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Study of Dynamical Behavior of a Prey-Predator Model when the Prey Population Affected by Multifactor

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ABSTRACT

In this work, a prey-predator model with Holling type II functional response is considered. The proposed model incorporates the cost of the fear of predators in prey, effect of environmental pollution and harvesting on prey population. Firstly, the details of model derivation of are given and then some of behaviors of the model solutions are proved, the existence criteria for each of the model equilibrium points are determined. For each of the equilibrium point, the sufficient and necessary conditions for being locally asymptotically stable are found. Via Lyapunov method, the sufficient conditions for global stability of the model equilibrium point are determined. Some numerical simulations are performed to discover the impact of the fear, harvesting population and environmental pollution on prey dynamics.

Keywords: Fear, Harvesting, Lyapunov method, Pollution, Stability analysis.
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دراسة السلوك الديناميكي لنموذج الفريسة–المفترس عندما يتأثر عدد الفرائس بعوامل متعددة

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الملخص

في هذا العمل، تم النظر في نموذج فريسة–مفترس مع استجابة وظيفية من النوع الثاني لهولنج. يتضمن النموذج المقترح تكلفة الخوف من الحيوانات المفترسة في الفريسة، وتأثير التلوث البيئي والحصاد على أعداد الفرائس. أولاً، تم تقديم تفاصيل اشتقاق النموذج ثم تم إثبات بعض سلوكيات حلول النموذج، وتم تحديد معايير الوجود لكل من نقاط توازن النموذج. لكل نقطة توازن، تم إيجاد الشروط الكافية والضرورية للاستقرار

المحلي المقارب. من خلال طريقة Lyapounv، تم تحديد الشروط الكافية للاستقرار العالمي لنقطة توازن النموذج. تم إجراء بعض المحاكاة العددية لاكتشاف تأثير الخوف والحصاد والتلوث البيئي على ديناميكيات الفرائس.

INTRODUCTION

Prey predator model is critical in mathematical ecology. Many mathematician authors considered mathematical models through differential equations on a variety of problems arising in ecological interaction between species [(1-3)]. Population dynamics may be affected by many factor like; fear, harvesting, toxicity, stage structure, delay, harvesting, cannibalism, anti-predator skills, refuge, infectious illness, and other population-affecting elements of the natural environment [(4-11)].

Physiological changes of prey population may be caused by predation fears and these physiological changes may reduce the reproduction of prey population. In [(3)] the authors showed that the bird reduce forty percent less offspring by predation fears. In [(4)] the authors considered a predator–prey model incorporating fear effect on reproduction of prey individuals, in their study they noticed that the fear has no impact on the stability of the model when the system incorporate bilinear functional response, but the system become stable under fear effect, if it incorporate the Holling type II functional response, based on their system in [(8)] the authors considered a prey predator with fear effect and harvesting cooperation, they discussed the stability, Hopf-bifurcation and Bogdanov –Takens bifurcation the system. In[(11)] the authors considered an ecological model with fear effect and prey refuge, they showed that effect of both factor can stabilize the system. For more results about fear effect, see [(2, 12-14)].

In the recent decades, the impact of harvesting activities on organisms and ecosystems are globally concerned. Several types of harvesting in predator-prey systems are already being formed and studied, mathematician authors have added many terms to the predator or prey density. The most common of these harvesting terms are; nonzero constant functions; functions of linear harvesting rate or nonlinear Michaelis-Menten type of predator harvesting [(15-20)]. Models studied with one of these harvesting forms exhibit far richer dynamics compared to the models with no harvesting. In [(18)] the authors studied the effect the harvesting in form of linear function on an eco-epidemiological model incorporating prey refuge.in their model a simple change of model dynamics happened. In [(17)] the authors consider a predator– prey system with a nonlinear Michaelis–Menten type of predator harvesting. They argue that harvesting with nonlinear form is more realistic and reasonable than modeling constant effort harvesting or linear harvesting. In their study the complexity of the model dynamics with this form of harvesting effect is demonstrated.

In recent years, ecological models incorporating the toxicant that emitted into the environment from external sources, as well as formed by precursors of biological species, have been proposed and analyzed by several researchers on [(21-24)]. In particular. The authors in[(24)] proposed a mathematical model to study the effect of pollution on natural stable two species communities. In their work, effects of a toxicant simultaneously on growth rate and carrying capacity of the species have not been considered. However, the authors in[(23)] proposed an ecological model to study the effects of environmental pollution on single-species and predator-prey system by assuming that the intrinsic growth rate of species decreases as the uptake concentration of the toxicant increases, while its carrying capacity decreases with the environmental concentration of the toxicant. The authors in [(21)] proposed an ecological model to study the survival of resource-dependent competing species. They assumed that competing species and its resource are affected simultaneously by a toxicant emitted into the environment from external sources as well as formed by precursors of competing species. Inspired by the aforementioned works, the aim of the current work is to develop Lotka-Volterra prey-predator system with Holling type II functional response, by incorporating the impact of effect of fear, harvesting and pollution on the dynamics of the prey population. In the next section the derivation of the model is given and some results regarding to the model are proved. In the third section, the bounded and the permanence of the model solutions studid.in the section four, the existence conditions of all feasible equilibrium points are found. In section five and six, stability analysis (local, as well as globally) of the model is studied. In section seven, the model numerically solved. Finally in section eight, a brief conclusion on the total work is given.

To derive the proposed model, the following assumptions are assumed:

1. In the absence of predators, fear of predator, harvesting, and pollution, the prey population grows logistically.
2. The prey population is killed directly by predators with Holling type II functional response[1].
3. The prey population growth affected by fear from predator effect.
4. The prey population harvested with a nonlinear Michaelis–Menten type of prey harvesting [21].
5. Only the effect of pollution on prey population is taken in to consideration.

