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RESEARCH ARTICLE - MATHEMATICS

The action of Hom space and tensor product of Lie algebras

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Abstract

The primary goal of this study is to explore the Triple Action of Representation Lie Algebras (ATHS) with unique characteristics on Lie algebra using Schur's lemma, which states that the action of the tensor product of Lie algebras representation has interesting property. This involves examining the interaction of Lie algebra representations, focusing on specific actions within Lie algebras. A new approach to the tensor product and direct sum for Lie algebra representations has been introduced. Additionally, this study explores the relationship between actions of Hom space and tensor product. The main goal is to achieve results on Lie algebra representations by using a new structure of six representations and their corresponding actions.

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***Keywords**:* Tensor product, direct sum, dual of representation of Lie algebras, action of Lie Algebra.

1. Introduction and Preliminaries

Lie algebras Li are defined as a vector space over a field, equipped with a bilinear multiplication called the bracket, which must satisfy: bilinear, Jacobi identity and anti-symmetric [1], [2]. Let (Li_1, [,]) and (Li_2, [,]) be two Lie algebras over the same field k, a homomorphism to Lie algebras from Li into Li_2 is a linear function such that y ([p, q]) = y(p) y(q) – y(q) y(p), for all p, q Є Li_1, if in y is differentiable and inclusive, then y is called a Lie algebra isomorphism [3].

A Lie algebra Li can be represented by a real or complex vector space m, which is finite-dimensional, through a homomorphism Li → gl(M) of Lie algebras. This representation, denoted by y: Li → gl(M), is characterized by being a linear function that fulfills the condition y([p, q]) = y(p) y(q) – y(q) y(p) [4], [5]. Two Lie algebras, Li_1 and Li_2, are being discussed along with their respective representations, y_1 and y_2, which operate on vectors M_1 and M_2. The tensor product of these representations, denoted as $y_1 \otimes y_2$, creates a new representation of $y_1 \times y_2$ that acts on the combined vector $M_1 \otimes M_2$. This operation is defined as $y_1 \otimes y_2(p, q) = y_1(p) \otimes I + I \otimes y_2(q)$ for all p in Li_1 and q in Li_2 [6], [7], [8]. If Li_1=Li_2, then let y_1 and y_2 symbolize vector spaces M_1 and M_2, respectively. The tensor product of y_1 and y_2, written as y_1 y_2, serves as a representation of Li acting on the vector space M_1, M_2. This can be defined as: (y_1 y_2)(p) = y_1(p) I + I y_2(p) [9]. If yM_1: Li→gl(M_1) and yM_2: Li→gl(M_2) are two representations for the Lie algebra

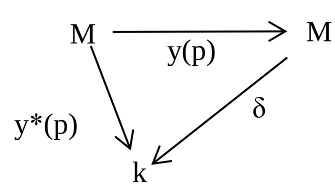
Li, then a direct sum of these representations is defined as $yM_1 \oplus yM_2$: Li→gl ($M_1 \oplus M_2$). This means that $(yM_1 \oplus yM_2)(p) = (yM_1(p), yM_2(p))$ [10]. If y represents the Lie algebra Li acting on a vector space M of finite dimensions, the dual form of y is when Li acts on the dual space M*, where defined as $y^*(p) = -(y(p))^{tr}$. This dual representation is also known as the contragradient representation [11], [12]. A multiplicative Hom-Lie algebra is defined as a set comprising a vector space M, a skew-symmetric bilinear map (bracket) [p, q], and the linear map yM: M →M that maintains the bracket. The Lie algebra must satisfy the Hom-Jacobi identity in relation to y. A Hom-Lie algebra (M, [p, q], y) is classified as a regular Hom-Lie algebra if y is the reversible map [13]. Note: $gl(n,\mathbb{R})$ is general linear algebra is the space for all n × n real matrices where trace = 0 and $so(n, \mathbb{R})$ is the special orthogonal algebra such that: $so(n, \mathbb{R}) = \{ \mathcal{X} \in gl(n,\mathbb{R}): \mathcal{X} + \mathcal{X}^{tr} = 0\}$ where $\mathcal{X}$ is a matrix for n × n degree. See [14] [15]

2. The result of the action for representation of Hom-space and tensor product

Schur's lemma is defined as the idea of the operation on the tensor product of two Lie algebra representations. In this lemma, it is assumed that y_1 and y_2 are two representations of Lie algebras Li that act on finite-dimensional vector spaces M_1 and M_2 respectively. An action is then defined for Li on the space of linear transformations from M_1 to M_2, denoted as Hom(M_1, M_2), where y: Li → gl(Hom(M_1, M_2)), $y(p)\vartheta = y_1(p)\vartheta - \vartheta\, y_2(p)$, for all p in Li and q in Hom(M_1, M_2), with Hom(M_1, M_2) being considered as an equivalence between M_2 and M_1. see [16] [17].

Proposition (2.1) [18]

Let y:Li→gl(M) is the Lie algebra Li representation effect on the k-finite dimensional vector space M, then y*: Li→ gl(M*) the dual representation on M* which is given by $y^*(p) = \delta(y(p))$, for all p Є Li, where δ: M → k see the figure -1.



