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RESEARCH ARTICLE - MATHEMATICS

On Hermitian Solution Equations-Type operators Associated with Hilbert Domain

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Abstract

This work began by addressing the essential and sufficient requirements that must be met in order for a solution to the operator equation $ßX\Phi = \Omega = \Phi^* X ß^*$ system to exist, where these operators are linear and bounded on a Hilbert space. Subsequently, the general solution for this system was formulated, and the work concluded by discussing the existence of the Hermitian solution under specific conditions.

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1. Introduction

The operator equation is recognized to have many uses in linear systems, control theory and other fields, See [1]. In 1966, Douglas [2] created his famous equation $ßX = \Phi$ and stipulated some requirements for the existence of a solution to this equation. Then, many scientists became interested in related equations (see [3 - 10]). Mitra [11] studied matrix equations $ßX = \Phi$, $ßX\Phi = \Omega$ and the system of equations $ßX = \Omega, X\Phi = D$ in 1976, and during this study, he addressed the essential and enough circumstances for the existence of the Hermitian solution. In 1995, H. Liping, and Z. Qingguang obtained a set of necessary and sufficient conditions for the existence of a solution to the matrix equation $ßX\Phi + CYD = \Omega$, in addition to representing the solution based on the g-inverse [12]. In 2008, Q. Xu, L. Sheng, and Y. Gu explored the possibility of solving the operator equation $ßX\Phi^* - \Phi X^* ß^* = \Omega$ for the adjointable operators between Hilbert C*-module. The results were applied to the study of positive real solutions and positive solutions of the operator equation $ßX\Phi = \Omega$ [13]. In 2023, Hu, and Yuan [14] obtained an expression for the general common solution of the equation system $ßX = \Phi, X\Omega = D$ after addressing the essential and enough circumstances for the existence of this solution. In 2024, Shi Lei, Wang Qing-Wen, Xie Lv-Ming and Zhang Xiao-Feng [15] deduced the essential and enough circumstances for solving the generalized commutative quartic matrix equation $ßX\Phi = \Omega$, and then presented the general solution formula for this equation. In 2017, Vosough and Moslehian [16] provided some descriptions for the presence of a solution to the system $ßX\Phi = \Phi, \Phi Xß = \Phi$ equations, and then presented representations for some forms of the solution to this system. After them, Xiao and Guoxing [17] came in 2018 and expanded the above system to the $ßX\Phi = \Omega, \Phi Xß = \Omega$ system and studied its related issues. In this work, we will study some issues related to an equation system based on system $ßX\Phi = \Omega, \Phi Xß = \Omega$, which will be mentioned in third part of this work. Now we will move on to the second section of this work to address a set of introductions, definitions, and theories that will be useful in the third section. Mathematicians have studied many operator equations and various systems; see, for example, [18 - 29]. Each of $\mathcal{F}$ and $\mathcal{H}$ represents a Hilbert space in this work, and the collection of linear operators with limits from $\mathcal{F}$ to $\mathcal{H}$ is represented by $\mathcal{L}(\mathcal{F},\mathcal{H})$. When $\mathcal{F} = \mathcal{H}$, it is expressed as $\mathcal{L}(\mathcal{F})$, and $R(ß)$ represent the ring ß, see [30,31]. If $ß \in \mathcal{L}(\mathcal{F},\mathcal{H})$ and $X \in \mathcal{L}(\mathcal{H},\mathcal{F})$ exist such that $< ßx, y > = < x, Xy >$, then X is referred to as adjoint operator of ß and represented by the letter $ß^*$, and ß is referred to as self-adjoint if $ß = ß^*$ [32]. While if X represents a solution to equation $ßXß = ß$, It is referred to as the inner inverse of ß and represented by the letter $ß^-$, As is well known, $ß \in \mathcal{L}(\mathcal{F},\mathcal{H})$ is regular if and only if ß has a closed range. Therefore, it exists if and only if ß has a closed range. Furthermore, to this, if X represents a solution to equation

