2025,Volume 2 , Issue 6 , PSIJK (33-39)

Pure Sciences International Journal of Kerbala

New Type of Generalized Continuous Functions in Proximity Topological Space

Entessar Jaber Abed Noor[a*], Yiezi Kadham Mahdi Altalkany[b]

[a] *Department of Mathematic, College of Education For Women, University, Kufa, Najaf, Iraq.*
[b] *Department of Mathematic, College of Computer Sciences and Mathematics University of Kufa, Najaf, Iraq.*

P A P E R I N F O

Received: 23 December 2024
Accepted: 16 February 2025
Published: 30 June 2025

Keywords: proximity Space,Nearness, Topology generated by a by aproximity space

Abstract

The paper deals with the importance of the continuum theory as a mathematical concept and its great role in solving many problems. Since the term proximity is a modern term, we were able in this research to answer a very important question: Is it possible to study types of continuity in proximity space like the types found in the usual topological space? In this research, we were able to find a type of continuity (δ_ω −continuous) and obtain many results, relying on proximity theories and the different definitions of proximity space, and we were able to link them to the new definition that was established.

1. INTRODUCTION

Continuity is almost as old as general topology. Both notions are firstly mentioned by Fr`echet, topological structure in 1906 [2]. The importance of continuity in general topology is that the continuous image of a connected set is connected which is important in digital image [2, 4]. In 1909, Riesz introduced in his "theory of enchantment" proximity structures [1,4]

Then, specifically in1952 Efremovič rediscovered the subject (see Nainpally and Warrack, 1970; Engelking, 1977) [1,4]. They (Riesz and Efremovič) defined and developed the axioms of relationship between sets in a metric space by stating “A is near to B i.e. $A\delta B$) if and only if $D(A,B) = \inf\{d(x,y): x \in A, y \in B\} = 0$ [4]. Further, the notion of δ −neighborhood started from, Efremovič recast his axiomatization in term of strong inclusion. Lodato [8,9] and [6,7] Leader have worked with weaker axioms those of Efremovič proximity space that enable them to define an arbitrary topology on the underlying set. It must be mentioned that L.A. Al Swidi, has presented many studies and researches in the field of studying proximity space [5,12].

*Corresponding Author Institutional Email:
entessarjaber@gmail.com (Entessar Jaber Abed Noor)

Generally speaking, [3] topology representsit (K. Kuratowski) a closeness between points and sets, while proximity (V.A. Efremovič) is a closeness between sets. Topology less than proximity which have more structure ones carry but carry less structure than metric ones[2].Thus a notion of proximal continuity has been proposed, with the property of preserving the closed of two sets. Simple continuity is weaker than the proximity continuity since it preserves nearness of a point and a set [2]. Furthermore, [3] proximity relations are important solution problems based on human perception.

This paper is an attempt to present a new type and a new definition of continuity function in proximity space.

2. PRELIMINARIES OF PROXIMITY SPACE

In this section, proximity space will be introduced in addition to presenting important definition and proposition.

2.1 Definition [5, 6, 12]

On the family $P(\chi)$ a relation δ called a proximity on χ of if it satisfies the following conditions all subsets of a set χ:

$(P1)$ If $A \,\delta\, \Gamma$, then $\Gamma \,\delta\, A$;

$(P2)$ $A \,\delta\, (\Gamma \cup H)$ if and only if either $A \,\delta\, \Gamma$ or $A \,\delta\, H$;

$(P3)$ $\chi \,\bar{\delta}\, \emptyset$;

$(P4)$ $\{x\}\ \delta\ \{x\}$ for each $x \in \chi$;
$(P5)$ if $A\overline{\delta}\,\Gamma$, then there exist $\mathrm{E} \in P(\chi)$ such that $A\overline{\delta}\,\mathrm{E}$ and $\chi - \mathrm{E}\,\overline{\delta}\,\Gamma$, $(i.e\ \forall \mathrm{E}, A\delta \mathrm{E}\ or\ \Gamma\delta\chi - \mathrm{E} \Longrightarrow A\delta\Gamma$ (see [12]). We called the pair $(\chi;\ \delta)$ a proximity space. If $(P4)$ is instead of $(P'(4))\{\mathrm{P}\}\delta\ \{o\}$ if and only if $\mathrm{P} = o$, then the realtion δ is called a separated proximity, $(\chi;\ \delta)$ called a separated proximity space.

2.2 Example [5, 6]

Discrete and indiscrete proximity were defined as

(i) If we defined $A\delta_1\Gamma$ if and only if $A \cap \Gamma \neq \emptyset$, then δ_1 is discrete proximity on χ.

(ii) If $A\delta_2\Gamma$ for every $\mathrm{A} \neq \emptyset \neq \Gamma$ and $A, \Gamma \subseteq \chi$, then δ_2 is indiscrete proximity on χ.

2.3 Definition [5]

If δ_1 and δ_2 are two elements of class P of all proximities that are defined on a set χ,inclusion is defined as

$\delta_1 > \delta_2$ if and only if $A\ \delta_1\Gamma$ implies $A\ \delta_2\Gamma$ .In such a case we say $that$ δ_1 is finer $then$ δ_2, or $\delta_2$is coarser than δ_1.