Theorem (2.2)

Let M_j,j=1,2,3,4,5,6 is six representation of Lie algebra acting on vector spaces M_j respectively and let $Hom_k(Hom_k(M_4^*, M_5 \oplus M_6), (Hom_k(M_2, M_3^*) \oplus Hom_k(M_1, M_3^*))$ be k-vector space of all linear maps from $(Hom_k(M_4^*, M_5 \oplus M_6)$ into $(Hom_k(M_2 \oplus M_1, M_3^*)$. define δ: Li→ gl($Hom_k(Hom_k(M_4^*, M_5 \oplus M_6), (Hom_k(M_2 \oplus M_1, M_3^*)))$) such that:

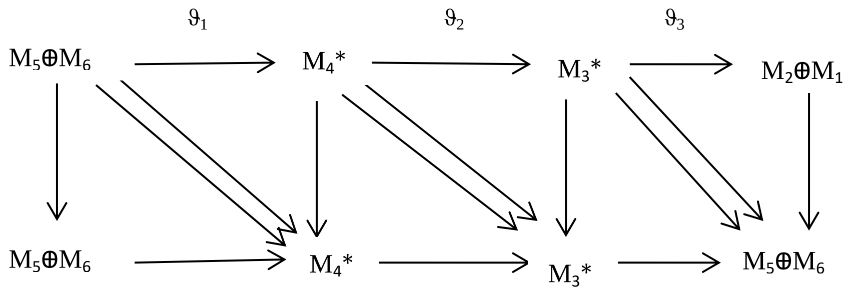
$\delta(p)\, \vartheta = ((y_1 \oplus y_2)(p)\, \vartheta_1 - \vartheta_1 y_3^*(p)\, \vartheta_2 - \vartheta_2(y_4^*(p)\, \vartheta_3 - \vartheta_3\, (y_6 \oplus y_5)(p))$.

Then the δ is a representation of Lie algebra on $Hom_k(Hom_k(M_4^*, M_5 \oplus M_6), (Hom_k(M_2, M_3^*) \oplus Hom_k(M_1, M_3^*))$.

Proof:

The following **figure-2** can be used to show that the action for Lie algebra Li to $Hom_k(Hom_k(M_4^*, M_5 \oplus M_6), (Hom_k(M_2, \oplus M_1, M_3^*))$ is as follows:

$\delta(p)\vartheta = ((y_1 \oplus y_2)(p)\, \vartheta_1 - \vartheta_1 y_3^*(p)\, \vartheta_2 - \vartheta_2(y_4^*(p)\, \vartheta_3 - \vartheta_3\, (y_6 \oplus y_5)(p))$



Now to prove δ is a representat f Lie algebra, we t prove tha a homomorphism of Lie algebra into $gl(Hom_k(Hom_k(M_4^*, M_5 \oplus M_6), (Hom_k(M_2 \oplus M_1, M_3^*)))$ by definition of Lie algebra homomorphism we must to prove δ is a linear function and satisfies the following:

$y([p,q])\vartheta = y(p)\,(y(q)\vartheta) - y(q)\,(y(p)\vartheta)$ for all $p, q \in Li$

i.e to prove:

$\delta(\alpha p + \beta q)\,\vartheta = \alpha\,\delta(p)\,\vartheta + \beta\,\delta(q)\,\vartheta$

Now: $\delta(\alpha p + \beta q)\,\vartheta = [(y_1 \oplus y_2)\,(\alpha p + \beta q)\vartheta_1 - \vartheta_1 y_3^*(\alpha p + \beta q)]\vartheta_2 - \vartheta_2\,[y_4^*\,(\alpha p + \beta q)\,\vartheta_3 - \vartheta_3(y_6 \oplus y_5)\,(\alpha p + \beta q)]$

$= [\,(\alpha\,(y_1 \oplus y_2)\,(p) + \beta(y_1 \oplus y_2)\,(q))\,\vartheta_1 - \vartheta_1(\alpha\, y_3^*(p) + \beta y_3^*(q))]\,\vartheta_2 - \vartheta_2\,[(\alpha\, y_4^*(p) + \beta y_4^*(q))\,\vartheta_3 - \vartheta_3((\alpha\,(y_6 \oplus y_5)\,(p) + \beta(y_6 \oplus y_5)\,(q))]$

$= [(\alpha\,(y_1 \oplus y_2)\,(p))\,\vartheta_1 - \vartheta_1(\alpha\, y_3^*(p)) + ((\beta(y_1 \oplus y_2)\,(q))\,\vartheta_1 - \vartheta_1\,(\beta y_3^*)(q))]\,\vartheta_2 - \vartheta_2[(\alpha\, y_4^*(p))\,\vartheta_3 - \vartheta_3(\alpha\,(y_6 \oplus y_5)\,(p)) + ((\beta\, y_4^*(q))\,\vartheta_3 - \vartheta_3\,(\beta(y_6 \oplus y_5))(q))]$

$[\alpha\,((y_1 \oplus y_2)\,(p))\,\vartheta_1 - \vartheta_1(y_3^*(p))) + \beta((y_1 \oplus y_2)\,(q))\,\vartheta_1 - \vartheta_1\,(y_3^*)(q)))]\,\vartheta_2 - \vartheta_2\,[(\alpha(y_4^*(p))\,\vartheta_3 - \vartheta_3((y_6 \oplus y_5)\,(p))) + \beta(y_4^*(q))\,\vartheta_3 - \vartheta_3\,((y_6 \oplus y_5))(q)))]$

$= \alpha\,\delta(p) + \beta\,\delta(q)$, then it is linear function

Now to prove $\delta([p, q])\vartheta = \delta(p)\,(\delta(q)\vartheta) - \delta(q)\,(\delta(p)\vartheta)$