$Xß X = X$, then X is referred to as generalized inverse of ß [33]. If X is the only generalized inverse of ß and satisfies $(ßX)^* = ßX$ and $(Xß)^* = Xß$ then it is referred to as a Moore-Penrose generalized inverse and represented by the letter $ß^+$, noting that $ßß^+ = P_{\overline{R(ß)}}$ *and* $ß^+ß = P_{\overline{R(ß^*)}}$, see [34-36]. If there is Φ self-adjoint operator such that $ß\Phi = 0$ and $ß^+\Phi = \Omega$, so we write $ß \preccurlyeq \Phi$, suppose that $ß \overset{*}{\leq} \Phi$ this means $ßß^* = \Phi ß^*, ß^*ß = ß^*\Phi$ it is known that, for $ß, \Phi \in \mathcal{L}(\mathcal{F})$ when this spaces operators are all self-adjoint, $ß \preccurlyeq \Phi$ if and only if $ß \overset{*}{\leq} \Phi$, see [37].

2. Essential theories supplied the prerequisites.

Theorem 2.1.[38]: let $ß, \Phi, \Omega \in \mathcal{L}(\mathcal{F})$ are regular, then the subsequent requirements are equivalent:

(i) Exist $X \in \mathcal{L}(\mathcal{F})$ a solution of the equation $ßX\Phi = \Omega$

(ii) $R(\Omega) \subseteq R(ß), R(\Omega^*) \subseteq R(\Phi^*)$,

(iii) $ßß^+\Omega\Phi^+\Phi = \Omega$.

In this case, $X = ß^+\Omega B^+ + U - ß^+ßUBB^+$. Where $U \in \mathcal{L}(\mathcal{F})$ is arbitrary.

Theorem 2.2 [39]: let $ß, \Phi \in \mathcal{L}(\mathcal{F})$ and $\overline{\Psi}$ denote the closure of a space Ψ.

(i) $ßß^* = \Phi ß^* \Leftrightarrow ß = \Phi P_{\overline{R(ß^*)}} \Leftrightarrow ß = \Phi Q$ for some $Q \in \mathcal{L}(\mathcal{F})$, *where* $\mathcal{L}(\mathcal{F})$ the space of all orthogonal projections;

(ii) $ß^*ß = ß^*\Phi \Leftrightarrow ß = \Phi P_{\overline{R(ß)}} \Leftrightarrow ß = Q\Phi$ for some $Q \in \mathcal{L}(\mathcal{F})$, *where* $\mathcal{L}(\mathcal{F})$ the space of all orthogonal projections;

(iii) $ß \overset{*}{\leq} \Phi \Leftrightarrow \Phi = ß + P_{\overline{N(ß^*)}}\Phi P_{\overline{N(ß)}}$;

(iv) $ß \overset{*}{\leq} \Phi \Leftrightarrow ß = P_{\overline{R(ß)}}\Phi = \Phi P_{\overline{R(ß^*)}} = P_{\overline{R(ß)}}\Phi P_{\overline{R(ß^*)}}$.

Theorem 2.3 [40]: let $ß, \Omega \in \mathcal{L}(\mathcal{F})$ and ß is regular, then the subsequent requirements are comparable:

(i) there is $X \in \mathcal{L}(\mathcal{F})$ a solution of the equation $ßX = \Omega$

(ii) $ßß^+\Omega = \Omega$

(iii) $R(\Omega) \subseteq R(ß)$.

The general solution in this instance is $X = ß^+\Omega + (I - ß^+ß)T$, where $T \in \mathcal{L}(\mathcal{F})$ is random.

Theorem 2.4 [40]: Let $ß, \Phi, \Omega, \mathcal{D} \in \mathcal{L}(\mathcal{F})$ and $ß, \Phi, K = \Phi^*(I - ß^+ß)$ are regular. Then, system $ßX = \Omega, X\Phi = \mathcal{D}$ have a hermitian solution $X \in \mathcal{L}(\mathcal{F})$ if and only if $ßß^+\Omega = \Omega$, $\mathcal{D}\Phi^+\Phi = D$, $ß\mathcal{D} = \Omega\Phi$ and $ß\Omega^*, \Phi^*\mathcal{D}$ are Hermitian. The general Hermitian solution in this instance is $= ß^+\Omega + (I - ß^+ß) K^+(\mathcal{D}^* - \Phi^*ß^+\Omega) + (I - ß^+ß)(I - K^+K)\left(ß^+\Omega + (I - ß^+ß)K^+(\mathcal{D}^* - B^*ß^+\Omega)\right)^* + (I - ß^+ß)(I - K^+K)U(I - K^+K)^*(I - ß^+ß)^*$, where $U \in \mathcal{L}(\mathcal{F})$ is hermitian.