2.4 Proposition [4, 5, 12]

Let $(\chi;\ \delta)$ be a proximity space. Then
(a) if $A\ \delta\ \Gamma$ and $\Gamma\ \delta\ C$, then $A\ \delta\ C$;
(b) if $A\ \bar{\delta}\ \Gamma$ and $C \subseteq \Gamma$, then $A\bar{\delta}C$;
(c) if there exists a point $\mathrm{P} \in X$ such that $A\ \delta\ \{\mathrm{P}\}$ and $\{\mathrm{P}\}\ \delta\ \Gamma$, then $A\ \delta\ \Gamma$;
(d) if $A \cap \Gamma \neq \emptyset$, then $A\ \delta\ \Gamma$;
(e) $A\bar{\delta}\emptyset$ for every $A \subseteq \chi$;
(f) if $A\ \delta\ \Gamma$, then $A \neq \emptyset$ and $\Gamma \neq \emptyset$.

2.5 Proposition [7]

The axiom (P5 in Definition 1.1] is equivalent to any of the following statements:

(i) If $A\bar{\delta}B$, then there are sets M and N such that $A\bar{\delta}M, B\bar{\delta}N$;

(ii) If $A\bar{\delta}B$, then there are sets M and N such that $A\bar{\delta}\chi - M, X - N\bar{\delta}B$ and $M\bar{\delta}N$;

(iii) If $A\bar{\delta}B$, there are two sets M and N such that $A\bar{\delta}\chi - M$,and $B\bar{\delta}\chi - N$, $M \cap N = \emptyset$.

2.6 Definition [2, 6]

Let (χ, δ) be a proximity space, then for all $A, B \subset \chi$, B a proximity or δ -neighborhood of A and denoted that relation as $A \ll B$if and only if $A\bar{\delta}\chi - B$.

2.7 Theorem [2]

Let (X, δ) be a proximity space. Then the relation$\ll$ satisfies the following properties:
$(O1)$ $\chi \ll \chi$;
$(O2)$ If $\mathrm{A} \ll B$, then $\mathrm{A} \subset B$;
$(O3)$ $\mathrm{A} \subset B \ll C \subset D$ implies $\mathrm{A} \ll D$;
$(O4)$ $\mathrm{A} \ll B$ implies $\chi - B \ll \chi - \mathrm{A}$;
$(O5)$ $\mathrm{A} \ll B_k$ is true for $k = 1,2,\ldots,n$ if and only if $\mathrm{A} \ll \cap_{k=1}^{n} B_k$;
$(O6)$ If $A \ll B$, then there exists a set $N \subset X$ such that $A \ll N \ll B$. This is implies $A \ll int\,N \subset cl\,N \ll intB \subset clB$. (see Proximity Approach to Problem in Topology and Analysis). If δ is a separated proximity, then
$(O7)$ $\{\mathrm{P}\} \ll \chi - \{c\}$ if and only if $\mathrm{P} \neq c$.

2.8 Corollary [6]

If $A_\pi \ll B_\pi$,$\pi = 1, 2, \ldots\ n, then$

$$\bigcap_{\pi=1}^{n} A_\pi \ll \bigcap_{\pi=1}^{n} B_\pi \quad and \quad \bigcup_{\pi=1}^{n} A_\pi \ll \bigcup_{\pi=1}^{n} B_\pi$$

2.9 Remark [6]

The family all δ-neighborhoods of a set ω in proximity space(χ, δ) is $\mathcal{F}(\omega)$and δ-neighborhoods in general is not open set with respect to this topology.

2.10 Proposition [7,6]

Let (χ, δ) be a proximity space. Then
$(\iota 1)$ $\Gamma \in \mathcal{F}(\mathrm{A})$ implies $A \subset \Gamma$;
$(\iota 2)$ $\Gamma \in \mathcal{F}(A)$ implies $X - A \in \mathcal{F}(\chi - \Gamma)$;
$(\iota 3)$ If $A \subset \Gamma$, then $\mathcal{F}(A) \subset \mathcal{F}(\Gamma)$);
$(\iota 4\ )$ $\mathcal{F}(A \cup \Gamma) = \mathcal{F}(A) \cap \mathcal{F}(\Gamma)$;
$(\iota 5)$ If $\Gamma \in \mathcal{F}(A)$), then there exists a $C \in \mathcal{F}(A)$ such that $\Gamma \in \mathcal{F}(C)$);
(f) $\mathcal{F}(A) \cap \mathcal{F}(\Gamma)) \subset \mathcal{F}(A \cap \Gamma)$, where $\mathcal{F}(A) \cap \mathcal{F}(\Gamma) = \{C \cap D: C \in \mathcal{F}(A), D \in \mathcal{F}(\Gamma)\}$.

3. TOPOLOGY GENERATED BY a PROXIMITY

In this part, we will consider the topology on χ induced by a proximity on X, and we will study some definitions and elementary properties.

3.1 Theorem [2, 9, 10]

The family T_δ in a proximity space $(\chi,\ \delta)$ is called a topology on the set χ.

3.2 Remark [5]: If (χ, δ_J) is a proximity space, then it has a unique topology T_{δ_J}generated by δ_J.

3.3 Definition [5]

Let (χ, δ) be a proximity space. A subset $F \subset \chi$ is to be closed in χ if and only if $\mathrm{P}\ \delta\ F$ implies $\mathrm{P} \in F$. By T_δ denotes the family of complements of all the sets defined in such a way.

