P- ISSN 1991-8941 E-ISSN 2706-6703
2025,(19), (01):291– 298
Journal of University of Anbar for Pure Science (JUAPS) Open Access

Expansion of the Al-Zughair transformation and its developer for solving linear ordinary differential equations with constant coefficients

Zainab Nasir Abdul Razzaq , Ali Hassan Mohammed*

Department of Mathematics, Faculty of Education for Women, University of Kufa , Kufa, Iraq;
zainabn.alsanam@student.uokufa.edu.iq , prof.ali57hassan@gmail.com

ARTICLE INFO

Received: 05 / 07 /2024
Accepted: 05/ 09 /2024
Available online: 20/ 06 /2025

DOI: 10.37652/juaps.2024.151357.1283

Keywords:
The Expansion Al-Zughair transform, Zainab Al-Zughair transforms, solve linear ordinary differential equations with constant coefficients.

ABSTRACT

Differential equations with constant coefficients are a fundamental component of applied mathematics, effectively describing Complex and physical systems. Recently, new transformations have emerged to reduce the computational complexity and simplify these equations, making them easier to solve. These developments are beneficial to complicated and physical systems. As a result, more engineers and scientists have become equipped with the knowledge to address challenging problems involving intricate dynamic systems.

In this study, we enhance the efficiency of existing methods by applying the Zainab Al-Zughair transformation and extending Al-Zughair transformations to solve linear ordinary differential equations (LODEs) with constant coefficients, given a set of initial conditions. *n*-th order linear systems of LODEs with constant coefficients can be solved using the Zainab Al-Zughair and Expansion Al-Zughair transformations. In addition, these methods enable easy modification of initial conditions, simplification of equations, reduction of computational complexity, acceleration of the solution process, and deeper insights into the behavior of dynamic systems.

Introduction

Differential equations have been integral to sports analysis research since Newton's time and are widely used to understand physical and engineering sciences. One may argue that their applications extend beyond these fields to disciplines, such as economics, sociology, psychology, and medicine. In engineering and physical sciences, differential equations primarily define the relationships and governing principles of variable interactions.

The Laplace transform [1,2,3,4] is recognized as one of the key transformations for solving linear ordinary differential equations (LODEs) with constant coefficients, given some initial conditions. It follows the general formula:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

where $a_0, a_1 \ldots, a_n$ are constants.

In 2018, Sadiq [5,6] built on the definition of the Al-Zughair transform to develop another transformation, known as the expansion Al-Zughair transform. It was formulated using the general expression, as follows:

$$\square[f(x)] = \int_0^1 x^p . f(x)dx = F(p).$$

Although the expanded Al-Zughair transform was used to solve ordinary differential equations with variable coefficients, we successfully applied it to solve differential equations with varying orders and constant coefficients.

In this research, we apply the expanded Al-Zughair transform along with a newly introduced transformation, the Zainab Al-Zughair transform [7], whose general formula is as follows:

$$\mathcal{Z}Z\left(f(x)\right) = \int_0^1 x^{1/\mathfrak{z}} f(x)dx \quad ; \mathfrak{z} \in R/[-1,0].$$

*Corresponding author at: Department of Mathematics, Faculty of Education for Women, University of Kufa, Kufa, Iraq
ORCID: https:// https://orcid.org/0000-0000-0000-0000 ,
Tel: +964 07804445555
Email: zainabn.alsanam@student.uokufa.edu.iq

These transformations are used to solve the ordinary differential equations with constant coefficients, given initial conditions.

The core focus of this study is the Expanded Al-Zughair transform and the Zainab Al-Zughair transform.

Finally, both transformations were used to solve linear systems of *n*-th order ordinary differential equations with constant coefficients.

1. Expanded Al Zughair Transform of Other Derivatives

Considering ⍰ = ⍰ (lnx), we define the expansion of Al-Zughair transform for ⍰ as follows:

$$\square[\,⍰(lnx)] = \int_0^1 x^p.\,⍰\,(lnx)\,dx \quad , \; so$$

(1) $\square[\,⍰'\,(lnx)] = ⍰(0) - (p+1)\,\square[\,⍰(lnx)]$

Proof:

$$\square[\,⍰'\,(lnx)] = \int_0^1 x^{p+1}.\,\frac{⍰'(lnx)}{x}\,dx$$

Let $u = x^{p+1} \Rightarrow du = (p+1)\,x^p\,dx$,

and $dv = \frac{⍰'(lnx)}{x}\,dx \Rightarrow v = ⍰(lnx)$

$$\int_0^1 x^{p+1}.\,\frac{⍰'(lnx)}{x}\,dx = x^{p+1}.\,⍰(\mathrm{lnx})\Big|_0^1 - (p+1)\int_0^1 x^p.\,⍰(lnx)dx$$

$$= ⍰(0) - (p+1)\,\square[\,⍰(lnx)]$$

(2) $\square[\,⍰''\,(lnx)] = ⍰'(0) - (p+1)⍰(0) + (p+1)^2\,\square[\,⍰(lnx)]$

Proof:

$$\square[\,⍰''\,(lnx)] = \int_0^1 x^{p+1}.\,\frac{⍰''(lnx)}{x}\,dx$$

Let $u = x^{p+1} \Rightarrow du = (p+1)\,x^p\,dx$,

and $dv = \frac{⍰''(lnx)}{x}\,dx \Rightarrow v = ⍰'(lnx)$

$$\int_0^1 x^{p+1}.\,\frac{⍰''(lnx)}{x}\,dx = x^{p+1}.\,⍰'(\mathrm{lnx})\Big|_0^1 - (p+1)\int_0^1 x^p.\,⍰'(lnx)\,dx$$

$$= ⍰'(0) - (p+1)\,\square[\,⍰'(lnx)]$$

$$= y'(0) - (p+1)\big[⍰(0) - (p+1)\,\square[\,⍰(lnx)]\big]$$

$$= ⍰'(0) - (p+1)⍰(0) + (p+1)^2\,\square[\,⍰(lnx)]$$

(3) $\square[\,⍰'''\,(lnx)] = ⍰''(0) - (p+1)⍰'(0) + (p+1)^2\,⍰(0) - (p+1)^3\,\square[\,⍰(lnx)]$

Proof:

$$\square[\,⍰'''\,(lnx)] = \int_0^1 x^{p+1}.\,\frac{⍰'''(lnx)}{x}\,dx$$

Let $u = x^{p+1} \Rightarrow du = (p+1)\,x^p\,dx$,

and $dv = \frac{⍰'''(lnx)}{x}\,dx \Rightarrow v = ⍰''(lnx)$

$$\int_0^1 x^{p+1}.\,\frac{⍰'''(lnx)}{x}\,dx = x^{p+1}.\,⍰''(lnx)\Big|_0^1 - (p+1)\int_0^1 x^p.\,⍰''(lnx)\,dx$$

$$= ⍰''(0) - (p+1)\,\square[\,⍰''(lnx)]$$

$$= ⍰''(0) - (p+1)\big[⍰'(0) - (p+1)⍰(0) + (p+1)^2\,\square[\,⍰(lnx)]\big]$$

$$= ⍰''(0) - (p+1)⍰'(0) + (p+1)^2\,⍰(0) - (p+1)^3\,\square[\,⍰(lnx)]$$

(4) $\square\big[\,⍰^{(4)}\,(lnx)\big] = ⍰'''(0) - (p+1)\,⍰''(0) + (p+1)^2\,⍰'(0) - (p+1)^3⍰(0) + (p+1)^4\,\square[\,⍰(lnx)]$

Proof:

$$\square\big[⍰^{(4)}\,(lnx)\big] = \int_0^1 x^{p+1}.\,\frac{⍰^{(4)}(lnx)}{x}\,dx$$

Let $u = x^{p+1} \Rightarrow du = (p+1)\,x^p\,dx$,

and $dv = \frac{⍰^{(4)}(lnx)}{x}\,dx \Rightarrow v = ⍰'''(lnx)$

$$\int_0^1 x^{p+1}.\,\frac{⍰^{(4)}(lnx)}{x}\,dx = x^{p+1}.\,⍰'''(\mathrm{lnx})\Big|_0^1 - (p+1)\int_0^1 x^p.\,⍰'''(lnx)\,dx$$

$$= ⍰'''(0) - (p+1)\,\square[\,⍰'''(lnx)]$$

$$= ⍰'''(0) - (p+1)\big[⍰''(0) - (p+1)⍰'^{(0)} + (p+1)^2\,⍰(0) - (p+1)^3\,\square[\,⍰(lnx)]\big]$$

$$= ⍰'''(0) - (p+1)\,⍰''(0) + (p+1)^2\,⍰'(0) - (p+1)^3⍰(0) + (p+1)^4\,\square[\,⍰(lnx)]$$

Then, in general,

$$\text{□}[\text{⍰}^{n}(lnx)] = \text{⍰}^{(n-1)}(0) + (-1)(p+1)\text{⍰}^{(n-2)}(0) + (-1)^2(p+1)^2\text{⍰}^{(n-3)}(0) + \cdots + (-1)^{n-1}(p+1)^{n-1}\text{⍰}(0) + (-1)^{n}(p+1)^{n}\text{□}[\text{⍰}(lnx)]$$

Example 1: To solve the DE ⍰′+2 ⍰=sin(*lnx*)+cos(*lnx*) , ⍰(0)=0

We apply the expanded Al-Zughair transform on both sides, obtaining the following:

$$\text{□}[\text{⍰}'(lnx)] + 2\text{□}[\text{⍰}(lnx)] = \text{□}[\sin(lnx) + \cos(lnx)]$$

$$\text{□}\text{⍰}(0) - (p+1)[\text{⍰}(lnx)] + 2\text{□}[\text{⍰}(lnx)] = \frac{-1}{[(p+1)^2+1]} + \frac{(p+1)}{[(p+1)^2+1]}$$

$$-(p-1)\text{□}[\text{⍰}(lnx)] = \frac{p}{[(p+1)^2+1]}$$

$$\text{□}[\text{⍰}(lnx)] = \frac{-p}{(p-1)[(p+1)^2+1]}$$

$$\frac{-p}{(p-1)[(p+1)^2+1]} = \frac{A}{(p-1)} + \frac{Bp+C}{[(p+1)^2+1]}$$

$$= \frac{Ap^2+2Ap+2A+Bp^2-Bp+Cp-C}{(p-1)[(p+1)^2+1]}$$

A + B = 0

2A – B + C = –1

2A – C = 0

Hence, $A = \frac{-1}{5}$ $B = \frac{1}{5}$ $C = \frac{-2}{5}$

$$\wp^{-1}\left[\frac{-p}{(p-1)[(p+1)^2+1]}\right] = \wp^{-1}\left[\frac{\frac{-1}{5}}{(p-1)}\right] + \wp^{-1}\left[\frac{\frac{1}{5}p-\frac{2}{5}}{[(p+1)^2+1]}\right]$$