Therefore, the dynamics of above assumptions can be modeled mathematically through the following system of differential equations.

THE MATHEMATICAL MODEL

$$\begin{aligned}
\frac{dX}{dt} &= \frac{bX}{1+fY} - d_1X - c_1X^2 - \left[\frac{\alpha Y}{1+\alpha TX} + \frac{qE}{m_1+m_2X} + \sigma_1 W\right]X \\
\frac{dY}{dt} &= \frac{e\alpha XY}{1+\alpha TX} - d_2Y - c_2Y^2 \\
\frac{dZ}{dt} &= \pi - \mu_1 Z - \sigma_2 XZ \\
\frac{dw}{dt} &= \sigma_2 XZ - \mu_2 W
\end{aligned} \qquad \dots (1)$$

Where, $X(0) > 0, Y(0) > 0, Z(0) > 0, W(0) > 0$, $X(t)$ is the prey density, $Y(t)$ is the predator population number of first species, Z(t) is the environment concentration of toxicant at time t, W(t) be the toxicant concentration in the prey population at time and the parameter descriptions is given in Table 1.

The right-hand side of each equation in system (1), satisfy the Lipschitzian condition. Therefor the model solution is unique. Further, the time derivative of each X, Y, Z and W are zero or positive at Space YZW, Space XZW, Space XYW and Space XYZ, respectively. Therefore, if the solution initiates at a non-negative point, then the component $X, Y$and Z of the solution points of system 1, cannot cross any coordinates of the solution points. Hence components $X, Y$and Z of solution points is always non negative.

Table 1: Parameter description of system (1).

parameters	Description
b	Prey Birth rate in absence fears
f	Level of fear due to prey response to anti-predators
d_1,d_2	Natural death rate of prey and predator, respectively
c_1, c_2	Intraspecific competition rates
α	the predator's search efficiency for prey.
$e \leq 1$	Conversion efficiency from biomass of prey to biomass of predator
T	the predator's average handling time of prey.
q	catch ability coefficient of prey.
E	the effort made to harvest prey individual.
m_1, m_2	are suitable positive constant.
π	the exogenous input rate of the toxicant in the environment
μ_1	the natural depletion rate of the environmental pollution.
μ_2	the natural washout rate of the pollution from the organism.
σ_1	the rates at which prey population is decreasing due to pollution.
σ_2	uptake rate of pollution by organism.

BOUNDEDNESS AND PERSISTENCE

Regarding to boundedness and persistence of system (1), we proved the following lemma and theorem

Lemma 1. In system (1), the following hold:

a. $\lim\limits_{t\to\infty} Sup\ X(t) \leq \frac{1}{c_1}|b - d_1|$

b. $\lim\limits_{t\to\infty} Sup\ Y(t) \leq \frac{1}{c_1 c_2}|e\alpha|b - d_1| - c_1 d_2|$

c. $\lim\limits_{t\to\infty} Sup\ [Z(t) + W(t)] \leq \frac{\pi}{\mu}$

where, $\mu = Min\{\mu_1, \mu_2\}$

Proof: a/ From first equation of system (1), it gets:

$\frac{dX}{dt} \leq |b - d_1|X - c_1X^2$

So,

$\lim\limits_{t\to\infty} Sup\ X(t) \leq \frac{|b-d_1|}{c_1}$

b/ Applying part(a) at the second equation of system (1) then as $t \to \infty$, it gets:

$$\frac{dY}{dt} \leq \frac{1}{c_1}|e\alpha|b - d_1| - c_1 d_2|Y - c_2Y^2$$

Thus,

$\lim\limits_{t\to\infty} Sup\ Y(t) \leq \frac{1}{c_1 c_2}|e\alpha|b - d_1| - c_1 d_2|$

c/ Let $N = Z + W$, then:

$\frac{dN}{dt} \leq \pi - \mu N$

Thus,

$\lim\limits_{t\to\infty} Sup\ N \leq \frac{\pi}{\mu}$

This completes the proof.

Note 1. Above guarantees that all solutions of system (1), are bounded.

Definition 1. (1) The population of Prey and predator in System (1) is said to be permanent if there exist positive constants φ and ϕsuch that:

$\varphi \geq Max\left\{\lim\limits_{t\to\infty} Sup\ X(t), \lim\limits_{t\to\infty} Sup\ Y(t)\right\}$ and:

$Min\left\{\lim\limits_{t\to\infty} Inf\ X(t), \lim\limits_{t\to\infty} Inf\ Y(t)\right\} \geq \phi$

Theorem 1. If the following conditions are provided, then system 1, is permanent.

$$b > (1 + fY_m)\left[\frac{qE}{m_1} + d_1 + \alpha Y_{max} + \sigma_1\frac{\pi}{\mu}\right] \quad \ldots (2)$$

$$e\alpha X_{min} > d_2(1 + \alpha T X_{max}) \quad \ldots (3)$$

Where, $X_{Max} = \frac{1}{c_1}|b - d_1|, Y_{Max} = \frac{1}{c_1 c_2}|e\alpha|b - d_1| - c_1 d_2|$ and:

$X_{min} = \frac{1}{c_1}\left[\frac{b}{1+fY_m} - \frac{qE}{m_1} - d_1 - \alpha Y_m - \sigma_1 M\right]$

Proof. From lemma 1, we have $\lim\limits_{t\to\infty} Sup\ X(t) \leq X_{Max}, \lim\limits_{t\to\infty} Sup\ Y(t) \leq Y_{Max}$ $and\ \ \lim\limits_{t\to\infty} Sup\ W(t) \leq \frac{\pi}{\mu}$

So, as $t \to \infty$, it gets:

$\frac{dX}{dt} \geq X\left[\frac{b}{1+fY_{max}} - \frac{qE}{m_1} - d_1 - \alpha Y_m - \sigma_1\frac{\pi}{\mu}\right] - c_1X^2$