3.4 Proposition [7,5]

If G is a subset of a proximity space (χ, δ), then G is called open in topology T_δ if and only if $\{\mathrm{P}\}\ \bar{\delta}\ \chi - G$ for every $\mathrm{P} \in G$.

3.5 proposition [7,6]

For any two proximity relations δ_1, δ_2 in X, if $\delta_1 < \delta_2$, then $\mathrm{T}_{\delta_1} \subset \mathrm{T}_{\delta_2}$.

3.6 Proposition [7,11]

If A and Ω are subsets of a proximity space (χ, δ), then $\mathrm{A}\ \bar{\delta}\ \Omega$ implies:

(i) $\overline{\Omega} \subset \chi - \mathrm{A}$; and

(ii) (ii) $\Omega \subset int\,(\chi - \mathrm{A})$.

3.7 Proposition [4,11]

If $\bar{A}$ and int(A) detnote, respectively, the closure and the interior of the set A of a proximity space (X,δ) with respect to the topology T_δ, then

(i) $A \ll \Omega$ implies $\bar{A} \ll \Omega$

(ii) $A \ll \Omega$ implies $A \ll \text{int}(\Omega)$.

3.8 Theorem [11]

Let $\emptyset \neq Y \subset \chi$ *and* (χ,δ) be a proximity space, for $A, B \subset Y$ let $A\delta_Y B$ if and only if $A\ \delta B$.Then (Y,δ_Y) is a proximity space.

3.9 Proposition [5]

Let (χ,δ)be a proximity space, $\emptyset \neq Y \subset \chi$, then $\mathcal{F}_Y(A) = \{\Omega \subseteq Y, A \ll_Y \Omega\} = \mathcal{F}_\chi(A) \cap \{Y\}$.

3.10 Definition [6]

Let (χ,δ) be a proximity space, and let non empty set $Y \subset \chi$, the restriction on Y *of* the proximity δ and is denoted by δ_Y is defined on the subset Y of the set χ.The ordered pair (Y,δ_Y) is called the proximity subspace of the proximity space (X,δ).

4. Proximally of continuous functions

In this part, we will discuss the most important definition and special feature of continuity in proximity space.

4.1 Definition [1, 2, 3]

Let (χ,δ_χ) and $(\Upsilon,\delta_\Upsilon)$ be two proximity spaces. The mapping $f : \chi \to \Upsilon$ is said to be proximally or δ-continuous if $A\delta_\chi\eta$ implies $f(A)\delta_\Upsilon f(\eta)$ for every two sets $A, \eta \subset \chi$.

4.2 Proposition

If f is $\delta -$ continuous and onto function, then inverse image of each $T_{\delta_Y} -$ open set is $T_{\delta_\chi} -$open set.

4.3 Proposition [5]

Let a mapping $f : \chi \longrightarrow Y$, where (Y,δ) is a proximity space and let a relation on $P(\chi)$ of the set χ in the following way:

$A\delta^*\eta$ if and only if $f(A)\ \delta\ f(\eta)$

The inverse image of the proximity δ is relation δ^* defined in such a way and denoted by $f^{-1}(\delta)$.

4.4 Proposition [1, 4]

If $f : \chi \to Y$ and δ *is* a proximity on the set Y, then $f^{-1}(T_\delta) = T(f^{-1}(\delta))$.

4.5 Corollary [5]

If $f : \chi \to Y$ and if δ_1 and δ_2 are the proximities on Y for which $\delta_1 < \delta_2$, then $f^{-1}(\delta_1) < f^{-1}(\delta_2)$ hold.

4.6 Proposition [1, 2]

A mapping $f : \chi \to Y$ of a proximity (χ,δ_χ) into a proximity space $(Y,\delta_Y)$is δ-continuous if and only if for every two sets, $K \subset Y$, $R\bar{\delta}_Y K$ implies $f^{-1}(R)\ \bar{\delta}_\chi f^{-1}(K)$.

4.7 Corollary [5]

Let $f : \chi \to \Upsilon$ be a mapping from a set χ on a proximity space $(\Upsilon,\delta_\Upsilon)$, then$\delta_\chi = f^{-1}(\delta_\Upsilon)$) is the coarsest proximity on χ for which f is a δ-continuous mapping.

4.8 Corollary [5]

Let δ_1 and δ_2 be two proximities χ. The identity mapping $i: (\chi,\delta_1) \to (\chi,\delta_2)$from the set χ is a δ-continuous mapping if and only if $\delta_1 > \delta_2$.

5. δ_ω −CONTINUOUS FUNCTION

5.1 Definition

Let $f: (\chi,\delta_\chi) \to (Y,\delta_Y)$ be a mapping, the f is said to be δ_ω −continuous if and only if for all $\theta \in \chi, and$ for all $V \subset Y$, $f(\theta)\bar{\delta}_Y Y - V$, there exist $U \subset \chi, \theta\bar{\delta}_\chi \chi - U$, $f(U)\bar{\delta}_Y Y - V$.

5.2 Example

Let $f: (\chi,\delta_\chi) \to (Y,\delta_Y)$ such that $\chi = \{1,2\}, Y = \{3,4\}$ be a mapping, defined as follow $f(1) = 3$, $f(2) = 4$ for all $\theta \in \chi$, and the proximity relation defined on χ, Y respectively as follows:

$\delta_\chi: AS_\chi B \leftrightarrow A \cap B \neq \emptyset$ and $\delta_Y: A\delta_Y B \leftrightarrow A \cap B \neq \emptyset$, then

δ_X={(χ,χ),(χ,{1}),(χ,{2}),({1},{1}),({2},{2}),({1},χ),({2},χ)}, and

$\delta_Y = \{(Y,Y), (Y,\{3\}, (Y,\{4\}), (\{3\},Y), (\{4\},Y), (\{3\},\{3\}), (\{4\},\{4\})\}$.