$$= \frac{-1}{5}x^{-2} + \wp^{-1}\left[\frac{\left(\frac{1}{5}p+\frac{1}{5}\right)-\frac{3}{5}}{[(p+1)^2+1]}\right]$$

$$= \frac{-1}{5}x^{-2} + \wp^{-1}\left[\frac{\frac{1}{5}(p+1)}{[(p+1)^2+1]}\right] + \wp^{-1}\left[\frac{-\frac{3}{5}}{[(p+1)^2+1]}\right]$$

$$\text{⍰}(lnx) = \frac{-1}{5}x^{-2} + \frac{1}{5}\cos(lnx) + \frac{3}{5}\sin(lnx)$$

Example 2: To solve the DE ⍰′′ + ⍰= sin (2*lnx*), ⍰′(0) = ⍰(0) = 0

We use the expanded Al-Zughair transform, as follows:

$$\wp[\text{⍰}''(lnx)] + \wp[\text{⍰}(lnx)] = \wp[\sin(2lnx)]$$

$$\text{⍰}'(0) - (p+1)\text{⍰}(0) + (p+1)^2\,\text{□}[\text{⍰}(lnx)] + \wp[\text{⍰}(lnx)] = \frac{-2}{(p+1)^2+4}$$

$$\wp[\text{⍰}(lnx)] = \frac{-2}{[(p+1)^2+1][(p+1)^2+4]}$$

$$\frac{-2}{[(p+1)^2+1][(p+1)^2+4]} = \frac{Ap+B}{[(p+1)^2+1]} + \frac{Cp+D}{[(p+1)^2+4]}$$

$$= \frac{Ap^3+2Ap^2+5Ap+Bp^2+2Bp+5B+Cp^3+2Cp^2+2Cp+Dp^2+2Dp+2D}{[(p+1)^2+1][(p+1)^2+4]}$$

A + C = 0

2A + B +2C + D = 0

5A + 2B + 2C + 2D = 0

5B + 2D = −2

Hence, A = 0 $B = \frac{-2}{3}$ C = 0 $D = \frac{2}{3}$

$$\wp^{-1}\left[\frac{-p}{(p-1)[(p+1)^2+1]}\right] = \wp^{-1}\left[\frac{\frac{-2}{3}}{[(p+1)^2+1]}\right] + \wp^{-1}\left[\frac{\frac{2}{3}}{[(p+1)^2+4]}\right]$$

$$\text{⍰}(lnx) = \frac{2}{3}\sin(lnx) - \frac{1}{3}\sin(2lnx)$$

Example 3: To solve the DE ⍰′′ + 3 ⍰′+2⍰=$4x^2$, ⍰′(0) = 0, ⍰(0) = 0

We employ the expanded Al-Zughair transform, as follows:

$$\wp[\text{⍰}''(lnx)] + 3\wp[\text{⍰}'(lnx)] + 2\wp[\text{⍰}(lnx)] = \wp[4x^2]$$

$$\text{⍰}'(0) - (p+1)\text{⍰}(0) + (p+1)^2\,\text{□}[\text{⍰}(lnx)] + 3\text{⍰}(0) - 3(p+1)\text{□}[\text{⍰}(lnx)] + 2\wp[\text{⍰}(lnx)] = \frac{4}{p+3}$$

$$p(p-1)\,\wp[\text{⍰}(lnx)] = \frac{4}{p+3} \Rightarrow \wp[\text{⍰}(lnx)] = \frac{4}{p(p-1)(p+3)}$$

$$\frac{p^2-p-1}{p(p-1)(p+3)} = \frac{A}{p} + \frac{B}{p+3} + \frac{C}{p-1}$$

$$= \frac{A(p-1)(p+3) + Bp(p-1) + Cp(p+3)}{p(p-1)(p+3)}$$

$$= \frac{Ap^2+2Ap-3A+Bp^2-Bp+Cp^2+3Cp}{p(p-1)(p+3)}$$

A + B + C = 0

$2A - B + 3C = 0$

$3A = 4$

Hence $A = \frac{-4}{3}$ $B = \frac{1}{3}$ $C = 1$

$$\wp^{-1}\left[\frac{4}{p(p-1)(p+3)}\right] = \wp^{-1}\left[\frac{\frac{-4}{3}}{p}\right] +$$

$$\wp^{-1}\left[\frac{\frac{1}{3}}{p+3}\right] + \wp^{-1}\left[\frac{1}{p-1}\right]$$

$$\square(lnx) = \frac{-4}{3}x^{-1} + \frac{1}{3}x^{2} + x^{-2}$$

Example 4: To solve the DE $\square''' + 3\square'' + 3\square' + \square = -x$, $\square(0) = 1, \square'(0) = 0, \square''(0) = 0$