The left hand side is

$\delta([p,q])\,\vartheta = ((y_1 \oplus y_2)\,[p,q]\,\vartheta_1 - \vartheta_1\, y_3^*\,[p,q])\,\vartheta_2 - \vartheta_2\,(y_4^*[p, q]\,\vartheta_3 - \vartheta_3\,(y_6 \oplus y_5)[p,q])$

$= [((y_1 \oplus y_2)\,(p)\,(y_1 \oplus y_2)\,(q) - (y_1 \oplus y_2)\,(q)\,(y_1 \oplus y_2)\,(p))\,\vartheta_1 - \vartheta_1\,(y_3^*\,(p)\,y_3^*\,(q) -$

$y_3^*(q)\,y_3^*(p))]\,\vartheta_2 - \vartheta_2\,[(y_4^*(p)\,y_4^*(q) - y_4^*(q)\,y_4^*(p))\,\vartheta_3 - \vartheta_3\,((y_6 \oplus y_5)\,(p)\,(y_6 \oplus y_5))\,(q) - (y_6 \oplus y_5)\,(q)\,(y_6 \oplus y_5))\,(p))]$

$= [((y_1 \oplus y_2)\,(p)\,(y_1 \oplus y_2)\,(q)\,\vartheta 1 - (y_1 \oplus y_2)\,(q)\,(y_1 \oplus y_2)\,(p)\,\vartheta_1) - (\vartheta_1\, y_3^*(p)\,y_3^*(q) - \vartheta_1\, y_3^*(q)\,y_3^*(p)))]\vartheta_2 - \vartheta_2\,[((y_4^*(p)\,y_4^*(q)\,\vartheta_3 - y_4^*(q)\,y_4^*(p)\,\vartheta_3) - (\vartheta_3\,(y_6 \oplus y_5)\,(p)\,(y_6 \oplus y_5)\,(q) - \vartheta_3\,(y_6 \oplus y_5)\,(q)\,((y_6 \oplus y_5)\,(p)))]$

$= [((y_1 \oplus y_2)\,(p)\,(y_1 \oplus y_2))\,(q)\,\vartheta_1 - \vartheta_1 y_3^*(p) y_3^*(q)) - ((y_1 \oplus y_2)\,(q)\,(y_1 \oplus y_2)\,(p)\,\vartheta_1 - \vartheta_1\, y_3^*(q)\,y_3^*(p))]\,\vartheta_2 - \vartheta_2\,[(y_4^*(p)\,y_4^*(q)\vartheta_3 - \vartheta_3\,(y_6 \oplus y_5)\,(p)\,(y_6 \oplus y_5)\,(q)) - (y_4^*(q)\,y_4^*(p)\,\vartheta_3 - \vartheta_3\,(y_6 \oplus y_5)\,(q)\,(y_6 \oplus y_5)\,(p))]$

And the right hand side is:

$\delta(p)\,(\delta(q)\vartheta) - \delta(q)(\delta(p)\,\vartheta) = [(y_1 \oplus y_2)\,(p)\,(\delta\,(q)\,\vartheta_1 - (\delta(q)\,\vartheta_1) y_3^*(p)) - ((y_1 \oplus y_2)\,(q)\,(\delta(p)\,\vartheta_1) - (\delta(p))\,y_3^*(q)]\,\vartheta_2 - \vartheta_2\,[\,(y_4^*(p)\,(\delta\,(q)\,\vartheta_3) - (\delta(q)\,\vartheta_3)\,(y_6 \oplus y_5)(p)) - (y_4^*(q)\,(\delta(p)\,\vartheta_3) - (\delta(p))\,(y_6 \oplus y_5)\,(q)]$

$= [(y_1 \oplus y_2)\,(p)\,((y_1 \oplus y_2)(q)\,\vartheta_1 - \vartheta_1\, y_3^*(q)) - ((y_1 \oplus y_2)\,(q)\,\vartheta_1 - \vartheta_1\, y_3^*(q))\,y_3^*(p)) -$

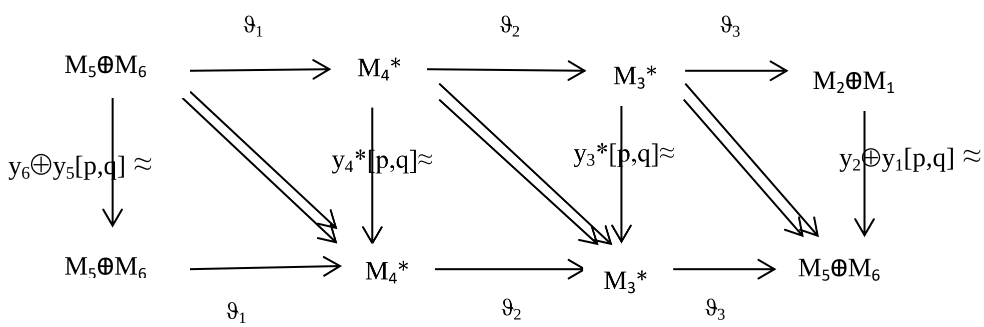
$(y_1\oplus y_2)(q)$ $((y_1\oplus y_2)$ (p) $\vartheta_1 - \vartheta_1$ $y_3^*(p)) - ((y_1\oplus y_2)$ (p) $\vartheta_1 - \vartheta_1$ $y_3^*(p))$ $y_3^*(q))]$ $\vartheta_2 - \vartheta_2$ $[y_4^*(p)$ $(y_4^*$ (q) $\vartheta_3 - \vartheta_3$ $(y_6\oplus y_5)$ $(q)) - (y_4^*(q)$ $\vartheta_1 - \vartheta_1$ $(y_6\oplus y_5)$ $(q))$ $(y_6\oplus y_5)$ $(p)) - y_4^*(q)$ $(y_4^*$ (p) $\vartheta_1 - \vartheta_1$ $(y_6\oplus y_5)$ $(p)) - (y_4^*$ (p) $\vartheta_1 - \vartheta_1$ $(y_6\oplus y_5)$ $(p))$ $(y_6\oplus y_5)$ $(q))]$