3. Main outcomes and analytical methodologies:

In this part of the work, we are studying the system of equation with the operator

$$ßX\Phi = \Omega = \Phi^*Xß^*, \tag{1}$$

where ß and Φ are known operators and X is an unknown operator. We begin with the following theorem that provides a few prerequisites for this system's solution to exist.

Theorem 3.1: If $ß, \Phi, \Omega \in \mathcal{L}(\mathcal{F})$ and $ß, \Phi$ are regular and $ß^+\Omega\Phi^+ = \Phi^{*+}\Omega ß^{*+}$, then the following are equivalent:

i) The operator equations system (1) is solvable.

ii) $\Omega = ßß^+\Omega ß^{*+}ß^* = \Phi^*\Phi^{*+}\Omega\Phi^+\Phi$

iii) $R(\Omega) \subseteq R(ß), R(\Phi^*)$ and $R(\Omega^*) \subseteq R(ß), R(\Phi^*)$

Proof: *(i)* ⇒ *(ii)*

Let's assume that the system operator equations (1) has a solution and that θ is a solution to this system, so we obtain

$$ß\theta\Phi = \Omega = \Phi^*\theta ß^* \tag{2}$$

consequently, we obtain that

$$R(\Omega) \subseteq R(ß), R(\Phi^*) \text{ and } R(\Omega^*) \subseteq R(ß), R(\Phi^*),$$

using this data along with Theorem 2.3, we obtain the following equations

$$\Omega = ßß^+\Omega,\ \Omega = \Phi^*\Phi^{*+}\Omega,\ \Omega^* = ßß^+\Omega^* \text{ and } \Omega^* = \Phi^*\Phi^{*+}\Omega^*.$$

therefore, we can express Ω in the following two forms

$$ßß^+\Omega ß^{*+}ß^* = ßß^+\Omega = \Omega = \Phi^*\Phi^{*+}\Omega = \Phi^*\Phi^{*+}\Omega\Phi^+\Phi \tag{3}$$

(ii) ⇒ *(iii)*

Let's assume (3) satisfy, this provides us directly with the following information

$$R(\Omega) \subseteq R(ß), R(\Phi^*) \text{ and } R(\Omega^*) \subseteq R(ß), R(\Phi^*) \tag{4}$$

(iii) ⇒*(i)*

Involving Theorem 2.3, it follows that

$$\Omega = ßß^+\Omega,\ \Omega = \Phi^*\Phi^{*+}\Omega,\ \Omega^* = ßß^+\Omega^* \text{ and } \Omega^* = \Phi^*\Phi^{*+}\Omega^*$$

by comparing these four equations and inserting one into another, we will obtain the following two equations

$$\Omega = ßß^+\Omega = ßß^+\Omega\Phi^+\Phi \text{ and } \Omega = \Phi^*\Phi^{*+}\Omega = \Phi^*\Phi^{*+}\Omega ß^{*+}ß^*.$$

By assumption, we have $ß^+\Omega\Phi^+ = \Phi^{*+}\Omega ß^{*+}$, and this changes the form of the two equations we recently obtained to become as follows

$$\Omega = ßß^+\Omega\Phi^+\Phi \text{ and } \Omega = \Phi^*ß^+\Omega\Phi^+ß^*$$

here we clearly observe that $ß^+\Omega\Phi^+$ is a solution to the system (1).

Now we will study the presence of the system (1) solution under the condition of two specific requirements in the next theorem.