We use the expanded Al-Zughair transform, as follows:

$$\wp[\square'''(lnx)] + 3\wp[\square''(lnx)] + 3\wp[\square'(lnx)] + \wp[\square(lnx)] = \wp[-x]$$

$$\square''(0) - (p+1)\square'(0) + (p+1)^2\square(0) - (p+1)^3\wp[\square(lnx)] + 3\square'(0) - 3(p+1)\square(0) + 3(p+1)^2\wp[\square(lnx)] + 3\square(0) - 3(p+1)\wp[\square(lnx)] + \wp[\square(lnx)] = \frac{-1}{p+2}$$

$$[p^2 + 2p + 1 - 3p - 3 + 3] + [-(p+1)^3 + 3(p+1)^2 - 3(p+1) + 1]\wp[\square(lnx)] = \frac{-1}{p+2}$$

$$[p^2 - p + 1] - p^3\wp[\square(lnx)] = \frac{-1}{p+2}$$

$$\Rightarrow -p^3\wp[\square(lnx)] = \frac{-1}{p+2} - [p^2 - p + 1]$$

$$-p^3\wp[\square(lnx)] = \frac{-1-p^3-2p^2+p^2+2p-p-2}{p+2}$$

$$\Rightarrow \wp[\square(lnx)] = \frac{p^3 + p^2 - p + 3}{p^3(p+2)}$$

$$\frac{p^3 + p^2 - p + 3}{p^3(p+2)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{(p+2)}$$

$$= \frac{Ap^3 + 2Ap^2 + Bp^2 + 2Bp + Cp + 2C + Dp^3}{p^3(p+2)}$$

$A+D=1$

$2A+B=1$

$2B+C=-1$

$2C=3$

Hence $A=\frac{9}{8}$ $B=\frac{-5}{4}$ $C=\frac{3}{2}$ $D=\frac{-1}{8}$

$$\wp^{-1}\left[\frac{p^3 + p^2 - p + 3}{p^3(p+2)}\right]$$

$$= \wp^{-1}\left[\frac{\frac{9}{8}}{p}\right] + \wp^{-1}\left[\frac{\frac{-5}{4}}{p^2}\right] + \wp^{-1}\left[\frac{\frac{3}{2}}{p^3}\right] + \wp^{-1}\left[\frac{\frac{-1}{8}}{(p+2)}\right]$$

$$\square(lnx) = \frac{9}{8}x^{-1} + \frac{5}{4}x^{-1}(lnx) + \frac{3}{4}x^{-1}(lnx)^2 - \frac{1}{8}x$$

Example 5: To solve the DE $\square^{(4)} - \square = 0$,

$\square(0) = \square''(0) = 1,\ \square'(0) = \square'''(0) = 0$

We consider the expanded Al-Zughair transform, as follows:

$$\wp[\square^{(4)}(lnx)] - \wp[\square(lnx)] = \wp[0]$$

$$\square'''(0) - (p+1)\square''(0) + (p+1)^2\square'(0) - (p+1)^3\square(0) + (p+1)^4\square[\square(lnx)] - \wp[\square(lnx)] = 0\ (-p^3 - 3p^2 - 4p - 2) + [(p+1)^4 - 1]\square[\square(lnx)] = 0$$

$$\square[\square(lnx)] = \frac{p^3 + 3p^2 + 4p + 2}{[(p+1)^4 - 1]}$$

$$\frac{p^3+3p^2+4p+2}{[(p+1)^2-1][(p+1)^2+1]} = \frac{Ap+B}{[(p+1)^2-1]} + \frac{Cp+D}{[(p+1)^2+1]}$$

$$= \frac{Ap^3+2Ap^2+2Ap+Bp^2+2Bp+2B+Cp^3+2Cp^2+Dp^2+2Dp}{[(p+1)^2-1][(p+1)^2+1]}$$

$A + C = 1$

$2A + B + 2C + D = 3$

$2A + 2B + 2D = 4$

$2B = 2$

Hence, $A = 1$ $B = 1$ $C = 0$ $D = 0$

$$\wp^{-1}\left[\frac{p^3 + 3p^2 + 4p + 2}{[(p+1)^2 - 1][(p+1)^2 + 1]}\right] = \wp^{-1}\left[\frac{p+1}{[(p+1)^2 - 1]}\right]$$

$\square(lnx) = \cosh(lnx)$

2. Zainab Al-Zughair Transform of Other Derivatives

Considering $\square = \square(\ln x)$, we define Zainab Al-Zughair transform for $\square$ as follows:

$$\widehat{\mathcal{Z}Z}[\square(lnx)] = \int_0^1 x^{1/z}.\square(lnx)\,dx\ ,\ so$$

$$(1)\ \widehat{\mathcal{Z}Z}[\square'(lnx)] = \square(0) - \left(\frac{1+z}{z}\right)\widehat{\mathcal{Z}Z}[\square(lnx)]$$

Proof:

$$\widehat{\mathcal{Z}\mathrm{Z}}[\, \square'(\,lnx\,)] = \int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square'(\,lnx\,)}{x}\, dx$$

