}\eta(J)]$, so $\chi_{ȷ} y_i \eta(J) \subseteq \eta(H)$. Thus, for each $m_t \subseteq J$, $\chi_{ȷ} y_i \eta(m_t) \subseteq \eta(H)$. Therefore, $\eta(\chi_t y_i m_t) = \eta(h)$, for some h$\subseteq H$. Then, $\chi_{ȷ} y_i m_t$-h$\subseteq ker\ \eta \subseteq H$ and hence, $\chi_{ȷ} y_i m_t \subseteq H$, for each $m_t \subseteq J$. Thus, $\chi_{ȷ} y_i \subseteq [H{:}_{\Re} J]$ but [H:$_{\Re} J$] is □-□- prime ideal, so either $\chi_{ȷ}^2 \subseteq [H{:}J]$ or $y_i^2 \subseteq [H{:}J]$. Thus, either $\chi_{ȷ}^2 J \subseteq H$ or $y_i^2 J \subseteq H$ so either $\chi_{ȷ}^2\eta(J) \subseteq \eta(H)$ or $y_i^2\eta(J) \subseteq \eta(H)$. Therefore, either $\chi_{ȷ}^2 \subseteq [\eta(H){:}\eta(J)]$ or $y_i^2 \subseteq [\eta(H){:}\eta(J)]\ \ is\ a$ □-fuzzy prime ideal $of\ \Re$. Hence, $\eta(H)\ is\ a$ □-□-quasi-prime submodule $of\,E^*$.

Proposition (3.4):

We let □ is a □-submodule of a □-module E. Then, □ $is\ a$ □- □-quasi- prime submodule $of\ E$ if and only if $[□{:}_E L]$ is a □-□-quasi- prime submodule of E for every L $of\ \Re$.

Proof:

We assume □ $is\ a$ □-□-quasi- prime submodule of E. We suppose $a_1 a_2$ □-singletons of $\Re$, and $e_r \subseteq$ E such that $a_1 a_2 e_r \subseteq$ [□:$_E$L]. Thus, $n_s a_1 a_2 e_r \subseteq$ □ for every $n_s \subseteq L$. However, □ is a □-□-quasi-prime submodule of E, that either $n_s a_1^2 e_r \subseteq$ □ or $n_s a_2^2 e_r \subseteq$ □ $\forall n_s \subseteq L$. Then, either $a_1^2 e_r \subseteq$ [□:$_E$L] is □-□-quasi-prime submodule of E.

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Conflict of Interest

Author/s should declare any relavent conflict of interest.

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بعض نتائج الوحدات الفرعية الأولية الضبابية شبه □

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الخلاصة:

تمت في هذا البحث دراسة موضوع الوحدة الفرعية شبه الضبابية T والتي تعتبر أحد أنواع النماذج الضبابية الجزئية. تم تقسيم هذا العمل إلى ثلاثة أقسام، لكل منها دراسة تبدأ بإعطاء مقدمة للموضوع ثم إعطاء التعريفات الأساسية ثم شرح تفاصيل موضوع T-quasi fuzzy prime submodule مدعمة بأمثلة قيمة ومهمة والملاحظات من المهم هو ترسيخ هذا المفهوم. ويتناول القسم الأخير إيجاد العلاقة بين هذا المفهوم T-شبه الوحدة الأولية الضبابية مع بعض المفاهيم الأخرى مثل مفهوم الوحدة الضبابية الضربية وأيضا مفهوم المثالية الضبابية القصوى. كما تم تعزيز هذا القسم بالعديد من الأمثلة والملاحظات والخصائص التي تثبت تلك العلاقة.

الكلمات المفتاحية : – T-نموذج أولي ضبابي، T-وحدة فرعية أولية ضبابية، نموذج ضبابي أقصى، وحدة فرعية ضبابية أولية، وحدة فرعية ضبابية الضرب.