Now if $x = 1, f(\theta) = f(1) = 3$, then

(i) if $V = \{3\}, f(1) = 3\bar{\delta}_Y Y - \{3\} = \{4\}$, there exist $U = \{1\}, \{1\}\bar{\delta}_\chi X - \{1\} = \{2\}$, and

$f(U) = f(\{1\}) = 3\bar{\delta}_Y Y - \{3\}$={4}.

(ii) If $V = \{4\}$, then $f(1) = 3\delta_Y Y - \{4\} = \{3\}$,

Let $x = 2, f(x) = f(2) = 4$, then

(iii) If $V = \{3\}$,$f(2) = 4\delta_Y Y - \{3\} = \{4\}$,

If $V = \{4\}$, then $f(2) = 4\bar{\delta}_Y Y - \{4\} = \{3\}$, there exist $U = \{2\}, 2\bar{\delta}_\chi \chi - \{1\}$,

$f(U) = f(\{2\}) = 4\bar{\delta}_Y Y - \{4\}$={3}, that's imply f is δ_ω −continuous function.

5.3 Remark

Notes that f in above example is proximity continuous function ($\delta -$ continuous) since $\{3\}\bar{\delta}_Y\{4\}$ and $f^{-1}\{3\}\bar{\delta}_\chi f^{-1}\{4\}$.

5.4 Remark:

If $f: (\chi, T_{\delta_\chi}) \to (Y, T_{\delta_Y})$ such that $T = T_{\delta_\chi}$ and f is δ_ω −continuous function, then f is continuous function since if f is δ_ω −continuous function, then [by Definition 5.1] for all $x \in \chi, V \subset Y$, such that $f(x)\bar{\delta}_Y Y - V, f(x) \ll_Y V$, there exist $U \subset X, x\bar{\delta}_\chi \chi - U, x \ll_X U$ and $f(U)\bar{\delta}_Y Y - V$, $f(U) \subset V$ imply $f$is continuity function.

But conversely, is not true as that example

5.5 Example:

Let: $(\chi,\delta_\chi) \to (Y,T_{\delta_Y})$, $T_\chi = T_{\delta_{I\chi}}$ is indiscrete proximity topological space defined on a set $\chi = \{\sigma,\mu\}$ and $T_{\delta_{DY}}$ is discrete proximity topological space and $T_Y = T_{\delta_{DY}}$ defined on a set $Y=\{\rho,\vartheta,\alpha\}$ such that $f(\sigma) = f(\mu) = \alpha$.

If $x = \sigma, f(\sigma) = \alpha$ and $V = \alpha$, then $f(\sigma) = \alpha\bar{\delta}_Y Y - \{\alpha\} = \{\rho,\vartheta\}$ since $T_{\delta_{DY}}$ is discrete proximity topological space, there isn't not exist $U \subset \chi$ such that $x\bar{\delta}_\chi \chi - U$, hence f is not δ_ω −continuous function.

5.6 Remark

If $f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$ is δ −continuous function, then f is δ_ω −continuous function, since if for all $x \in \chi, V \subset Y$, $f(x)\bar{\delta}_Y Y - V, f(x) \ll V$[by Definition2.6], then there exist $h \subset Y$such that $f(x) \ll h \ll V$ [by Theorem 2.7 (O6)], put $h = f(U)$, then $f(\theta) \ll f(U) \ll V$ imply $f(U) \ll V$, $f(U)\bar{\delta}_Y Y - V$ [by Definition2.6], and to prove $\theta\bar{\delta}_\chi \chi - U$ since $f(\theta)\bar{\delta}_Y Y - f(U)$ [by Definition 2.5] $f(\theta) \ll f(U)$, $f$is δ −continuous, then $f^{-1}(f(\theta))\bar{\delta}_\chi f^{-1}(Y - f(U))$[by Proposition 3.5] imply $\theta\bar{\delta}_\chi \chi - U, U \subseteq \chi$, then f is δ_ω −continuous function.

5.7 Proposition

$f\colon (\chi,\delta_\chi) \to (\gamma,\delta_\gamma)$ is δ_ω −continuous if and only if for all $x \in \chi$ and for all $V \subset \gamma, f(x) \ll V$, there exist $U \subset \chi, x \ll U, f(U) \ll V$.

Proof

Let $f\colon (\chi,\delta_\chi) \to (\gamma,\delta_\gamma)$ is δ_ω −continuous from proximity space χ to proximity space γ then for all $x \in \chi$, and for all $V \subset \gamma$, $f(x)\bar{\delta}_\gamma \gamma - V$ iff $f(x) \ll V$ [by Definition2.6], there exist $U \subset$, $x\bar{\delta}_\chi \chi - U$ iff $x \ll U, f(U)\bar{\delta}_\gamma \gamma - V$ and $f(U) \ll V$ [by Definition2.6].

5.8 Proposition

Let $f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$ is onto δ_ω −continuous function, then $f^{-1}\colon (Y,\delta_Y) \to (\chi,\delta_\chi)$ is δ_ω −continuous function.