Let $u = x^{\frac{1}{ʒ}+1} \Rightarrow du = (\frac{1+ʒ}{ʒ})\, x^{\frac{1}{ʒ}}\, dx$,

and $dv = \frac{\square'(\,lnx\,)}{x}\, dx \Rightarrow v = \square(\,lnx\,)$

$$\int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square'(\,lnx\,)}{x}\, dx = x^{\frac{1}{ʒ}+1}.\square(\,\mathrm{lnx}\,)\Big|_0^1 - (\frac{1+ʒ}{ʒ})\int_0^1 x^{\frac{1}{ʒ}}.\ \square(\,lnx\,)\, dx$$

$$= \square(0) - (\frac{1+ʒ}{ʒ})\, \widehat{\mathcal{Z}\mathrm{Z}}[\, \square(\,lnx\,)]$$

(2) $\widehat{\mathcal{Z}\mathrm{Z}}[\, \square''(\,lnx\,)] = \square'(0) - \left(\frac{1+ʒ}{ʒ}\right)\square(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \widehat{\mathcal{Z}\mathrm{Z}}[\square(\,lnx\,)]$

Proof:

$$\widehat{\mathcal{Z}\mathrm{Z}}[\, \square''(\,lnx\,)] = \int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square''(\,lnx\,)}{x}\, dx$$

Let $u = x^{\frac{1}{ʒ}+1} \Rightarrow du = (\frac{1+ʒ}{ʒ})\, x^{\frac{1}{ʒ}}\, dx$,

and $dv = \frac{\square''(\,lnx\,)}{x}\, dx \Rightarrow v = \square'(\,lnx\,)$

$$\int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square''(\,lnx\,)}{x}\, dx = x^{\frac{1}{ʒ}+1}.\square'(\,\mathrm{lnx}\,)\Big|_0^1 - (\frac{1+ʒ}{ʒ})\int_0^1 x^{\frac{1}{ʒ}}.\ \square'(\,lnx\,)\, dx$$

$$= \square'(0) - (\frac{1+ʒ}{ʒ})\, \widehat{\mathcal{Z}\mathrm{Z}}[\, \square'(\,lnx\,)]$$

$$=\square'(0) - (\frac{1+ʒ}{ʒ})\left[\square(0) - (\frac{1+ʒ}{ʒ})\, \widehat{\mathcal{Z}\mathrm{Z}}[\square(\,lnx\,)]\right]$$

$$= \square'(0) - \left(\frac{1+ʒ}{ʒ}\right)\square(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \widehat{\mathcal{Z}\mathrm{Z}}[\square(\,lnx\,)]$$

(3) $\widehat{\mathcal{Z}\mathrm{Z}}[\, \square'''(\,lnx\,)] = \square''(0) - \left(\frac{1+ʒ}{ʒ}\right)\square'(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \square(0) - \left(\frac{1+ʒ}{ʒ}\right)^3 \widehat{\mathcal{Z}\mathrm{Z}}[\, \square(lnx)]$

Proof:

$$\widehat{\mathcal{Z}\mathrm{Z}}[\, \square'''(\,lnx\,)] = \int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square'''(\,lnx\,)}{x}\, dx$$

Let $u = x^{\frac{1}{ʒ}+1} \Rightarrow du = (\frac{1+ʒ}{ʒ})\, x^{\frac{1}{ʒ}}\, dx$,

and $dv = \frac{\square'''(\,lnx\,)}{x}\, dx \Rightarrow v = \square''(\,lnx\,)$

$$\int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square'''(\,lnx\,)}{x}\, dx = x^{\frac{1}{ʒ}+1}.\square''(\,\mathrm{lnx}\,)\Big|_0^1 - (\frac{1+ʒ}{ʒ})\int_0^1 x^{\frac{1}{ʒ}}.\ \square''(\,lnx\,)\, dx$$

$$= \square''(0) - (\frac{1+ʒ}{ʒ})\, \widehat{\mathcal{Z}\mathrm{Z}}[\, \square''(\,lnx\,)]$$

$$= \square''(0) - (\frac{1+ʒ}{ʒ})\left[\square'(0) - \left(\frac{1+ʒ}{ʒ}\right)\square(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]\right]$$

$$= \square''(0) - \left(\frac{1+ʒ}{ʒ}\right)\square'(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \square(0) - \left(\frac{1+ʒ}{ʒ}\right)^3 \mathcal{Z}\mathrm{Z}[\, \square(lnx)]$$

(4) $\widehat{\mathcal{Z}\mathrm{Z}}\left[\square^{(4)}(lnx)\right] = \square'''(0) - \left(\frac{1+ʒ}{ʒ}\right)\square''(0) + \left(\frac{1+ʒ}{ʒ}\right)^2 \square'(0) - \left(\frac{1+ʒ}{ʒ}\right)^3 \square(0) + \left(\frac{1+ʒ}{ʒ}\right)^4 \widehat{\mathcal{Z}\mathrm{Z}}[\, \square(lnx)]$

Proof:

$$\widehat{\mathcal{Z}\mathrm{Z}}\left[\square^{(4)}(\,lnx\,)\right] = \int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square^{(4)}(\,lnx\,)}{x}\, dx$$