So, under condition (2), it gets:

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$$\lim_{t\to\infty} Inf\ X \geq \frac{1}{c_1}\left[\frac{b}{1+fY_m} - \frac{qE}{m_1} - d_1 - \alpha Y_m - \sigma_1 \frac{\pi}{\mu}\right] = X_{min} > 0$$

and then

$$\frac{dY}{dt} \geq Y\left[\frac{e\alpha X_{min}}{1+\alpha T X_{max}} - d_2\right] - c_2 Y^2$$

Under condition (3), it gets

$$\lim_{t\to\infty} Inf\ X \geq \frac{1}{c_2}\left[\frac{e\alpha X_{min}}{1+\alpha T X_{max}} - d_2\right] = Y_{min} > 0$$

$Max\{X_{min}, Y_{max}, M\} \geq max\left\{\lim_{t\to\infty} Sup\ X(t), \lim_{t\to\infty} Sup\ Y(t)\right\}$ and $min\left\{\lim_{t\to\infty} Inf\ X(t), \lim_{t\to\infty} Inf\ Y(t)\right\} \geq min\{X_{min}, Y_{min}\}$

This completes the proof.

EXISTENCE OF EQUILIBRIUM POINTS

In the third section, all the existence equilibrium points of system (1) has been founded, they are $E_1(0,0,\frac{\pi}{\mu_1},0)$, $E_2(\bar{X},0,\bar{Z},\bar{W})$, $E_3(X^*,Y^*,Z^*,W^*)$.

1. The extinct equilibrium point $E_1\left(0,0,\frac{\pi}{\mu_1},0\right)$ is always existing.
2. The predator free equilibrium point $E_2(\bar{X},0,\bar{Z},\bar{W})$, where,

$\bar{Z} = \frac{\pi}{\mu_1+\sigma_2\bar{X}}$, , $\bar{W} = \frac{\sigma_2\bar{X}\pi}{\mu_1\mu_2+\mu_2\sigma_2\bar{X}}$, and $\bar{X}$ is a positive root of the equation:

$$AX^3 + BX^2 + CX + D = 0 \qquad \dots (4)$$

Where

$A = -c_1 m_2 \sigma_2$

$B = bm_2\mu_2\sigma_2 - d_1 m_2\mu_2\sigma_2 - c_1 m_1 E\mu_2\sigma_2 - c_1 m_2\mu_1\mu_2 - \pi\sigma_1\sigma_2 m_2$

$C = bm_1\mu_2\sigma_{2_2}\mu_2\sigma_2 + bm_2\mu_1\mu_2 - d_1 m_1\mu_2\sigma_2 - d_1 m_2\mu_1\mu_2 - c_1 m_1\mu_1\mu_2 - q\mu_2\sigma_2 - \pi\sigma_1\sigma_2 m_1$

$D = bm_1 e\mu_1\mu_2 - d_1 m_1 E\mu_1\mu_2 - qE\mu_1\mu_2$

According to Descartes's rule, equation (4) has a unique positive root, if and only if one of the following cases holds:

$B > 0, D > 0$ and $C > 0$,

$C > 0, D > 0$ and $B < 0$ or

$B < 0, C < 0$ and $D > 0$

3. The coexistence equilibrium point $E_3(X^*,Y^*,Z^*,W^*)$ exists uniquely if in the following algebraic system there is a positive solution.

$$\frac{b}{1+fY} - d_1 - c_1 X - \frac{\propto Y}{1+\propto TX} - \frac{qE}{m_1+m_2 X} - \sigma_1 W = 0$$

$$\frac{e\alpha X}{1+\alpha TX} - d_2 - c_2 Y = 0$$

$$\pi - \mu_1 z - \sigma_2 XZ = 0$$

$$\sigma_2 XZ - \mu_2 W = 0$$

This algebraic system gives the following:

$$Y^* = \frac{1}{c_2(1+\alpha TX^*)}(e\alpha X^* - d_1(1+\alpha TX^*)) = t_1(X^*)$$

$$Z^* = \frac{\pi}{\mu_1+\sigma_2 X^*} = t_2(X^*)$$

$$W^* = \frac{\sigma_2 X^*\,\pi}{\mu_2(\mu_1+\sigma_1 X^*)} = t_3(X^*)$$

While X^*represents a positive root of the following equation:

$$T(X) = \frac{b}{1+f\,t_1(X)} - d_1 - c_1 X - \frac{\propto t_1(X)}{1+\propto TX} - \frac{qE}{m_1+m_2 X} - \sigma_1 t_3(X) = 0$$

suppose the following condition holds:

$$\frac{bc_2}{c_2-fd_1} + \frac{\alpha d_1}{c_2} > d_1 + \frac{qE}{m_1+m_2 X} \qquad \dots (5)$$

Then:

$\lim_{X\to\infty} T(X) < 0$ and $T(0) = \frac{bc_2}{c_2-fd_1} - d_1 + \frac{\alpha d_1}{c_2} - \frac{qE}{m_1+m_2 X} > 0$

Therefore, a positive root X^* exist, if condition (5) holds:

Further, $Y^* > 0$, if

$$\frac{e\alpha X^*}{(1+\alpha TX^*)} > d_1 \qquad \dots (6)$$

Consequently $E_3(X^*,Y^*,Z^*,W^*)$ exist if conditions (5) and (6) hold.

LOCAL STABILITY AND HOPF-BIFURCATION

To study the topological structure near (local stability) an equilibrium point (X,Y,Z,W) of system (1), the following transformation used:

$V_1(t) = X(t) - X$, $V_1(t) = Y(t) - Y, V_3(t) = Z(t) - Z$ $V_4(t) = W(t) - W$.