Proof

Let $y \in Y$ and $V \subset \chi, f^{-1}(y)\bar{\delta}_\chi \chi - V$ since f is onto, then there exist $x \in \chi$ such that $f^{-1}(y)=x$, $x\bar{\delta}_\chi \chi - V$. Since f is δ_ω −continuous function, then for all $x \in \chi, H \subset Y, f(x)\bar{\delta}_Y Y - H$, there exists $U \subset \chi$, $U=V$, $x\bar{\delta}_\chi \chi - U, x\bar{\delta}_\chi \chi - V$ $f(U)\bar{\delta}_Y Y - H$.

5.8 Proposition

Let$f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$, then f is δ_ω −continuous function if f for all $x \in \chi$, and for all $V \subset Y, V \in \mathcal{F}(f(x))$, there exist $U \subset \chi$ such that $U \in \mathcal{F}(\{x\}), V \in \mathcal{F}(f(U))$.

Proof:

Let $f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$ is δ_ω −continuous function, then for all $x \in \chi$, and for all $V \subset Y, f(x)\bar{\delta}_Y Y - V$ that mean $f(x) \ll V$ [by Definition 2.6] imply $V \in \mathcal{F}(f(x))$ [by Remark 2.9], there exist $U \subset X$ such that $x \in X$, $x\bar{\delta}_\chi \chi - U, U \in \mathcal{F}(\{x\}), f(U)\bar{\delta}_Y Y - V, V \in \mathcal{F}(f(U))$[Remark 2.9].

Conversely, let for all $\in X$, $V \subset Y$, $V \in \mathcal{F}(f(x))$[by Remark 2.9], then $f(x)\bar{\delta}_Y Y - V$there exist $U \subset X$ such that $U \in \mathcal{F}(\{x\}), x\bar{\delta}_\chi \chi$-$U$, $V \in \mathcal{F}(f(U))$ hence $f(U)\bar{\delta}_Y Y - V$, then f is δ_ω −continuous function

5.9 Proposition

If $f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$ is δ_ω −continuous function from (χ,T_{δ_χ}) to proximity space (Y,T_{δ_Y}) then $f$is continuous with respect topology T_{δ_χ} and T_{δ_Y}, $(T = T_\delta)$

Proof:

Let f is δ_ω −continuous function from proximity space X to proximity space Y then for all $x \in X$, *and* for all $V \subset Y$, $f(x)\bar{\delta}_Y Y - V$,that mean $f(x) \ll V$ [by Definition 2.6] and V is T_{δ_Y} −open set [by proposition 3.3], there exist $U \subset X$, $x\bar{\delta}_\chi \chi - U$[by proposition 3.3]$U$ *is* T_{δ_Y} −open set [by proposition 3.3],) that mean $x \ll U$ [by Definition 1.6] and $f(U)\bar{\delta}_Y Y - V$ and $f(U) \ll V$, *then* $f(U) \subset V$ [by theorem 2.7(2)] thus f is continuous function.

5.10 Remark

The Converse of [proposition 5.9] is not true since if $\chi = R, d(x,y) = |x - y|$, $\delta_\chi = \delta_d$, let take $A = N$, $B = \left\{\frac{(n+1)}{2n},\ n \in N\right\}$, $A\delta_\chi B$ *if and only if* $d(A,B) = 0$, where $d(A,B) = \inf\{d(x,y)\colon x \in A, y \in B\}$ and $T_{\delta_\chi} = T_{\delta_Y}$and let $A = \bar{A}, B = \bar{B}, A\bar{\delta}_Y B \iff \bar{A} \cap \bar{B} = \emptyset$ and since $\delta_\chi < \delta_Y$, then f is not δ_ω −continuous function.

5.11 Remark

If $f\colon (\chi,\delta_{I\chi}) \to (Y,\delta_Y)$, then f is not δ_ω −continuous function because, if f is any function from $(\chi,\delta_{I\chi})$ to any proximity space (Y,δ_Y) and for all $x \in \chi$, $V \subset Y$ such that $f(x)\,\bar{\delta}_Y Y - V$, $\delta_{I\chi}$ is indiscrete proximity space, then [by Example 2.2] for all $A, B \subset \chi, A\delta_\chi B$ if and only if $A \neq \emptyset, B \neq \emptyset$ and $\{x\} \neq \emptyset$ imply is near from any nonempty subset of X, there is not exist $U \subset X$ and $x\,\bar{\delta}_{IX} X - U$, finally f is not δ_ω −continuous function.

5.12 Proposition

I$f\colon (\chi,\delta_\chi) \to (Y,\delta_Y)$ is constant function from proximity space (χ,δ_χ) to proximity space (Y,δ_Y), then $f$is δ_ω −continuous function.