Theorem 3.2: If $ß, \Phi,\ \Omega \in \mathcal{L}(\mathcal{F})$ such that Ω is regular and $R(ß), R(\Phi^*) \subseteq R(\Omega), R(\Omega^*)$ then the sentences that follow are interchangeable

- *(i)* There is a solution like $X \in \mathcal{L}(\mathcal{F})$ for the operator equations system (1)
- *(ii)* $\Omega \overset{*}{\leq} ßX\Phi$, $\Omega \overset{*}{\leq} \Phi^*Xß^*$

Proof: *(i)* ⇒ *(ii)*

We impose X is resolution of the operator equations system (1), then we get

$$ßX\Phi - \Omega = 0 \text{ and } \Phi^*Xß^* - \Omega = 0. \tag{5}$$

The assumption provides us with the following data $R(ß), R(\Phi^*) \subseteq R(\Omega), R(\Omega^*)$,
based on this data and using Theorem 2.3, we will obtain new data represented in the following forms

$$\Omega\Omega^+ß = ß,\ \Omega^*\Omega^{*+}ß = ß,\ \Omega\Omega^+\Phi^* = \Phi^* \text{ and } \Omega^*\Omega^{*+}\Phi^* = \Phi^*. \tag{6}$$

The equation $ßX\Phi - \Omega = 0$ in (5), and $\Omega\Omega^+ß = ß$ in (6), and after compensating the second in the first, will provide us with a new equation represented in the form

$$P_{\overline{R(\Omega)}}ßX\Phi - \Omega = ßX\Phi - \Omega = 0$$

this means
$$P_{\overline{R(\Omega)}}ßX\Phi = \Omega. \tag{7}$$

Equation $\Phi^*Xß^* - \Omega = 0$ in (5), and $\Omega^*\Omega^{*+}ß = ß$ in (6), using the same substitution method as in the previous step, will provide us a new equation that is shown in the form of

$$\Phi^*X\text{ß}^*P_{\overline{R(\Omega^*)}} - \Omega = \Phi^*X\text{ß}^* - \Omega = 0$$

it means $$\Phi^*X\text{ß}^*P_{\overline{R(\Omega^*)}} = \Omega \quad (8)$$

by the same way $\text{ß}X\Phi - \Omega = 0$ in (5), and $\Omega^*{\Omega^*}^+\Phi^* = \Phi^*$ in (6), will provide us with a new equation represented in the form

$$\text{ß}X\Phi P_{\overline{R(\Omega^*)}} - \Omega = \text{ß}X\Phi - \Omega = 0$$

this implies $$\text{ß}X\Phi P_{\overline{R(\Omega^*)}} = \Omega \quad (9)$$

in a similar way $\Phi^*X\text{ß}^* - \Omega = 0$ in (5), and $\Omega\Omega^+\Phi^* = \Phi^*$ in (6), also by repeating the same previous steps, provide us a new equation that is shown in the form of

$$P_{\overline{R(\Omega)}}\Phi^*X\text{ß}^* - \Omega = \Phi^*X\text{ß}^* - \Omega = 0$$

it means $$P_{\overline{R(\Omega)}}\Phi^*X\text{ß}^* = \Omega \quad (10)$$

also $\text{ß}X\Phi - \Omega = 0$ in (5), and $\Omega\Omega^+\text{ß} = \text{ß}$ and $\Omega^*{\Omega^*}^+\Phi^* = \Phi^*$ in (6) will provide us with a new equation represented in the form

$$P_{\overline{R(\Omega)}}\text{ß}X\Phi P_{\overline{R(\Omega^*)}} - \Omega = \text{ß}X\Phi - \Omega = 0$$

it implies $$P_{\overline{R(\Omega)}}\text{ß}X\Phi P_{\overline{R(\Omega^*)}} = \Omega. \quad (11)$$

Also $\Phi^*X\text{ß}^* - \Omega = 0$ in (5), and $\Omega^*{\Omega^*}^+\text{ß} = \text{ß}$ and $\Omega\Omega^+\Phi^* = \Phi^*$ in (6), will provide us with a new equation represented in the form

$$P_{\overline{R(\Omega)}}\Phi^*X\text{ß}^*P_{\overline{R(\Omega^*)}} - \Omega = \Phi^*X\text{ß}^* - \Omega = 0$$

this means $$P_{\overline{R(\Omega)}}\Phi^*X\text{ß}^*P_{\overline{R(\Omega^*)}} = \Omega. \quad (12)$$