Then the following linear system is obtained:

$$\frac{dV(t)}{dt} = J(X,Y,Z,W)V(t)$$

Where, $V(t)=\begin{pmatrix}V_1(t)\\V_2(t)\\V_3(t)\\V_4(t)\end{pmatrix}$ and $J(X,Y,Z,W)=$

$$\begin{bmatrix}\frac{b}{1+fY}-d_1-2c_2X-\frac{\alpha Y}{(1+\alpha TX)^2}-\frac{qEm_1}{(m_1+m_2X)^2}-\sigma_1 W & \frac{-bfX}{(1+fY)^2}-\frac{\alpha X}{1+\alpha TX} & 0 & -\sigma_1 X\\ \frac{e\alpha Y}{(1+\alpha TX)^2} & \frac{e\alpha X}{1+\alpha TX}-d_2-2c_2Y & 0 & 0\\ -\sigma_2 Z & 0 & -(\mu_1+\sigma_2 X) & 0\\ \sigma_2 Z & 0 & \sigma_2 X & -\mu_2\end{bmatrix}$$

The local stability conditions for E_1,E_2 and E_3 of the system (1) are established in Theorem 2, Theorem3 and Theorem 4, respectively.

Theorem 2. $E_1\left(0,0,\frac{\pi}{\mu_1},0\right)$ is locally asymptotically stable if and only if:

$$b<d_1+\frac{qE}{m_1} \qquad \dots (7)$$

Proof. $J(E_1)$ has only the following eigenvalues:

$\lambda_1=b-d_1-\frac{qE}{m_1}$, $\lambda_2=-d_2<0$, $\lambda_3=-\mu_1<0$ and $\lambda_4=-\mu_2<0$

λ_1 is negative if and only if condition (7) holds and this completes the proof.

Note 2. Since the eigenvalues of $J(0,0,\frac{\pi}{\mu_1},0)$ are always real. So, there is no possibility for undergoing hopf-bifurcation near $E_1(0,0,\frac{\pi}{\mu_1},0)$.

Theorem 3.

i. If $E_2(\bar{X},0,\bar{Z},\bar{W})$ is exist, it is locally asymptotically stable Provided that:

$$d_2(1+\alpha T\bar{X})>e\alpha\bar{X} \qquad \dots (8)$$

$$b<min\left\{R_4,\frac{R_6}{\mu_1\mu_2+\mu_2\sigma_2\bar{X}}\right\} \qquad \dots (9)$$

$$R_6+bR_5+b(\mu_1+\mu_2+\sigma_2\bar{X})R_4 < R_4R_5+b(\mu_1\mu_2+\mu_2\sigma_2\bar{X})+b^2(\mu_1+\mu_2+\sigma_2\bar{X}) \qquad \dots (10)$$

ii. System (1)exhibits a Hopf bifurcation near $E_2(\bar{X},0,\bar{Z},\bar{W})$ if the parameter values satisfy condition(8) holds and the following:

$$(R_1-b)\left(R_2-b(\mu_1+\mu_2+\sigma_2\bar{X})\right)+b(\mu_1\mu_2+\mu_2\sigma_2\bar{X})-R_3=0 \qquad \dots (11)$$

$$d_1+2c_2\bar{X}+\frac{qEm_1}{(m_1+m_2\bar{X})^2}+\sigma_1\bar{W}>b \qquad \dots (12)$$

Where R_1,R_2 and R_3 are given in the proof.

Proof. i. The eigenvalue in the Y-direction of $J(E_2)$ is: $\lambda_Y=\frac{e\alpha\bar{X}}{1+\alpha T\bar{X}}-d_2$

and the eigenvalues of $J(E_2)$ in the $X-$direction, $Z-$direction, and $W-$direction are roots of the equation:

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \qquad \dots (13)$$

Where:

$A_1=R_1-b$, $A_2=R_2-b(\mu_1+\mu_2+\sigma_2\bar{X})$ and $A_3=R_3-b(\mu_1\mu_2+\mu_2\sigma_2\bar{X})$

With:

$$R_1=d_1+2c_2\bar{X}+\frac{qEm_1}{(m_1+m_2\bar{X})^2}+\sigma_1 W^*+\mu_1+\mu_2+\sigma_2\bar{X}$$

$$R_2=\left(d_1+2c_2\bar{X}+\frac{qEm_1}{(m_1+m_2\bar{X})^2}+\sigma_1\bar{W}\right)(\mu_1+\mu_2+\sigma_2\bar{X})+\mu_2(\mu_1+\sigma_2\bar{X})+\sigma_1\sigma_2\bar{Z}\bar{X}$$

$$R_3=\mu_2\left(d_1+2c_2\bar{X}+\frac{qEm_1}{(m_1+m_2\bar{X})^2}+\sigma_1\bar{W}\right)(\mu_1+\sigma_2\bar{X})+\sigma_2\mu_1\bar{Z}$$

Due to conditions (8-10) guarantee that λ_Y is negative and all Routh-Hurwize criteria $A_1>0, A_3>0$ and $A_1A_2>A_3$,that is all the roots of (13) have negative part.