Proof:

Let f is constant function and for all $x \in X$, $f(x) = \alpha, \alpha \in Y$and let $V \subset Y$such that $f(x) = \alpha\,\bar{\delta}_Y Y - V$, then for all $U \subset \chi$ and since f is constant $f(U) = f(x) = \alpha$, then $f(U)\bar{\delta}_Y Y - V$, f is δ −continuous since for all $A\delta_\chi B \to f(A)\delta_Y f(B)$ and $\{x\}\delta_Y\{x\}$ for all $x \in X$ [by Definition 2.1 P4], To prove $x\bar{\delta}_\chi X - U$, since $f(U)\bar{\delta}_Y Y - V$, $f$is δ −continuous $f^{-1}(f(U))\,\bar{\delta}_Y\, f^{-1}(Y - V)$ [by

Proposition 3.5], $f^{-1}(f(x))\bar{\delta}_Y X - f^{-1}(V)$, $x\,\bar{\delta}_\chi \chi - f^{-1}(V)$, put $f^{-1}(V) = U$ that's imply $x\,\bar{\delta}_\chi \chi - U$, hence f δ_ω −continuous function.

5.13 Proposition

If $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$is δ_ω −continuous function,$A \subseteq X$, then $f_{/A}: (A, \delta_A) \to (Y, \delta_Y)$ is δ_ω −continuous function.

Proof

If $x \in A$ and $V \subset Y$ so $x \in X$ and because $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous function then for all $x \in X, V \subset Y, f(x)\bar{\delta}_Y Y - V$there exist $U \subset \chi$, $x\bar{\delta}_\chi \chi - U$, then $x \ll_\chi U$ hence, $x\bar{\delta}_A \chi - U$[by Theorem 3.9] and $\mathcal{F}_A(\{x\}) = \mathcal{F}_X(\{x\}) \cap A$ [by proposition 2.7],$A \cap U \in \mathcal{F}_A(\{x\}), \{x\} \ll_A A \cap U, \{x\}\ \bar{\delta}_A\ A - (A \cap U)$, since $A - (A \cap U) \subset \chi - U$, since$f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous and $A \cap U \subset U$,then$f(U))\bar{\delta}_Y Y - V$, hence $f(A \cap U))\bar{\delta}_Y Y - V$ which mean $f_{/A}: (A, \delta_A) \to (Y, \delta_Y)$ is δ_ω −continuous function.

5.14 Proposition

If $f: (X, \delta_X) \to (Y, \delta_Y)$ is $\delta -$ continuous such that δ_X, δ_Y are discrete relations, then f is δ_ω −continuous function.

Proof

Let $x \in \chi$ and $V \subset Y, f(x)\ \bar{\delta}_Y Y - V$, so because f is $\delta -$ continuous, then $x\,\bar{\delta}_\chi \chi - f^{-1}(V)$, let $f^{-1}(V) = U$, then $x\,\bar{\delta}_X \chi - U$. To prove $f(U)\ \bar{\delta}_Y Y - V$,if $f(U)\delta Y - V$, then $f(f^{-1}(V)\delta_Y Y - V$ and then $V\delta Y - V$ but $V \cap (Y - V) = \emptyset$ and this contradiction, then f is δ_ω −continuous function.

5.15 Proposition

Let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous function and $\mathrm{K} \subset Y$, $(\mathrm{K}, \delta_\mathrm{K})$is proximity subspace of proximity space(Y, δ_Y), *then* $f: (\chi, \delta_\chi) \to (\mathrm{K}, \delta_\mathrm{K})$ is δ_ω −continuous function.

Proof:

let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous function., then for all $x \in \chi, V \subseteq Y, f(x)\bar{\delta}_Y Y - V$, since $(\mathrm{K}, \delta_\mathrm{K})$ is proximity subspace of Y, then $V \subset \mathrm{K} \subset Y$, then $\mathrm{K} - V \subset Y - V$, also [by Theorem 3.8] $f(x)\bar{\delta}_\mathrm{K}\mathrm{K} - V$, and since f is $\delta\omega$ −continuous, then there exist $U \subset X, x\bar{\delta}_X X - U$,$f(U)\bar{\delta}_Y Y - V$ imply $f(U)\bar{\delta}_\mathrm{K}\mathrm{K} - V$ [by proposition 1.4(b)] hence $f: (\chi, \delta_\chi) \to (\mathrm{K}, \delta_\mathrm{K})$ is δ_ω −continuous function.

4.16 Remark

Let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous function and $(H, \delta_{/H})$ is proximity subspace of proximity space (χ, δ_χ) and $(K, \delta_{/K})$ is proximity subspace of (Y, δ_Y), then $f: (H, \delta_{/H}) \to (K, \delta_{/K})$ is δ_ω −continuous function since if $x \in H$ and let $V \subset K$, $f(x)\bar{\delta}_{/K}K - V$, $f(x) \ll_K K$.

To prove $f: (H, \delta_{/H}) \to (K, \delta_{/K})$ is δ_ω −continuous function. Since f is δ_ω −continuous, then there exist $U \subset \chi$ such that $x \ll_\chi U$ [by Definition 5.1] and since $H \subset \chi, H - U \subset \chi - U, x\,\bar{\delta}_H H - U$ and $f(U)\bar{\delta}_Y Y - V$ but $V \subset K \subset Y$, $K - V \subset Y - V$, then $f(U)\bar{\delta}_{/K}K - V$, hence $f: (H, \delta_{/H}) \to (K, \delta_{/K})$ is δ_ω −continuous function.

5.17 Proposition

Let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ is δ_ω −continuous function, then there exist two disjoint sets $C, D \subseteq Y$such that $C^c \ll f(\varphi)^c$.
and $D^c \ll V$, for all $\varphi \subset X$, $x\bar{\delta}_\chi \chi - \varphi$ and $V \subset Y$, $f(x)\bar{\delta}_Y Y - V$.