$= [((y_1\oplus y_2)$ (p) $(y_1\oplus y_2)$ $(q))$ $\vartheta_1 - (y_1\oplus y_2)$ (p) ϑ_1 $y_3^*(q) - (y_1\oplus y_2)$ (q) ϑ_1 $y_3^*(p) + \vartheta_1(y_3^*(q)$ $(y_1\oplus y_2)(p)) - (((y_1\oplus y_2)$ (q) $((y_1\oplus y_2)$ $(p))$ $\vartheta_1 + (((y_1\oplus y_2)$ (q) ϑ_1 $y_3^*(p) + (y_1\oplus y_2)$ $(p))$ ϑ_1 $y_3^*(q) - \vartheta_1(y_3^*(p)$ $y_3^*(q))]$ $\vartheta_2 - \vartheta_2[(y_4^*$ (p) $y_4^*(q))$ $\vartheta_1 - y_4^*(p)$ ϑ_1 $(y_6\oplus y_5)$ $(q) - y_4^*(q)$ ϑ_1 $(y_6\oplus y_5)$ $(p) + \vartheta_1((y_6\oplus y_5)$ (q) $y_4^*(p)) - ((y_4^*(q)$ $(y_4^*(p))$ $\vartheta_1 + ((y_4$ (q) ϑ_1 $(y_6\oplus y_5)$ $(p) + y_4^*(p))$ ϑ_1 $(y_6\oplus y_5)$ $(q) - \vartheta_1((y_6\oplus y_5)$ (p) $(y_6\oplus y_5)$ $(q))]$

$= [((y_1\oplus y_2)$ (p) $(y_1\oplus y_2))$ (q) $\vartheta_1 - \vartheta_1 y_3^*(p)y_3^*(q)) - ((y_1\oplus y_2)(q)$ $(y_1\oplus y_2)$ (p) $\vartheta_1 - \vartheta_1$ $y_3^*(q)$ $y_3^*(p))]$ $\vartheta_2 - \vartheta_2$ $[(y_4^*(p)$ $y_4^*(q)\vartheta_3 - \vartheta_3$ $(y_6\oplus y_5)(p)$ $(y_6\oplus y_5)$ $(q)) - (y_4^*(q)$ $y_4^*(p)$ $\vartheta_3 - \vartheta_3(y_6\oplus y_5)$ (q) $(y_6\oplus y_5)$ $(p))]$

Then the left hand side is equal to right hand side, which means that δ is a representation of Lie algebra Li.

The following **figure-3** can be used to show that δ is the Lie algebra homomorphism.



Remark (2.3)

The map δ above is called action of Lie algebra on Hom_k $(Hom_k(M_4^*,M_5\oplus M_6),(Hom_k(M_2\oplus M_1, M_3^*))$.

Theorem (2.4)

Let δ: $Li \to gl$ $((M_4^*\otimes M_5\oplus M_6)$ $\otimes((M_2\oplus M_1\otimes M_3^*)))$ which is the representation of Li acting on the vector space $(M_4^*\otimes M_5\oplus M_6)$ $\otimes((M_2\oplus M_1\otimes M_3^*))$ and defined by:

$\delta(p)$ $((V_4^*\otimes V_5\oplus V_6)\otimes(V_3^*\otimes V_2\oplus V_1)) = [((y_2\oplus y_1)$ $(p)\otimes I + I\otimes y_3^*(p))]$ $I + I\otimes[(y_4^*$ $(p)\otimes I + I\otimes$ $(y_5\oplus y_6)$ $(p))]$ $((V_4^*\otimes V_5\oplus V_6)\otimes((V_3^*\otimes V_2\oplus V_1))$

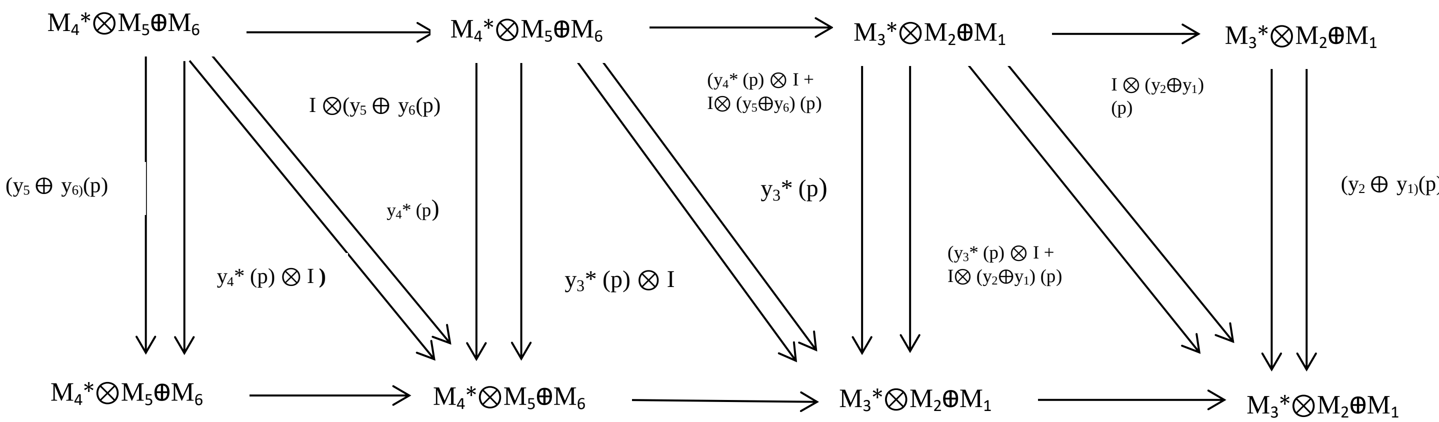
$(\delta(p)$ $((V_4^*\otimes V_5\oplus V_6)\otimes(V_3^*\otimes V_2\oplus V_1)))(r) = [((y_2\oplus y_1)(p)$ $(V_3^*(r)\otimes V_2\oplus V_1) + (V_2\oplus V_1)(r)\otimes y_3^*(p)V_3^*] + [y_4^*(p)$ $(V_4^*\otimes(r)V_5\oplus V_6) + (V_5\oplus V_6)(r)\otimes(y_5\oplus y_6)(p)$ $V_4^*]$.