Proof. ii. Suppose at the parameter p equation satisfied, the eigenvalues of $J(E_2)$ satisfy the following equation:

$$(\lambda^2+A_2)(\lambda+A_1)=0$$

Where, A_2 is positive under condition (12). And hence the roots of Eq. (13) are:

$\lambda_1(p)=i\sqrt{A_2}$, $\lambda_2(f_3)=-i\sqrt{A_2}$, and $\lambda_3(p)=-A_1$

However, there exists a neighborhood$N(\delta,p)$ of p such that for all values of f in (δ,p) , these roots of Eq.13can be written in general as

$\lambda_1=a(p)+ib(p)$, $\lambda_2=a(p)-ib(p)$, and $\lambda_3=-A_3(p)$

Thus, by substituting$\lambda_1=a(f)+ib(f)$ in (13), it is obtained that:

$$\begin{cases} D_1(p)\frac{da(p)}{df} - D_2(p)\frac{db(p)}{dp} + D_3(p) = 0 \\ D_1(p)\frac{db(p)}{dp} + D_2(p)\frac{da(p)}{dp} + D_4(p) = 0 \end{cases} \quad \dots (14)$$

Where:

$D_1(p) = 3\left[(a(p))^2 - (b(p))^2\right] + 2A_2(p)a(p) + B_2(p)$

$D_2(p) = 6a(p)b(p) + 2A_2(p)b(p)$

$D_3(p) = \left[(a(p))^2 - (b(p))^2\right]\frac{dA_3(p)}{dp} + a(p)\frac{dB_2(p)}{dp} + C_2(p)$

$D_4(p) = 2a(p)b(p)\frac{dA_3(p)}{dp} + b(p)\frac{dB_2(p)}{dp}$

Thus, by solving the linear system (14) for the unknown $\frac{dRee(\lambda_1(p))}{dp}$,it gets:

$$\frac{dRee(\lambda_1(p))}{dp} = \frac{da(p)}{dp} = -\frac{D_2(p)D_4(p)+D_1(p)D_3(p)}{[D_1(p)]^2+[D_2(p)]^2}$$

it is easy to verify that:

$$\left[\frac{dRee(\lambda_1(p))}{dp}\right]_{f=f_3} \neq 0$$

The proof is completed.

Theorem 4. Suppose that $E_3(X^*,Y^*,Z^*,W^*)$ is exist, then it is locally asymptotical stable if:

$$max\left\{\frac{\alpha^2TY^*}{(1+\alpha TX^*)^2} + \frac{qEm_2}{(m_1+m_2X^*)^2}, \frac{e\alpha}{(1+\alpha TX^*)^2}\right\} < c_2 \quad \dots(15)$$

$$\left|-c_2 + \frac{\alpha^2TY^*}{(1+\alpha TX^*)^2} + \frac{qEm_2}{(m_1+m_2X^*)^2}\right| > \frac{bf}{(1+fY^*)^2} + \frac{\alpha}{1+\alpha TX^*} + \sigma_1 \quad \dots (16)$$

$$\mu_1 > \sigma_2(Z^* - X^*) \quad \dots (17)$$

$$\mu_2 > \sigma_2(X^* + Z^*) \quad \dots (18)$$

Proof.

$$J(X^*,Y^*,Z^*,W^*) = \begin{bmatrix} -c_2X + \frac{\alpha^2TX^*Y^*}{(1+\alpha TX^*)^2} + \frac{qEm_2X^*}{(m_1+m_2X^*)^2} & \frac{-bfX^*}{(1+fY^*)^2} - \frac{\alpha X^*}{1+\alpha TX^*} & 0 & -\sigma_1X^* \\ \frac{e\alpha Y^*}{(1+\alpha TX^*)^2} & -c_2Y^* & 0 & 0 \\ -\sigma_2Z^* & 0 & -(\mu_1+\sigma_2X^*) & 0 \\ \sigma_2Z^* & 0 & \sigma_2X^* & -\mu_2 \end{bmatrix}$$

If λ is an eigenvalue of $J(X^*,Y^*,Z^*,W^*)$, from the Theorem of Gerschgorin, λ lies within at least one of the following Gershgorin discs.

$$\left|\lambda + c_2X^* - \frac{\alpha^2TX^*Y^*}{(1+\alpha TX^*)^2} - \frac{qEm_2X^*}{(m_1+m_2X^*)^2}\right| \leq \frac{bfX^*}{(1+fY^*)^2} + \frac{\alpha X^*}{1+\alpha TX^*} + \sigma_1X^*$$

$$|\lambda + c_2Y^*| \leq \frac{e\alpha Y^*}{(1+\alpha TX^*)^2}$$

$$|\lambda + \mu_1 + \sigma_2X^*| \leq \sigma_2Z^*$$

$$|\lambda + \mu_2| \leq \sigma_2Z^* + \sigma_2X^*$$

Then, under due the given conditions (15-18) in this theorem, all the eigenvalue has negative real part. This completes the proof.

Note 3. Because of complexity, we did not determine the bifurcation parameter for hop bifurcation around$E_3(X^*,Y^*,Z^*,W^*)$, but in section seven we will do it numerically.

GLOBAL STABILITY

Global stability means that any trajectories finally tend to the attractor of the system, regardless of initial conditions. Therefore, Most of biological systems, especially prey predator system, are needed to be globally stable. The global asymptotically stable stability GAS for each $E_1, E_2,$ and E_3 of the system1 is established in Theorem 5, Theorem 6, and Theorem 7, respectively.

Theorem 5. $E_1\left(0,0,\frac{\pi}{\mu_1},0\right)$ is GAS, if there exist a small positive number ϵ, such that:

$$b + \epsilon\frac{\pi\sigma_2}{\mu_1} < d_1 + \frac{qE}{m_1+m_2X_{max}} \quad \dots (19)$$

Proof. Consider the function $L_1(X,Y,Z,W) = X + Y + \epsilon\left[Z - \frac{\pi}{\mu_1} - \frac{\pi}{\mu_1}\ln\left(\frac{\mu_1Z}{\pi}\right)\right] + \epsilon W$

It is clear that, $L_1(X,Y,Z,W) > 0$, for $(X,Y,Z,W) \in R_+^4$ and $L_1\left(0,0,\frac{\pi}{\mu_1},0\right) = 0$.