Proof

Since f is δ_ω −continuous then for all $x \in X$, and for all $V \subset Y$, $f(x)\bar{\delta}_Y Y - V$there exist $\varphi \subset X, x\bar{\delta}_\chi \chi - \varphi$,$f(\varphi)\bar{\delta}_Y Y - V$ and [by proposition 2.4(ii)] there exist two disjoint sets $C, D \subseteq Y$ [by Proposition 2.5(iii)], $f(\varphi)\bar{\delta}_Y Y - C, f(\varphi) \subset C, C^c \subset f(\varphi)^c$,hence $C^c \ll f(\varphi)^c$ and $Y - V\bar{\delta}_Y Y - D$, then $Y - D\bar{\delta}_Y Y - V$ imply $D^c \subset V, D^c \ll V$.

5.18 Proposition

Let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$is δ_ω −continuous, then for all $x \in X, V \subset Y$, then the following is satisfies:

1. $\overline{V^c} \subseteq \chi - f(\varphi)$ $[\overline{V^c} \subseteq (f(\varphi))^c$,
2. $V^c \subseteq (f(\varphi)^c)^\circ$ [V is open],
3. $\overline{f(\varphi)}\ \bar{\delta}_Y Y - V$ for some $\varphi \subseteq \chi$ and for all $V \subseteq Y$,
4. $f(\varphi) \ll int(V)$ $[f(\varphi) \ll \overline{(V^c)}^c]$.

Proof:

1. Since $f$is δ_ω −continuous, then for all$x \in \chi, V \subseteq Y, f(x)\bar{\delta}_Y Y - V$ there exist $\varphi \subset \chi$ such that$x \in \chi$, $x\bar{\delta}_\chi \chi - \varphi$ and $f(\varphi)\bar{\delta}_Y Y - V$[by Proposition 3.4] $\overline{Y - V} \subset \chi - f(\varphi)$, then $\overline{V^c} \subset (f(\varphi))^c$.

2. By [Proposition 3.4] $\chi - V \subset int\,(\chi - f((\varphi))$, then $V^c \subseteq (f(\varphi)^c)^\circ$,
3. Since $f(\varphi) \ll V$[ByProposition 3.6(i)],$\overline{f(\varphi)} \ll V$ and $\overline{f(\varphi)}\ \bar{\delta}_Y Y - V$ [By Proposition 3.6(ii)].
4. Since $f(\varphi) \ll V$, *then* $f(\varphi) \ll int(V)$ and since [$int\ A = (\overline{A^c})^c$] then $[f(\varphi) \ll \overline{(V^c)}^c]$.

5.19 Proposition

Let δ_χ,δ_Y be two proximity relations on the set χ, $f: (\chi, \delta_X) \to (Y, \delta_Y), \chi = Y$ is identity function of a set χ, then f is δ_ω −continuous function if and only if $\delta_\chi > \delta_Y$.

Proof:

Let f is δ_ω −continuous, then for all $x \in \chi$, and for all $V \subset Y$, $f(x)\ \bar{\delta}_Y Y - V$ there exist $U \subset \chi$, $x\bar{\delta}_\chi \chi - U, x \in U$, then $f(x)\bar{\delta}_Y f(\chi - U) = f(\chi) - f(U)$ but $f(\chi) = Y$ [f is identity], let $f(U) = V$ hence$f(x)\bar{\delta}_Y Y - V$ for all $x \in \chi$, then $\delta_\chi > \delta_Y$, Conversely, let $\delta_\chi > \delta_Y$ and for all $x \in \chi, V \subset Y$, $f(x)\bar{\delta}_Y Y - V$ but $f(x) = x, f^{-1}(Y) = \chi, x\bar{\delta}_\chi \chi - f^{-1}(V)$, now let $f^{-1}(V) = U \subset \chi$, hence there exist $U \subset \chi, x\bar{\delta}_\chi \chi - U, x \in U$ and $f(U) = \cup \{f(x), x \in U\}$, then f is δ_ω −continuous function.

5.20 Proposition

If $f: (\chi, T_{\delta_\chi}) \to (Y, T_{\delta_Y})$ is onto δ_ω −continuous, then for every open set V in T_{δ_Y} there exist open set U in T_{δ_χ} such that $f(U) \subset V$.

Proof

Let f is δ_ω −continuous , $V \in T_{\delta_Y}$ since f is onto then for all $y \in V \subset Y, y\bar{\delta}_Y Y - V$ since f is onto there exist $x \in \chi$, such that $y = f(x)$ and since V is open, then $y = f(x)\ \bar{\delta}_Y Y - V$ [by Proposition 3.4] and since f is δ_ω −continuous function, then there exist $U \subset \chi$ such that $x\bar{\delta}_\chi \chi - U$, $x \in U$ imply U is open [by Proposition 2.3] and $f(U)\bar{\delta}_Y Y - V$, hence $f(U) \subset V$.

5.21 Remake

If $f: (\chi, T_{\delta_\chi}) \to (Y, T_{\delta_Y})$ is open function, then not necessary f is δ_ω −continuous function as explained in that example.