Therefore, by referring back to Theorem 2.2, and applying relations (7), (9), and (11) to it, we arrive at the following equation

$$\Omega \overset{*}{\leq} \text{ß}X\Phi.$$

Consequently, by referring back to Theorem 2.2, and applying Relations (8), (10), and (12) to it, we arrive at the following equation

$$\Omega \overset{*}{\leq} \Phi^*X\text{ß}^*$$

(ii) ⇒ *(i)*

Let's assume we have the following data

$$\Omega \overset{*}{\leq} \text{ß}X\Phi \text{ and } \Omega \overset{*}{\leq} \Phi^*X\text{ß}^* \quad (13)$$

using relations in (13) consecutively with Theorem 2.2, we obtain

$$P_{\overline{R(\Omega)}}\text{ß}X\Phi = \Omega = \Phi^*X\text{ß}^*P_{\overline{R(\Omega^*)}} \quad (14)$$

respectively. The assumption provides us with the following data $R(\text{ß}) \subseteq R(\Omega), R(\Omega^*)$, based on this data and using Theorem 2.3, we will obtain new data represented in the following forms

$$\Omega\Omega^{+}ß = ß \text{ and } \Omega^{*}\Omega^{*+}ß = ß \tag{15}$$

now we substitute 1 and 2 in (14) with their equivalent values in (15), resulting in the following

$$ßX\Phi = \Omega = \Phi^{*}Xß^{*}$$

this means that X represents a solution to the system (1).

The following theorem presents the general solution formula for system (1) according to the generalized inverse and under certain conditions.

Theorem 3.3: If $ß, \Phi \in \mathcal{L}(\mathcal{F})$ are regular, $\Omega \in \mathcal{L}(\mathcal{F})$ and $\Omega \overset{*}{\leq} ß, \Phi$, and $ß^{*} \overset{*}{\leq} \Phi$ then the general solution to System (1) is

$$X = ß^{-}\Omega\Phi^{-} + \left(\Phi^{*}(I - ß^{-}ß)\right)^{-}(\Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*})ß^{*-} + W - (\Phi^{*}(I - ß^{-}ß))^{-}\Phi^{*}(I - ß^{-}ß)Wß^{*}ß^{*-}$$

$$-ß^{-}ß[\left(\Phi^{*}(I - ß^{-}ß)\right)^{-}(\Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*})ß^{*-} + W - \left(\Phi^{*}(I - ß^{-}ß)\right)^{-}\Phi^{*}(I - ß^{-}ß)Wß^{*}ß^{*-}]\Phi\Phi^{-},$$

where $W \in \mathcal{L}(\mathcal{F})$ is random.

Proof: From the assumption, we get $R(\Omega) \subseteq R(ß), R(\Omega^{*}) \subseteq R(\Phi^{*})$, we notice here that the condition of Theorem 2.1, which requires the presence of a solution for Equation $ßX\Phi = \Omega$, has been satisfied. Therefore, we can use this Theorem 2.1 to construct the solution for Equation $ßX\Phi = \Omega$ in the following manner

$$X = ß^{-}\Omega\Phi^{-} + Z - ß^{-}ßZ\Phi\Phi^{-} \tag{16}$$

where $Z \in \mathcal{L}(\mathcal{F})$ is arbitrary. Now we will consider X mentioned in (16) as a solution to the equation

$$\Omega = \Phi^{*}Xß^{*},$$

so, and after compensating for X with its equivalent, we will get

$$\Omega = \Phi^{*}(ß^{-}\Omega\Phi^{-} + Z - ß^{-}ßZ\Phi\Phi^{-})ß^{*}$$

from him, we obtain

$$\Omega = \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*} + \Phi^{*}Zß^{*} - \Phi^{*}ß^{-}ßZ\Phi\Phi^{-}ß^{*} \tag{17}$$