Further,

$$\frac{dL_1}{dt} = \frac{bX}{1+fY} - d_1X - c_1X^2 - \frac{\alpha(1-e)XY}{1+\alpha TX} - \frac{qEX}{m_1+m_2X} - \sigma_1XW$$

$$-d_2Y - c_2Y^2 - \epsilon\mu_2W - \epsilon\mu_1\left(Z - \frac{\pi}{\mu_1}\right)^2 + \sigma_2\epsilon\frac{\pi}{\mu_1}X$$

Accordingly,

$\frac{dL_1}{dt} \leq \left(b + \epsilon\frac{\pi\sigma_2}{\mu_1} - d_1 - \frac{qE}{m_1+m_2X}\right)X - \epsilon\mu_1\left(Z - \frac{\pi}{\mu_1}\right)^2$

Thus, $\frac{dL_1}{dt}$ is negative under condition (19), and this completes the proof.

Theorem 6. If $E_2(\bar{X},0,\bar{Z},\bar{W})$ is exist, then it is GAS if,

$$\bar{X} < \frac{d_2}{\alpha} \quad \dots (20)$$

$$F_1 > max\left\{\frac{3b^2f^2}{2\sigma_2}, \frac{3\sigma_2^2\pi^2}{2\mu_1^2\mu_2}\right\} \quad \dots (21)$$

$$F_2 > max\left\{\frac{\sigma_2^2\bar{X}^2}{\mu_2}, \frac{3\sigma_2^2\pi^2}{2\mu_1^2F_1}\right\} \quad \dots (22)$$

$$F_1 = c_1 - \frac{qEm_2}{m_1(m_1+m_2\bar{X})} \quad and \quad F_2 = \sigma_2\bar{X} - \mu_1$$

Proof. Consider the function:

$L_2(X,Y,Z,W) = X - \bar{X} - \bar{X}\ln\left(\frac{X}{\bar{X}}\right)Y + \frac{1}{2}(Z - \bar{Z})^2 + \frac{1}{2}(W - \bar{W})^2$

It is clear that,

$L_2(X,Y,Z,W) > 0$, for $(X,Y,Z,W) \in R_+^4$ and $L_2(\bar{X},0,\bar{Z},\bar{W}) = 0$. Further,

$\frac{dL_2}{dt} = (X - \bar{X})\left[\frac{b}{1+fY} - d_1 - c_1X - \frac{\alpha Y}{1+\alpha TX} - \frac{qE}{m_1+m_2X} - \sigma_1W\right]$

$+ \frac{e\alpha XY}{1+\alpha TX} - d_2Y - c_2Y^2 + (Z - \bar{Z})(\pi - \mu_1Z - \sigma_2XZ)$

$+(W - \bar{W})(\sigma_2XZ - \mu_2W)$

Due to condition (20), it gets:

$\frac{dL_2}{dt} \leq -\frac{1}{3}F_1(X - \bar{X})^2 - \frac{bfY(X-\bar{X})}{1+fY} - c_2Y^2$

$-\frac{1}{3}F_1(X - \bar{X})^2 - \sigma_2Z(X - \bar{X})(Z - \bar{Z}) - \frac{1}{2}F_2(Z - \bar{Z})^2$

$-\frac{1}{3}F_1(X - \bar{X})^2 + \sigma_2Z(X - \bar{X})(W - \bar{W}) - \frac{1}{2}\mu_2(W - \bar{W})^2$

$-\frac{1}{2}F_2(Z - \bar{Z})^2 + \sigma_2\bar{X}(Z - \bar{Z})(W - \bar{W}) - \frac{1}{2}\mu_2(W - \bar{W})^2$

Thus, $\frac{dL_2}{dt} < 0$, if conditions (21), (22) holds and This completes the proof.

Theorem 7. If $E_3(X^*,Y^*,Z^*,W^*)$ is exist, then it is GAS if,

$$F_3 > max\left\{\frac{3b^2f^2}{2k_2c_2}, \frac{3\sigma_2^2\pi^2}{2\mu_1^2\mu_2}\right\} \quad \dots (23)$$

$$F_4 > max\left\{\frac{\sigma_2^2X^{*2}}{\mu_2}, \frac{3\sigma_2^2\pi^2}{2\mu_1^2F_3}\right\} \quad \dots (24)$$

Where:

$F_3 = k_1\left(c_1 - \frac{qEm_2}{m_1(m_1+m_2X^*)} - \frac{\alpha^2TY^*}{1+\alpha TX^*}\right), F_4 = \sigma_2X^* - \mu_1$ and the positive constants k_1 and k_2 satisfy the equation:

$$k_1 = \frac{e}{1+\alpha TX^*}k_2$$

Proof. Consider the functions:

$L_1(X,Y,Z,W) = k_1\left[X - X^* - X^*ln\left(\frac{X}{X^*}\right)\right] + k_2\left[Y - Y^* - Y^*ln\left(\frac{Y}{Y^*}\right)\right] + \frac{1}{2}(Z - Z^*)^2 + \frac{1}{2}(W - W^*)^2$

Clearly, $L_2(X,Y,Z,W) > 0$, for $(X,Y,Z,W) \in R_+^4$ and $L_2(X^*,Y^*,Z^*,W^*) = 0$. Further,

$\frac{dL_2}{dt} = \frac{k_1(X-X^*)}{X}\left[\frac{bX}{1+fY} - d_1X - c_1X^2 - \frac{\alpha XY}{1+\alpha TX} - \frac{qEX}{m_1+m_2X} - \sigma_1XW - k_1\left(\frac{b}{1+fY^*} - d_1 - c_1X^* - \frac{\alpha Y^*}{1+\alpha TX^*} - \frac{qE}{m_1+m_2\bar{X}} - \sigma_1W^*\right)\right]$