Then the representation of Lie algebra on $((M_4^*\otimes M_5\oplus M_6)\otimes((M_2\oplus M_1\otimes M_3^*))$ becomes:

$(\delta(p)$ $((V_4^*\otimes V_5\oplus V_6)\otimes(V_3^*\otimes V_2\oplus V_1)))(r) = [-(V_2\oplus V_1)((y_2\oplus y_1)(p)$ $(r))V_3^* + (V_2\oplus V_1)(r)$ $y_3^*(p)V_3^* - V_4^*(y_4^*(p)$ $(r))V_5\oplus V_6 + V_4^*(r)$ $(y_5\oplus y_6)$ $(V_5\oplus V_6)$

Proof:

The following figure-4 shows that the action of Lie algebra Li on tensor product as follows:



To show that δ is a representation of Li acting on $gl((M_4^*\otimes M_5\oplus M_6) \otimes((M_2\oplus M_1\otimes M_3^*))$.

Where $\delta(p) ((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))$ is the linear map.

I.e to prove: $(\delta(\alpha p + \beta q)) ((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1)) = (\alpha\delta(p))$

$((V_4^*\otimes V_5\oplus V_6) \otimes(V3^*\otimes V2\oplus V1)) + (\beta\delta(q)) ((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))$

$= [(y_2\oplus y_1)(\alpha p + \beta q)\otimes I + I \otimes y_3^*(\alpha p + \beta q)]\otimes I + I \otimes [y_4^*(\alpha p+ \beta q)\otimes I +I\otimes(y_6\oplus y_5)$

$(\alpha p+ \beta q)] ((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))$

$= ([(\alpha(y_2\oplus y_1)(p) + \beta(y_2\oplus y_1) (q)) \otimes I+ I \otimes (\alpha y_3^*(p) + \beta y_3^*(q)))(V_3^*\otimes V_2\oplus V_1)] \otimes I + I \otimes [(\alpha y_4^*(p) + \beta y_4^*(q)) \otimes I+ I \otimes (\alpha (y_6\oplus y_5)(p) + \beta (y_6\oplus y_5)(q))) (V_4^*\otimes V_5\oplus V_6)]$

$=[(\alpha(y_2\oplus y_1) (p) \otimes I+\beta((y_2\oplus y_1) (q) \otimes I) +(I \otimes \alpha y_3^*(p)+ I \otimes \beta y_3^*(q))] (V_3^*\otimes V_2\oplus V_1)] \otimes I+ I \otimes [(\alpha y_4^*(p) \otimes I+\beta y_4^* (q) \otimes I) + (I \otimes \alpha (y_6\oplus y_5) (p) + I \otimes \beta (y_6\oplus y_5) (q)) (V_4^*\otimes V_5\oplus V_6)]$

$[(\alpha(y_2\oplus y_1) (p)V_2\oplus V_1)\otimes V_3^*+(\beta((y_2\oplus y_1) (q)V_2\oplus V_1) \otimes(V_3^*\otimes V_2\oplus V_1) \otimes(\alpha y_3^*(p)V_3^*) + V_2\oplus V_1\otimes(\beta y_3^*(q)V_3^*)] +[(\alpha y_4^*(p)V_4^*) \otimes V_5\oplus V_6 +(\beta(y_4^*(q)V_4^*) \otimes V_5\oplus V_6+ V_4^*\otimes(\alpha (y6\oplus y5) (p)V_5\oplus V_6) + V_4^* \otimes(\beta (y_6\oplus y_5) (q)V_5\oplus V_6)]$

$= [((\alpha(y_2\oplus y_1) (p)V_2\oplus V_1) \otimes V_3^* +V_2\oplus V_1 \otimes(\alpha y_3^*(p) V_3^*)) + ((\beta((y_2\oplus y_1) (q)V_2\oplus V_1) \otimes V_3^* +V_2\oplus V_1 \otimes(\beta y_3^*(q) V_3^*)] + [((\alpha y_4^*(p)V_4^*) \otimes V_5\oplus V_6 +V_4^*\otimes(\alpha (y_6\oplus y_5) (p)V_5\oplus V_6)+(\beta(y_4^*(q)V_4^*) \otimes V_5\oplus V_6 +V_4^* \otimes(\beta (y_6\oplus y_5) (q)V_5\oplus V_6))]$