Let $u = x^{\frac{1}{ʒ}+1} \Rightarrow du = (\frac{1+ʒ}{ʒ})\, x^{\frac{1}{ʒ}}\, dx$,

and $dv = \frac{\square^{(4)}(\,lnx\,)}{x}\, dx \Rightarrow v = \square'''(lnx)$

$$\int_0^1 x^{\frac{1}{ʒ}+1}.\ \frac{\square^{(4)}(\,lnx\,)}{x}\, dx = x^{\frac{1}{ʒ}+1}.\square'''(\,\mathrm{lnx}\,)\Big|_0^1 - (\frac{1+ʒ}{ʒ})\int_0^1 x^{\frac{1}{ʒ}}.\ \square'''(\,lnx\,)\, dx$$

$$= \square'''(0) - (\frac{1+ʒ}{ʒ})\, \widehat{\mathcal{Z}\mathrm{Z}}[\, \square'''(\,lnx\,)]$$

$$=\square'''(0)-\left(\frac{1+z}{z}\right)\left[\square''(0)-\left(\frac{1+z}{z}\right)\square'(0)+\left(\frac{1+z}{z}\right)^2\square(0)-\left(\frac{1+z}{z}\right)^3\widehat{ZZ}[\square(lnx)]\right]$$

$$=\square'''(0)-\left(\frac{1+z}{z}\right)\square''(0)+\left(\frac{1+z}{z}\right)^2\square'(0)-\left(\frac{1+z}{z}\right)^3\square(0)+\left(\frac{1+z}{z}\right)^4\widehat{ZZ}[\square(lnx)]$$

Then, in general

$$\widehat{ZZ}[\square^{n}(lnx)]=\square^{(n-1)}(0)+(-1)\left(\frac{1+z}{z}\right)\square^{(n-2)}(0)+(-1)^2\left(\frac{1+z}{z}\right)^2\square^{(n-3)}(0)+\cdots+(-1)^{n-1}\left(\frac{1+z}{z}\right)^{n-1}\square(0)+(-1)^{n}\left(\frac{1+z}{z}\right)^{n}\widehat{ZZ}[\square(lnx)]$$

Example 1: To solve the DE $\square'-\square=1,\ \square(0)=0$

We use the Zainab Al-Zughair transform, as follows:

$$\widehat{ZZ}[\square'(lnx)]-\widehat{ZZ}[\square(lnx)]=\widehat{ZZ}[1]$$

$$\square(0)-\left(\frac{1+z}{z}\right)\widehat{ZZ}[\square(lnx)]-\widehat{ZZ}[\square(lnx)]=\frac{z}{1+z}$$

$$-\left(\frac{1+2z}{z}\right)\widehat{ZZ}[\square(lnx)]=\frac{z}{1+z}$$

$$\widehat{ZZ}[\square(lnx)]=z\cdot\left[\frac{-z}{(1+z)(1+2z)}\right]$$

$$\left[\frac{-z}{(1+z)(1+2z)}\right]=\frac{A}{(1+z)}+\frac{B}{(1+2z)}=\frac{A+2Az+B+Bz}{(1+z)(1+2z)}$$

A+B=0

2A+B = −1

Hence, A = −1 B = 1

$$\left(\widehat{ZZ}\right)^{-1}\left[\frac{-z}{(1+z)(1+2z)}\right]=\left(\widehat{ZZ}\right)^{-1}\left[\frac{-z}{(1+z)}\right]+\left(\widehat{ZZ}\right)^{-1}\left[\frac{z}{(1+2z)}\right]$$

$$\square(lnx)=-1+x$$

Example2: To solve the DE $\square'+3\square=cos(lnx)$, $\square(0)=1$

We use the Zainab Al-Zughair transform, as follows:

$$\widehat{ZZ}[\square'(lnx)]+3\widehat{ZZ}[\square(lnx)]=\widehat{ZZ}[cos(lnx)]$$

$$\square(0)-\left(\frac{1+z}{z}\right)\widehat{ZZ}[\square(lnx)]+3\widehat{ZZ}[\square(lnx)]=\frac{z(1+z)}{(1+z)^2+z^2}$$

$$1+\left(\frac{-1-z+3z}{z}\right)\widehat{ZZ}[\square(lnx)]=\frac{z(1+z)}{(1+z)^2+z^2}$$

$$\left(\frac{-1+2z}{z}\right)\widehat{ZZ}[\square(lnx)]=\frac{z(1+z)}{(1+z)^2+z^2}-1$$

$$\left(\frac{-1+2z}{z}\right)\widehat{ZZ}[\square(lnx)]=\frac{z+z^2-1-2z-2z^2}{(1+z)^2+z^2}$$

$$\widehat{ZZ}[\square(lnx)]=\left(\frac{-1-z-z^2}{(1+z)^2+z^2}\right)\cdot\left(\frac{-z}{1-2z}\right)$$

$$\widehat{ZZ}[\square(lnx)]=z\cdot\left(\frac{1+z+z^2}{(1-2z)[(1+z)^2+z^2]}\right)$$

$$\left(\frac{1+z+z^2}{(1-2z)[(1+z)^2+z^2]}\right)=\frac{A}{(1-2z)}+\frac{Bz+C}{[(1+z)^2+z^2]}=\frac{A+2Az+2Az^2+Bz-2Bz^2+C-2Cz}{(1+2z)(1+3z)}$$