5.22 Example

Let $f: \left(\chi, T_{\delta_\chi}\right) \to (Y, \delta_Y)$ such that f is identity function, $\chi = \{m, j, k\} = Y$, T_{δ_χ} is normal proximity topological space, $T_{\delta_X} = \{X, \emptyset, \{m\}, \{j, k\}\}$ and δ_Y is discrete relation defined on Y, clearly f is open function but is not δ_ω −continuous since if $x = j, f(j) = j$, let V={j}, $f(j)\bar{\delta}_Y Y - \{j\} = \{m, k\}$ since $\{j\} \cap \{m, k\} = \emptyset$ [by Example 1.2 (i)], there exist $U = \{j, k\} \subset X$, $j\bar{\delta}_\chi \chi - U = \chi - \{j, k\} = \{m\}$ but $f(\{j, k\})\delta Y - \{j\} = \{m, k\}$, then f is not δ_ω −continuous.

5.23 Proposition

Let $f: (\chi, \delta_\chi) \to (Y, \delta_Y)$ δ_ω −continuous, then the following statements are holds for all $x \in \chi, U, \mathrm{E} \subset \chi$ and $V, H \subset Y$:

i. There exist nonempty set E subset of χ such that $x \ll_\chi \chi - \mathrm{E} \ll_\chi$ U,
ii. There exist nonempty set H subset of Y such that, $f(U) \ll_Y Y - H \ll_Y V$.

Proof:

i. since f is δ_ω −continuous, then for all $x \in \chi, V \subset Y, f(x)\bar{\delta}_Y Y - V$, there exist $U \subset \chi, x\bar{\delta}_X \chi - U$, then there exist E $\subset \chi$ [by Definition 2.1.P5] such that $\{x\}\bar{\delta}_\chi$E, $x \ll_X \chi - \mathrm{E}$ [by Definition 2.6] and $\chi - \mathrm{E}\ \bar{\delta}_\chi\ \chi - \mathrm{U}$ imply $\chi - \mathrm{E} \ll \mathrm{U}$ [by Definition 2.6], then $x \ll_\chi\ \chi - \mathrm{E} \ll_X \mathrm{U}$ [by Definition 2.6].
ii. since f is δ_ω −continuous, then for all $x \in \chi, V \subset Y, f(x)\bar{\delta}_Y Y - V$, there exist $U \subset \chi, x\bar{\delta}_\chi \chi - U$, $f(U)\bar{\delta}_Y Y - V$, then there exist $\emptyset \neq H \subset Y$, $f(U)\bar{\delta}_Y H$ and $f(U) \ll_Y Y - H$ [by Definition 2.6] and $Y - H\bar{\delta}_Y Y - V, Y - H \ll_Y V$[by Definition 2.6] imply $f(U) \ll_Y Y - H \ll_Y V$.

6. CONCLUSION

δ_ω −continuous is a new type of continuity in the space of topological proximity. The lesson of continuity of each element of domain depending on definition of the neighborhood in the space of topological proximity.
This kind has got qualification of special condition and it is related to or connected with continuity in topological space and continuity within the topologicl proximity space.

7. REFERENCES

1. Anna Di Concilio, Proximity: a powerful tool in extension theory, function spaces, hyperspaces, Boolean algebra and point-free geometry, article January 2009, DOI: 10.1090/ conm /486/09508.
2. Agata Caserta, Strong Proximal Continuity and Convergence, Handawi publishing Corporation, volume 2013, Article ID 412796, 10 pages.
3. A. Kandil, O. A. El-Tantawy, On $\mathcal{G}-$ Proximity Spaces, Appl. Math. Inf. Sci. Lett. 4, No. 2,79-84(2016).
4. Longin Latecki, Semi-proximity continuous functions in digital images, Pattern Recognition Letters 16(1995)1175-1187.
5. L.A. Al Swidi, Raghad Almohammed, New Concepts via Topological Proximity Space, AIP Conf. Proc. 3097, 080014(2024) .
6. Cluster Outer Point via Proximity Space R Almohammed, LAA Jaber Iraqi Journal of Science 863-872,
7. Radoslav Dimitrijevic`, Proximity and uniform spaces, UNi·su, Februara 2010, PP:1-20.
8. S. Leader, Local proximity space, Math. Ann. 169(1957), 275-281.
9. S. Leader, Extension based on proximity boundedness, Math. Z. 108(1969), 137-144.
10. M. W. Lodato, On topological induced generalized proximity relation I, Proc. Amer. J. Math. Soc. 15 (1964), 417-422.
11. M. W. Lodato, On topological induced generalized proximity relation II, Pacific J. Math.17 (1966), 131-135.
12. J. F. Peters and P. Wasilewski, Foundations of near sets, Inform. Sci. 179(2009), 3091- 3109.
13. YKM Altalkany, L.A. Al Swidi, Focal Function in i –Topological Space via Proximity Space.

Abstract Arabic

نظراً لأهمية الاستمراريه كمفهوم رياضي ودورها الكبير في حل الكثير من المشاكل الرياضيه ولكون فضاء القرب من المصطلحات الحديثه تمكنا في هذا البحث من الاجابه على سؤال مهم جدا هو هل يمكن وضع ودراسة أنواع من الاستمراريه كالانواع الموجودة في الفضاء التبولوجي الاعتيادي؟ وقد تمكنا في هذه الورقه من إيجاد نوع من الاستمراريه أطلق عليه (δ_{ω} – continuous) واستطعنا الحصول على العديد من النتائج معتمدين بذلك على نظريات القرب وخواص فضاء القرب المخلتفه والحالات الخاصه للتبولوجي المتولد بواسطة فضاء القرب.