$= (\alpha\delta(p) (V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1) + (\beta\delta(q)) (V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1)$

And second, we prove:

$\delta([p, q])((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))=\delta(p)(\delta(q)((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))) - \delta(q)(\delta(p)((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1)))$

Thus, the left hand side is:

$\delta ([p, q]))((V_4^*\otimes V_5\oplus V_6) \otimes(V_3^*\otimes V_2\oplus V_1))=(((y_2\oplus y_1) ([p, q]) \otimes I+I\otimes y_3^* ([p, q]))\otimes I+ I\otimes (y_4^* ([p, q]) \otimes I +I \otimes(y_5\oplus y_6) ([p, q])))((V_4^*\otimes V_5\oplus V_6) \otimes((V_3^*\otimes V_2\oplus V_1))$

$= [(y_2\oplus y_1) (p) y_2\oplus y_1) (q) - y_2\oplus y_1) (q) y_2\oplus y_1) (p)) \otimes I+ I\otimes (y_3^*(p) y_3^*(q) - y_3^*(q) y_3^*(p))] \otimes I + I \otimes[(y_4^*(p)y_4^*(q) - y_4^*(q)y_4^*(p)) \otimes I + I \otimes((y_6\oplus y_5) (p) (y_6\oplus y_5) (q) - (y_6\oplus y_5) (q) (y_6\oplus y_5)(p)]$

$= [((y_2\oplus y_1)(p)((y_2\oplus y_1)(q)V_2\oplus V_1)\otimes V_3^* - ((y_2\oplus y_1)(q)((y_2\oplus y_1)(p)V_2\oplus V_1)\otimes V_3^* + (V_2\oplus V_1\otimes (y_3^*(p)\, y_3^*(q)V_3^* - (V_2\oplus V_1\otimes (y_3^*(q)\, y3^*(p)V_3^*)] + [(y_4^*(p)(y_4^*(q)\, V_4^*)\otimes V_5\oplus V_6) - (y_4^*(q)(y_4^*(p)V_4^*)\otimes V_5\oplus V_6 + (V_4^*\otimes ((y_5\oplus y_6)(p)(y_5\oplus y_6)(q)V_5\oplus V_6) - (V_4^*\otimes((y_5\oplus y_6)(q)(y5\oplus y6)(p)V_5\oplus V_6)]$

$= [((y_2\oplus y_1)(p)((y_2\oplus y_1)(q)V_2\oplus V_1)\otimes V_3^* + (V_2\oplus V_1\otimes (y_3^*(p)\, y_3^*(q)V_3^*) - ((y_2\oplus y_1)(q)((y_2\oplus y_1)(p)V_2\oplus V_1)\otimes V_3^* + (V_2\oplus V_1\otimes(y_3^*(q)\, y_3^*(p)V_3^*)] + [(y_4^*(p)(y_4^*(q)\, V_4^*)\otimes V_5\oplus V_6)+(V_4^*\otimes ((y_5\oplus y_6)(p)(y_5\oplus y_6)(q)V_5\oplus V_6) - ((y_4^*(q)(y_4^*(p)V_4^*)\otimes V_5\oplus V_6) + (V_4^*\otimes((y_5\oplus y_6)(q)(y_5\oplus y_6)(p)V_5\oplus V_6)]$

And the right hand side is $\delta(p)\,(\delta(q)((V_4^*\otimes V_5\oplus V_6)\otimes(V_3^*\otimes V_2\oplus V_1))) - \delta(q)\,(\delta(p)((V_4^*\otimes V_5\oplus V_6)\otimes(V3^*\otimes V2\oplus V1))$

$= [(y_2\oplus y_1)(p)(\delta(q)V_2\oplus V_1 \otimes(V_3^*\otimes V_2\oplus V_1)\otimes y_3^*(q)V_3^* - (y_2\oplus y_1)(q)(\delta(p)V_2\oplus V_1 \otimes V_3^*) - (\delta(p)V_2\oplus V_1 \otimes V_3^*)\, y_3^*(q)] + [y_4^*(p)\,(\delta(q)\,(V_4^*\otimes V_5\oplus V_6)+V_4^*\otimes(y_5\oplus y_6)(q)\, V_5\oplus V_6 - y_4^*(q)(\delta(p)V_4^*\otimes V_5\oplus V_6) - (\delta(p)\,(V_4^*\otimes V_5\oplus V_6)(y_5\oplus y_6)(q)]$

$= [(y_2\oplus y_1)(p)((y_2\oplus y_1)(q)V2\oplus V_1 \otimes V_3^*+V_2\oplus V_1 \otimes y_3^*(q)V_3^*) + ((y_2\oplus y_1)(q)V_2\oplus V_1 \otimes V_3^*+ V_2\oplus V_1\otimes y_3^*(q)V_3^*)\, y_3^*(p) - (y_2\oplus y_1)(q)((y_2\oplus y_1)(p)V_2\oplus V_1 \otimes V_3^* - V_2\oplus V_1\otimes y_3^*(p)V_3^*) - ((y_2\oplus y_1)(p)V_2\oplus V_1 \otimes V_3^* - V_2\oplus V_1\otimes y_3^*(p)V_3^*)\, y_3^*(q)] + [y_4^*(p)(y_4^*(q)\,(V_4^*\otimes V_5\oplus V_6) +V_4^*\otimes(y_5\oplus y_6)(q)V_5\oplus V_6)+(y_4^*(q)V_4^*\otimes V_5\oplus V_6 + V_4^*\otimes(y_5\oplus y_6)(q)V_5\oplus V_6)(y_5\oplus y_6)(p) - y_4^*(q)(y_4^*(p)\,(V_4^*\otimes V_5\oplus V_6) - V_4^*\otimes(y_5\oplus y_6)(p)V_5\oplus V_6) - (y_4^*(p)\,(V_4^*\otimes V_5\oplus V_6) - V_4^*\otimes (y_5\oplus y_6)(p)V_5\oplus V_6)(y_5\oplus y_6)(q)]$