A+C=1

2A + B−2C = 1

2A−2B = 1

Hence, $A=\frac{7}{10}$ $B=\frac{2}{10}$ $C=\frac{3}{10}$

$$\left(\widehat{ZZ}\right)^{-1}\left[\frac{1+z+z^2}{(1-2z)[(1+z)^2+z^2]}\right]=\left(\widehat{ZZ}\right)^{-1}\left[\frac{\frac{7}{10}z}{(1-2z)}\right]+\left(\widehat{ZZ}\right)^{-1}\left[\frac{\frac{2}{10}z^2+\frac{3}{10}z}{[(1+z)^2+z^2]}\right]$$

$$=\left(\widehat{ZZ}\right)^{-1}\left[\frac{\frac{7}{10}z}{(1-2z)}\right]+\left(\widehat{ZZ}\right)^{-1}\left[\frac{\left(\frac{3}{10}z^2+\frac{3}{10}z\right)-\frac{1}{10}z^2}{[(1+z)^2+z^2]}\right]$$

$$=\left(\widehat{ZZ}\right)^{-1}\left[\frac{\frac{7}{10}z}{(1-2z)}\right]+\left(\widehat{ZZ}\right)^{-1}\left[\frac{\frac{3}{10}z(1+z)}{[(1+z)^2+z^2]}\right]+\left(\widehat{ZZ}\right)^{-1}\left[\frac{-\frac{1}{10}z^2}{[(1+z)^2+z^2]}\right]$$

$$\square(lnx)=\frac{7}{10}x^{-3}+\frac{3}{10}\cos(lnx)+\frac{1}{10}\sin(lnx)$$

Example 3: To solve the DE $\square''+\square=x$,

$\square(0)=1,\ \square'(0)=0$

We apply the Zainab Al-Zughair transform, as follows:

$$\widehat{ZZ}[\square''(lnx)]+\widehat{ZZ}[\square(lnx)]=\widehat{ZZ}[x]$$

$$\square'(0)-\left(\frac{1+z}{z}\right)\square(0)+\left(\frac{1+z}{z}\right)^2\widehat{ZZ}[\square(lnx)]+\widehat{ZZ}[\square(lnx)]=\frac{z}{(1+2z)}$$

$$\left[\frac{(1+\mathfrak{z})^2+\mathfrak{z}^2}{\mathfrak{z}^2}\right]\widehat{\mathcal{Z}\mathrm{Z}}[\square(\,lnx\,)]=\frac{\mathfrak{z}}{(1+2\mathfrak{z})}+\frac{1+\mathfrak{z}}{\mathfrak{z}}$$

$$\widehat{\mathcal{Z}\mathrm{Z}}[\square(\,lnx\,)]=\frac{3\mathfrak{z}^2+3\mathfrak{z}+1}{\mathfrak{z}(1+2\mathfrak{z})}\cdot\frac{\mathfrak{z}^2}{[(1+\mathfrak{z})^2+\mathfrak{z}^2]}$$

$$\mathcal{Z}\mathrm{Z}[\square(\,lnx\,)]=\mathfrak{z}\cdot\left[\frac{3\mathfrak{z}^2+3\mathfrak{z}+1}{(1+2\mathfrak{z})[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]$$

$$\left[\frac{3\mathfrak{z}^2+3\mathfrak{z}+1}{(1+2\mathfrak{z})[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]=\frac{A}{(1+2\mathfrak{z})}+\frac{B\mathfrak{z}+C}{[(1+\mathfrak{z})^2+\mathfrak{z}^2]}$$

$$=\frac{A+2A\mathfrak{z}+2A\mathfrak{z}^2+B\mathfrak{z}+2B\mathfrak{z}^2+C+2C\mathfrak{z}}{(1+2\mathfrak{z})[(1+\mathfrak{z})^2+\mathfrak{z}^2]}$$

A + C = 1

2A + B + 2C = 3

2A + 2B = 3

Hence, $A=\frac{1}{2}$ $B=1$ $C=\frac{1}{2}$

$$\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{3\mathfrak{z}^2+3\mathfrak{z}+1}{(1+2\mathfrak{z})[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]$$

$$=\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{\frac{1}{2}\mathfrak{z}}{(1+2\mathfrak{z})}\right]$$

$$+\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{\mathfrak{z}^2+\frac{1}{2}\mathfrak{z}}{[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]$$

$$=\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{\frac{1}{2}\mathfrak{z}}{(1+2\mathfrak{z})}\right]+\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{\frac{1}{2}\mathfrak{z}^2+\frac{1}{2}\mathfrak{z}}{[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]$$

$$+\left(\widehat{\mathcal{Z}\mathrm{Z}}\right)^{-1}\left[\frac{\frac{1}{2}\mathfrak{z}^2}{[(1+\mathfrak{z})^2+\mathfrak{z}^2]}\right]$$

$$\square(\,lnx\,)=\frac{1}{2}x+\frac{1}{2}\cos(lnx)-\frac{1}{2}\sin(lnx)$$

Example 4: To solve the DE $\square''-\square'-\square=x\ cosh(lnx)$, $\square(0)=0$, $\square'(0)=1$