$+k_2(Y - Y^*)\left[\frac{e\alpha X}{1+\alpha TX} - d_2 - c_2Y\right]$

$+(Z - \bar{Z})[\pi - \mu_1Z - \sigma_2XZ - (\pi - \mu_1Z^* - \sigma_2X^*Z^*)]$

$+(W - W^*)[\sigma_2XZ - \mu_2W - (\sigma_2X^*Z^* - \mu_2Z^*)]$

$\leq -\frac{1}{3}F_3(X - X^*)^2 - k_1\frac{bf(Y-Y^*)(X-X^*)}{(1+fY)(1+fY^*)} - c_2k_2(Y - Y^*)^2$

$-\frac{1}{3}F_3(X - X^*)^2 - \sigma_2Z(X - X^*)(Z - Z^*) - \frac{1}{2}F_4(Z - Z^*)^2$

$-\frac{1}{3}F_3(X - X^*)^2 + \sigma_2Z(X - X^*)(W - W^*) - \frac{1}{2}\mu_2(W - W^*)^2$

$-\frac{1}{2}F_4(Z - Z^*)^2 + \sigma_2X^*(Z - Z^*)(W - W^*) - \frac{1}{2}\mu_2(W - W^*)^2$

So, $\frac{dL_2}{dt}$ is negative due to the conditions (23,24), and this complete the proof.

NUMERICAL SOLUTION

In order to discover the impact of fear, harvesting and toxicant on system (1), some numerical simulations are performed; all the simulations are carried out through Runge- kutta method of order six [21], using MATLAB. it is observed that system (1) approaches $E_3(1.34,8.77,41.74,58.25)$, when system (1) with the parameter values in (25) as illustrated in Fig1. Also Fig.1 illustrates the analytical finding regarding to stability for E_3, because the parameter valued in (25) satisfies the conditions in Theorem 4 and Theorem 7.

these parameter value satisfy stability conditions for the coexistence equilibrium point, therefore:

$b = 1.5, f = 0.01, d_1 = 0.01, c_1 = 0.01, \alpha = 0.1, T = 1$
$q = 0.01, E = 0.01, m_1 = 1, m_2 = 1, \sigma_1 = 0.01, e = 0.8$ … (25)
$d_2 = 0.01, c_2 = 0.01, \pi = 1, \mu_1 = 0.01, \sigma_2 = 0.01, \mu_2 = 0.01$

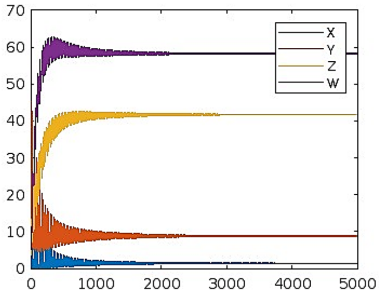

Fig. 1: Trajectory of System (1) when parameter values are as given by (25).

If, we solve system (1) with decreasing the level fear to ($f = 0.001$) and fixed other parameter values as given by (25), then dynamics of system (1) shows periodic solution near the coexistence equilibrium point; see Fig. 2.

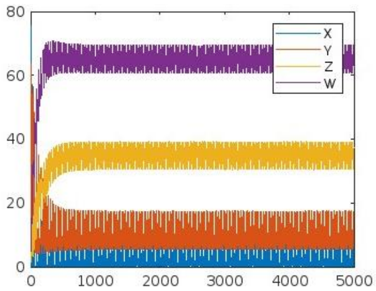

Fig. 2: Periodic oscillations of system (1), when $f = 0.001$ with the rest of parameters are given by (25).

Also if the uptake rate of pollution by organism, decreased to ($\sigma_2 = 0.001$) and fixed other parameter values as given by (25), then system (1) shows larger periodic solution near the coexistence equilibrium point; see Fig.3.

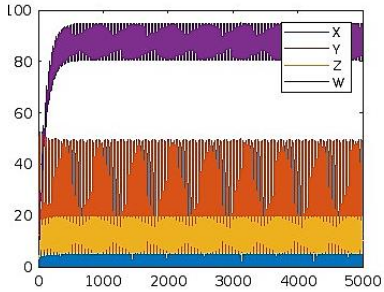

Fig. 3: Time series diagram shows large periodic oscillations around E_3, when $\sigma_2 = 0.001$with therest of parameters are given by (25).

But, if the parameter value of catch ability coefficient of prey and the effort made to harvest prey individual increased to ($q = 0.1$) and ($E = 0.1$) and fixed other parameter values as given by (25), then trajectories of system (1) show Small periodic solution near the coexistence equilibrium point; see Fig.4.

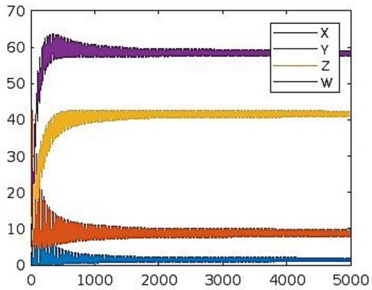

Fig. 4: Small periodic oscillations around E_3, when$q = 0.1, E = 0.1$ with the rest of parameters are given by (25).

Not that the above figures show that dynamics of the model population may induce a transition from the a stability situation to the state where the populations oscillate periodically or induce a transition from oscillate periodically situation to the stable situation.

DISCUSSION AND CONCLUSIONS

In this paper, it has been proposed an ecological system consisting of two interacting species (prey and predator). The reproduction of prey species was affected by fear from predators and the effect of harvesting and environmental pollution on prey population are also considered. The boundedness of the model solutions is guaranteed. Two criteria that make system (1) permanent are determined. All the criteria for locally as well as globally stability for the model equilibrium points are found. It is observed that criteria for both persistence and stable status, including parameters relative to the fear of predators in prey, environmental pollution and harvesting on prey population. Numerical computation showed that dynamics of the model population may induce a transition from the a stability situation to the state where the populations oscillate periodically or induce a transition from oscillate periodically situation to the stable situation if we change the parameter values of level fear, uptake rate of pollution catch ability coefficient of prey and the effort made to harvest prey individual.