$= [((y_2\oplus y_1)(p)((y_2\oplus y_1)(q)V_2\oplus V_1)\otimes V_3^* + (V_2\oplus V_1\otimes (y_3^*(p)\, y_3^*(q)V_3^*) - ((y_2\oplus y_1)(q)(((y_2\oplus y_1)(p)V_2\oplus V_1)\otimes V_3^* + (V_2\oplus V_1\otimes(y_3^*(q)\, y_3^*(p)V_3^*)] + [(y_4^*(p)(y_4^*(q)V_4^*)\otimes V_5\oplus V_6)+(V_4^*\otimes ((y_5\oplus y_6)(p)(y_5\oplus y_6)(q)V_5\oplus V_6) - ((y_4^*(q)(y_4^*(p)V_4^*)\otimes V_5\oplus V_6) + (V_4^*\otimes((y_5\oplus y_6)(q)(y_5\oplus y_6)(p)V_5\oplus V_6)]$

Then the left hand side is equal to the right hand side, therefore $\delta([p, q])\,((V_4^*\otimes V_5\oplus V_6)\otimes(V_3^*\otimes V_2\oplus V_1))$ is Lie algebra homomorphism and then it is a representation of Lie algebra.

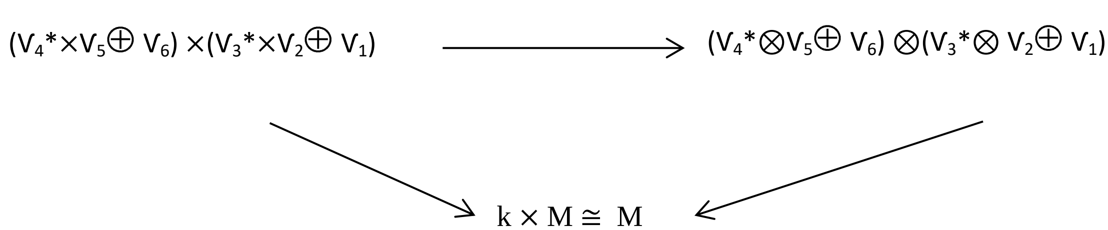
Remark (2.5) [19] [20]

In a broad sense, let n represent a positive integer.

$M^{***}\ldots n \cong M$, then n is the even positive integer.

$M^{***}\ldots n \cong M^*$, then is the odd positive integer.

Define δ by $\delta(V_4^*,V_3^*) = V_4^*(V_4)\, V_3$, for all $V_4 \in M_4$ and $k(V_4)= V_3$



By the universal property, there exists the unique ϕ define by:

ϕ $((M_4^*\otimes M_5\oplus M_6)\otimes(M_3^*\otimes M_2\oplus M_1)) = M_4^*(V_4)k(V_4)$, for all $V_4 \in M_4$, $V_3 = k(V_4) \in M_3$, $k \in Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$.

Theorem (2.6)

suppose that y_j, j = 1,2,3,4,5,6 are matrix representations of ATHS action of Lie algebra Li acting in the vector spaces M_j, j = 1,2,3,4,5,6 then ATHS action of Lie algebra on:

$((M_4^*\otimes M_5\oplus M_6)\otimes(M_3^*\otimes M_2\oplus M_1))$ is satisfied:

$((M_4^*\otimes M_5\oplus M_6)\otimes(M_3^*\otimes M_2\oplus M_1)) \cong Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$.

Proof:

here is a reversible linear mapping and an intertwining mapping of the action for the Li in:

$((M_4^*\otimes M_5\oplus M_6)\otimes(M_3^*\otimes M_2\oplus M_1))$ into $Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$ such that:

$\phi\,((y_3^* \otimes (y_2\oplus y_1))\otimes(y_4^* \otimes(y_5\oplus y_6)))\,(p) = ((y_3^*\, \vartheta_1\, y_2\oplus y_1)\, \vartheta_2\, (y_4^*\, \vartheta_3\, y_5\oplus y_6))\,(p)$

$= ((y_3^*(p)\, \vartheta_1\, y_2(p) \oplus y_1(p))\, \vartheta_2\, (y_4^*(p)\, \vartheta_3\, y_5(p) \oplus y_6(p)))$,

By remark (2.5) we have:

$(M_4^*\otimes M_5\oplus M_6) \cong Hom_k(M_4^*,M_5\oplus M_6)$, let $M_4^*\otimes M_5\oplus M_6 = M$ ……1

Let $M^{*\prime} = M_3^*\otimes M_2\oplus M_1$, $M \otimes M^{*\prime} \cong Hom_k(M,M')$.......2 By 1 and 2:

$((M_4^*\otimes M_5\oplus M_6)\otimes(M_3^*\otimes M_2\oplus M_1)) \cong Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$.