by the assumption we possess $ß^{*} \overset{*}{\leq} \Phi$, this implies that

$$ß^{*} = \Phi\Phi^{-}ß^{*} \tag{18}$$

by using relation (18), the formula of equation (17) will be transformed into this form

$$\Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*} + \Phi^{*}Zß^{*} - \Phi^{*}ß^{-}ßZß^{*} = \Omega$$

this gives us

$$\Phi^{*}(I - ß^{-}ß)Zß^{*} = \Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*} \tag{19}$$

the equation (19) that we recently formed represents the formula mentioned in Theorem 2.1, where Z is its solution. Therefore, using Theorem 2.1, it will be Z in the form of

$$Z = \left(\Phi^{*}(I - ß^{-}ß)\right)^{-}(\Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*})ß^{*-} + W - (\Phi^{*}(I - ß^{-}ß))^{-}\Phi^{*}(I - ß^{-}ß)Wß^{*}ß^{*-}$$

where W is arbitrary. Now we will replace Z that appeared in (16) with its equivalent to obtain

$X = ß^{-}\Omega\Phi^{-} + \left(\Phi^{*}(I - ß^{-}ß)\right)^{-}(\Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*})ß^{*-} + W - (\Phi^{*}(I - ß^{-}ß))^{-}\Phi^{*}(I - ß^{-}ß)Wß^{*}ß^{*-}$

$-ß^{-}ß[\left(\Phi^{*}(I - ß^{-}ß)\right)^{-}(\Omega - \Phi^{*}ß^{-}\Omega\Phi^{-}ß^{*})ß^{*-} + W - (\Phi^{*}(I - ß^{-}ß))^{-}\Phi^{*}(I - ß^{-}ß)Wß^{*}ß^{*-}\ ]\Phi\Phi^{-}.$

The following theorem discusses the prerequisites for system (1)'s hermitian solution to exist.

Theorem 3.4: if $ß, \Phi \in \mathcal{L}(\mathcal{F})$ are regular, $\Omega \in \mathcal{L}(\mathcal{F})$, $\Omega \overset{*}{\leq} \Phi \overset{*}{\leq} ß$ and $ß\Phi^{*+}\Omega$ and Ω are hermitian. System (1) then has a hermitian solution.

Proof: From the assumption and by using Theorem 2.2, we get $\Omega = ßP_{\overline{R(\Omega^{*})}}$, in this case we have $R(\Omega) \subseteq R(ß)$, so by using Theorem 2.3 we get

$$ßß^{+}\Omega = \Omega \tag{20}$$

so, by using (20), and since Ω is hermitian, we get $\Omega = \Omega ß^{*+}ß^{*}$. If we express A, B, C, D as $ß, ß^{*}, \Omega\Phi^{+}, \Phi^{*+}\Omega$ respectively, from this relation and the assumptions above we get

$$AA^{+}C = ßß^{+}\Omega\Phi^{+} = \Omega\Phi^{+} = C,\ DB^{+}B = \Phi^{*+}\Omega ß^{*+}ß^{*} = \Phi^{*+}\Omega = D, \tag{21}$$

and $AD = ß\Phi^{*+}\Omega = \Omega\Phi^{+}ß^{*} = CB,\ AC^{*} = ß(\Omega\Phi^{+})^{*} = ß\Phi^{*+}\Omega,\ B^{*}D = ß\Phi^{*+}\Omega$ (22)

from (21) and (22), then by using Theorem 2.4, then the system $AX = C, XB = D$ has hermitian solution, i.e.

$$ßX = \Omega\Phi^{+} \text{ and } Xß^{*} = \Phi^{*+}\Omega \tag{23}$$

has hermitian solution. Suppose θ is hermitian solution for system (23) then we get

$$ß\theta = \Omega\Phi^{+} \text{ and } \theta ß^{*} = \Phi^{*+}\Omega. \tag{24}$$

Now we need to verify that θ represents a solution to the equation $ßX\Phi = \Omega$, assisted (24) and $\Omega = \Omega\Phi^{+}\Phi$ $\left(\text{since } \Omega \overset{*}{\leq} \Phi\right)$, then we get

$$ß\theta\Phi = \Omega\Phi^{+}\Phi = \Omega \tag{25}$$

we will also verify that θ is a solution to the equation $\Phi^{*}Xß^{*} = \Omega$, taking in account equation (24) and $\Omega = \Phi^{*}\Phi^{*+}\Omega$, (since $\Omega \overset{*}{\leq} \Phi$ and Ω is hermitian) then we get

$$\Phi^{*}\theta ß^{*} = \Phi^{*}\Phi^{*+}\Omega = \Omega, \tag{26}$$

from (25), (26) note that θ is the system's hermitian solution (1).