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REFERENCES

1. Fatah S, Mustafa A, Amin S. Predator and n-classes-of-prey model incorporating extended Holling type II functional response for n different prey species. AIMS MATHEMATICS. 2023;8(3):5779-88. https://doi.org/10.3934/math.202329

2. He M, Li Z. Stability of a fear effect predator–prey model with mutual interference or group defense. Journal of Biological Dynamics. 2022;16(1):480-98. https://doi.org/10.1080/17513758.2022.2091800

3. Zanette LY, White AF, Allen MC, Clinchy M. Perceived predation risk reduces the number of offspring songbirds produce per year. Science. 2011;334(6061):1398-401. https://doi.org/10.1126/science.1210908

4. Cong P, Fan M, Zou X. Dynamics of a three-species food chain model with fear effect. Communications in Nonlinear Science and Numerical Simulation. 2021;99:105809. https://doi.org/10.1016/j.cnsns.2021.105809

5. Jamil ARM, Naji RK. Modeling and Analyzing the Influence of Fear on the Harvested Modified Leslie-Gower Model. Baghdad Science Journal. 2023;20(5):1701-. https://dx.doi.org/10.21123/bsj.2023.ID

6. Muhammed JO, Hassan KA. Bifurcation Analysis in a Discrete-Time Prey-Predator System with Crowly-Martin Functional Response. Tikrit Journal of Pure Science. 2023;28(5):129-46. https://doi.org/10.25130/tjps.v28i5.1588

7. Naji RK, Mustafa AN. The dynamicics of a prey-predator model with the existence of disease and pollution. J Math Comput Sci. 2013;3(1):94-123.

8. Pal S, Pal N, Samanta S, Chattopadhyay J. Effect of hunting cooperation and fear in a predator-prey model. Ecological Complexity. 2019;39:100770. https://doi.org/10.1016/j.ecocom.2019.100770

9.Tan Y, Cai Y, Yao R, Hu M, Wang W. Complex dynamics in an eco-epidemiological model with the cost of anti-predator behaviors. Nonlinear Dynamics. 2022;107(3):3127-41. https://doi.org/10.1007/s11071-021-07133-4

10. Wang X, Zanette L, Zou X. Modelling the fear effect in predator–prey interactions. Journal of mathematical biology. 2016;73(5):1179-204. https://doi.org/10.1007/s00285-016-0989-1

11. Zhang H, Cai Y, Fu S, Wang W. Impact of the fear effect in a prey-predator model incorporating a prey refuge. Applied Mathematics and Computation. 2019;356:328-37. http://dx.doi.org/10.1016/j.amc.2019.03.034.

12. Lavanya R, Vinoth S, Sathiyanathan K, Njitacke Tabekoueng Z, Hammachukiattikul P, Vadivel R. Dynamical Behavior of a Delayed Holling Type-II Predator-Prey Model with Predator Cannibalism. Journal of Mathematics. 2022;2022(1):4071375. https://doi.org/10.1155/2022/4071375

13. Liu J, Lv P, Liu B, Zhang T. Dynamics of a Predator-Prey Model with Fear Effect and Time

Delay. Complexity. 2021;2021(1):9184193. https://doi.org/10.1155/2021/9184193

14. Pal S, Tiwari PK, Misra AK, Wang H. Fear effect in a three-species food chain model with generalist predator. Math Biosci Eng. 2024;21(1):1-33. http://dx.doi.org/10.3934/mbe.2024001

15. Bergland H, Wyller J, Burlakov E. Pasture–livestock dynamics with density-dependent harvest and changing environment. Natural Resource Modeling. 2019;32(4):e12213. https://doi.org/10.1111/nrm.12213

16. Brauer F, Soudack A. Stability regions and transition phenomena for harvested predator-prey systems. Journal of Mathematical Biology. 1979;7(4):319-37. https://doi.org/10.1007/BF00275152

17. Hu D, Cao H. Stability and bifurcation analysis in a predator–prey system with Michaelis–Menten type predator harvesting. Nonlinear Analysis: Real World Applications. 2017;33:58-82. https://doi.org/10.1016/j.nonrwa.2016.05.010

18. MUSTAFA AN, AMIN SF. A harvested modified Leslie-Gower predator-prey model with SIS-disease in predator and prey refuge. Journal of Duhok University. 2019;22(2):174-84. https://doi.org/10.26682/sjuod.2019.22.2.20

19. Wuhaib S. Continuous Threshold Harvesting Intermediate Predator in Food Chain Model. Tikrit Journal of Pure Science. 2017;22(11):96-101. https://doi.org/10.25130/tjps.v22i11.922

20. Wuhaib S, Mansour M. Dynamic Prey Predator Model and multiple forms of Harvest of Infected Prey. Tikrit Journal of Pure Science. 2019;24(4):4. https://doi.org/10.25130/tjps.v24i4.406.

21. Agarwal M, Devi S. A resource-dependent competition model: Effects of toxicants emitted from external sources as well as formed by precursors of competing species. Nonlinear Analysis: Real World Applications. 2011;12(1):751-66. http://dx.doi.org/10.1016/j.nonrwa.2010.08.003

22. Dubey B, Hussain J. Modelling the interaction of two biological species in a polluted environment. Journal of Mathematical Analysis and Applications. 2000;246(1):58-79. https://doi.org/10.1006rjmaa.2000.6741

23. Freedman H, Shukla J. Models for the effect of toxicant in single-species and predator-prey systems. Journal of mathematical biology. 1991;30(1):15-30. https://doi.org/10.1007/BF00168004

24. Huaping L, Zhien M. The threshold of survival for system of two species in a polluted environment. Journal of Mathematical Biology. 1991;30:49-61. https://doi.org/10.1007/bf00168006