Theorem (2.7)

Let y_j: Li$\to$ gl(M), y_j^*: Li $\to$ gl(M*), for j = 1,2,3,4,5,6, and the action to Lie algebra Li in $Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$ is given by arepresentation δ, such that: $\delta(p)\, \vartheta = (y_1(p)\oplus y_2(p)\, \vartheta_1 - \vartheta_1\, y_3^*(p))\, \vartheta_2^* - \vartheta_2^*\, (y_4^*(p)\, \vartheta_3 - \vartheta_3\, (y_5(p) \oplus y_6(p)))$, for all p Є Li

then the action to a Lie algebra Li on $(Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))^*$ is also given by representation δ^*, such that:

$\delta^*(p)\, \vartheta^* = \vartheta_2^{**}\, (\vartheta_1^*\, y_1^*(p) \oplus y_2^*(p) - y_3^{**}(p)\, \vartheta_1^*) - (\vartheta_3^*\, y_4^{**}(p) - (y_5^*(p) \oplus y_6^*(p))\, \vartheta_3^*)\, \vartheta_2^{**}$.

Proof:

Let action of Lie algebra Li on $Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1))$ is the representation δ: Li$\to$gl $(Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))$, such that:

$\delta(p)\vartheta = (y_1(p)\oplus y_2(p)\vartheta_1 - \vartheta_1\, y_3^*(p))\vartheta_2 - \vartheta_2(y_4^*(p)\, \vartheta_3 - \vartheta_3\, (y_5(p) \oplus y_6(p)))$, for all $p \in$ Li, $\alpha \in (Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))$

To prove that δ^*: Li$\to$gl$(Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))^*$ is the representation to the Lie Algebra Li, such that:

$\delta^*(p)\, \vartheta^* = \vartheta 2^*\, (\vartheta_1^* y_1^*(p) \oplus y_2^*(p) - y_3^{**}(p)\vartheta_1^*) - (\vartheta_3^*\, y_4^{**}(p) - (y_6^*(p) \oplus y_5^*(p))\, \vartheta_3^*)\, \vartheta_2^*$

for all p Є Li, α^* Є $(Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))^*$.

$(\delta(p)\, \vartheta^*) = (y_1(p) \oplus y_2(p)\, \vartheta_1 - \vartheta_1\, y_3^*(p))\, \vartheta_2 - \vartheta_2\, (y_4^*(p)\, \vartheta_3 - \vartheta_3\, (y_5(p) \oplus y_6(p)))^*$

$= \vartheta_2^* (\vartheta_1^* y_1^* (p) \oplus y_2^* (p) - y_3^{**} (p) \vartheta_1^*) - (\vartheta_3^* y_4^{**} (p) - (y_6^*(p) \oplus y_5^*(p)) \vartheta_3^*) \vartheta_2^*$

For all $p \in Li$ and $\vartheta_2^*: (Hom_k(M_4^*,M_5\oplus M_6)^* \rightarrow Hom_k(M_3^*,M_2\oplus M_1)$, $\vartheta_3^*:M_4^{**}\rightarrow M_5^*\oplus M_6^*$ and $\vartheta_1^*: M_3^{**}\rightarrow M_2^*\oplus M_1^*$

So $(\delta [p, q] \vartheta)^* = (\delta(p) \vartheta \delta(q) \vartheta - \delta(q) \vartheta \delta(p) \vartheta)^*$

$= - \delta^* (p) \vartheta^* \delta^* (q) \vartheta^* + \delta^* (q) \vartheta^* \delta^* (p) \vartheta^*$

$= [\delta^* (q) \vartheta^* , \delta^* (p) \vartheta^*].$

Thus $\delta^* \vartheta^*$ be a representation from Li $\delta^* \vartheta^*$ is the Lie algebra homomorphism of Li.

Example (2.8)

If the $y_j: \mathbb{R}\rightarrow so(2,\mathbb{R}) \subset gl(2,\mathbb{R})$, for j =1,2,3,4,5,6, such that

$$y_1(p) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, y_2(p) = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}, y_3(p) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, y_4(p) = \begin{bmatrix} 0 & 0.4 \\ -0.4 & 0 \end{bmatrix},$$

$$y_5(p) = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0 \end{bmatrix}, y_6(p) = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

are six representation of Lie algebra on

$(Hom_k(Hom_k(M_4^*, M_5\oplus M_6), Hom_k(M_3^*, M_2\oplus M_1)))^*$ is:

$\delta^* (p) \vartheta^{**} = \vartheta_2^* (\vartheta_1^* y_1^* (p) \oplus y_2^* (p) - y_3^{**} (p) \vartheta_1^*) - (\vartheta_3^* y_4^{**} (p) - (y_5^*(p) \oplus y_6^*(p)) \vartheta_3^*) \vartheta_2^{**}$

$$=\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \right) \vartheta 1^* \left(- \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \right) \vartheta 2^* \left(- \begin{bmatrix} 0 & 0.4 \\ -0.4 & 0 \end{bmatrix} \right) \vartheta 3^* \left(\begin{bmatrix} 0 & -0.1 \\ 0.1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \vartheta 2^* \begin{bmatrix} 0 & -0.4 \\ 0.4 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -4.1 \\ -4.1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & -10 & 0 \\ 0 & -10 & 0 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 1.64 \\ 0 & 0 & -1.64 & 0 \\ 0 & -1.64 & 0 & 0 \\ -1.64 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-16.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{16 \times 16}$$

Conclusion

In this paper, we have given a historical background on the Lie algebra and proceeds to explain the concepts of Lie algebra representation and tensor product. It introduces a new six-representation structure based on proposed ideas, illustrated with graphs depicting the structure's flow, which laid the foundation for Schur's lemma. The paper then explores the application of the dual action of representation for Lie algebra alongside the tensor product to derive generalizations for cases of odd or even dimensions, as demonstrated in Example 2.8.

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