The next theory discusses the essential and enough conditions for the presence of System (1)'s hermitian solution under conditions different from those mentioned in the previous theory.

Theorem3.5: if $ß, \Phi \in \mathcal{L}(\mathcal{F})$ are regular, $\Omega \in \mathcal{L}(\mathcal{F})$ and $R(\Phi) \subseteq R(ß^{*})$ and $R(\Omega), R(\Omega^{*}) \subseteq R(ß)$, Thus, the requirements that follow are interchangeable:

(i) hermitian solution is present for system (1).

(ii) the system $\Phi^{*}X = \Omega ß^{*+},\ X\Phi = ß^{+}\Omega$ has as hermitian solution.

Proof: *(i)⇒(ii)*

Suppose that $\theta \in \mathcal{L}(\mathcal{F})$ is Hermitian solution of the system (1), then we get

$$ß\theta\Phi = \Omega = \Phi^{*}\theta ß^{*}. \tag{27}$$

Now we will take the left side of this system (27) and multiply it from the left by $ß^+$ to yield us

$$ß^+ß\theta\Phi = ß^+\Omega \tag{28}$$

from the assumption $R(\Phi) \subseteq R(ß^*)$ and using Theorem 2.3, we get $ß^*ß^{*+}\Phi = \Phi$ we substitute this into equation (28) to obtain

$$ß^+ß\theta ß^*\Psi^{*+}\Phi = ß^+\Omega, \tag{29}$$

Now we will take the right side of this system (27) and multiply it from the right by $ß^{*+}$ to yield us

$$\Omega ß^{*+} = \Phi^*\theta ß^*ß^{*+} \tag{30}$$

And our possession of $ß^*ß^{*+}\Phi = \Phi$ provides us with $\Phi^* = \Phi^*ß^+ß$, and by taking the equation that has recently developed for us and compensating it in (30), we will obtain

$$\Omega ß^{*+} = \Phi^*ß^+ß\theta ß^*ß^{*+} \tag{31}$$

in the proof of that theorem. We can clearly see that

$$(ß^+ß\theta ß^*ß^{*+})^* = ß^+ß\theta ß^*ß^{*+} \tag{32}$$

from (29), (31) and (32) which means that $ß^+ß\theta ß^*ß^{*+}$ is a hermitian solution for system $\Phi^*X = \Omega ß^{*+}$, $X\Phi = ß^+\Omega$.

(ii)⇒(i)

Taking θ is hermitian solution of the system $\Phi^*X = \Omega ß^{*+}$, $X\Phi = ß^+\Omega$, we get $\theta\Phi = ß^+\Omega$ and $\Phi^*\theta = \Omega ß^{*+}$. From the assumption, we have $R(\Omega), R(\Omega^*) \subseteq R(ß)$, we get $\Omega = ßß^+\Omega$, $\Omega^* = ßß^+\Omega^*$ so by using this after we multiply the last two equations consecutively by ß from the left and $ß^*$ from the right, and then substitute Ω and Ω^*, we will obtain new equations in the following form

$$ß\theta\Phi = ßß^+\Omega = \Omega \text{ and } \Phi^*\theta ß^* = \Omega ß^{*+}ß^* = \Omega$$

we find that a hermitian solution to system (1) is represented by θ.

4. Conclusions

Through this work, a general solution form for the operator equations system $ßX\Phi = \Omega = \Phi^*Xß^*$, based on the generalized inverse after identifying the necessary circumstances for the existence of this solution, founded on a set of definitions and theories found in previous works. Continuing this work, we propose for future studies to examine the Hermitian solution form for the same system.

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