We apply the Zainab Al-Zughair transform, as follows:

$$\widehat{\mathcal{Z}\mathrm{Z}}[\square''(lnx)]-\widehat{\mathcal{Z}\mathrm{Z}}[\square'(lnx)]-\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]$$
$$=\widehat{\mathcal{Z}\mathrm{Z}}[x\ cosh(lnx)]$$

$$\square'(0)-\left(\frac{1+\mathfrak{z}}{\mathfrak{z}}\right)\square(0)+\left(\frac{1+\mathfrak{z}}{\mathfrak{z}}\right)^2\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]$$
$$-y(0)$$
$$+\left(\frac{1+\mathfrak{z}}{\mathfrak{z}}\right)\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]-\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]$$
$$=\frac{\mathfrak{z}(1+2\mathfrak{z})}{(1+2\mathfrak{z})^2-\mathfrak{z}^2}$$

$$\left[\frac{1+3\mathfrak{z}+\mathfrak{z}^2}{\mathfrak{z}^2}\right]\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]=\frac{-1-3\mathfrak{z}-\mathfrak{z}^2}{(1+2\mathfrak{z})^2-\mathfrak{z}^2}$$

$$\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]=\frac{-\mathfrak{z}^2(1+3\mathfrak{z}+\mathfrak{z}^2)}{(1+3\mathfrak{z}+\mathfrak{z}^2)[(1+2\mathfrak{z})^2-\mathfrak{z}^2]}$$

$$\widehat{\mathcal{Z}\mathrm{Z}}[\square(lnx)]=\frac{-\mathfrak{z}^2}{[(1+2\mathfrak{z})^2-\mathfrak{z}^2]}$$

$$\square(lnx)=x\sinh(lnx)$$

Conclusion:

In this research, we examine the application of the expanded Zughair transform and the Zainab Al-Zughair transform in solving linear ordinary differential equations with fixed coefficients and specific initial conditions. These transformations extend traditional techniques, such as the Laplace transform and provide a robust foundation for deriving solutions to the differential equations.

Our findings demonstrate that the Zainab Al-Zughair transform and the expanded Al-Zughair transform are effective in solving linear ODEs. These transformations introduce new insights and methods that not only simplify the process of solving such equations, but also offer valuable approaches for addressing challenges across the physical sciences, engineering, and other disciplines.

References

[1] Khamsi, M. A. (2006). Laplace Transformation, Basic and Definitions and Results.2006) ، ").

[2] Braun, M., "Differential Equation and their Application ", 4 th ed. New York: Spring -Verlag, 1993.

[3] Kreyszig, E. (2007). Advanced Engineering Mathematics 9th Edition with Wiley Plus Set (Vol. 334). US: John Wiley & Sons.

[4] Ross, S. (2021). Introduction to ordinary differential equations.

[5] Mohammed A.H., Sadiq B.A., "AL-Zughair Transform", LAP LAMBERT Academic Publishing, 2017.

[6] Mohammed, A. H., Sadiq, B. A., & ALI, A. M. H. (2016). Solving new type of linear equations by using new transformation. European Academic Research, 4(8), 7035-7048.

[7] Mohammed, A. H., Abdul Razzaq, Z. N. A New (ZZ)^ Integral Transform and Its Applications in Differential Equations. in Press.

[8] Nayel, H. H., & AL-Jumaili, H. S. (2019). Fabrication of highly sensitive NH3 sensor based on mixed In2O3–AgxO Nanostructural Thin Films deposited on porous silicon. Journal of university of Anbar for Pure science, 13(2), 40-47.

توسعة تحويل الزغير ومطوره لحل المعادلات التفاضلية العادية الخطية ذات المعاملات الثابتة

زينب ناصر عبدالرزاق ، علي حسن محمد

قسم الرياضيات، كلية التربية للبنات، جامعة الكوفة، الكوفة، العراق؛

zainabn.alsanam@student.uokufa.edu.iq , prof.ali57hassan@gmail.com

الخلاصة:

أحد المكونات الأساسية للرياضيات التطبيقية هي المعادلات التفاضلية ذات المعاملات الثابتة. يمكن وصف كل من الأنظمة المعقدة والمادية بشكل جيد باستخدامه. وقد ظهرت مؤخرًا تحولات جديدة تساعد على تبسيط هذه المعادلات، وتسهيل حلها، وتقليل التعقيد الحسابي. تساعد هذه التطورات الباحثين والمهندسين على أن يكونوا مجهزين بشكل أفضل للتعامل مع صعوبات الأنظمة الديناميكية المعقدة.

تم في هذا البحث استخدام تحويلات الزغير الموسع وزينب الزغير لاكتشاف حل المعادلات التفاضلية الخطية العادية (L.O.D.Es) ذات المعاملات الثابتة وفق شروط أولية معينة. نحن ندرس استخدام هذه التحويلات، والتي تزيد من قوة التقنيات التقليدية، في حل الأنظمة الخطية والمعادلات التفاضلية. يظهر لمدى نجاح عمل زينب الزغير وموسعة الزغير في حل الأنظمة الخطية من الرتبة n ‎(L.O.D.Es)‎ ذات المعاملات الثابتة. ويتم التعامل مع الظروف الأولية بسهولة، وتصبح المعادلات أكثر وضوحًا، ويتم تقليل التعقيد الحسابي، ويتم تسريع عملية الحل، ويتم اكتساب فهم أعمق لسلوك الأنظمة الديناميكية.

الكلمات المفتاحية :تحويل الزغير الموسع، تحويل زينب الزغير، حل المعادلات التفاضلية الخطية العادية ذات المعاملات الثابتة