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New Subclasses of Regular and Bi-univalent Functions Based on Horadam polynomials

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ABSTRACT

In this paper, new subclasses of bi-univalent functions associated with Horadam polynomials are introduced and investigated. Additionally, the researchers find estimates of the first two coefficients of functions in these subclasses. Moreover, they obtained the Fekete-Szegö inequalities for these function classes. Besides, pertinent links of the results are provied with those considered in previously investigations.

1. INTRODUCTION

Let f be function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad . \qquad (1)$$

which belongs to A where A is the class of regular functions defined on the disk

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

with $f(0) = f'(0) - 1 = 0$ and let S be the subclass of A consisting of the form (1) which are also univalent in U. The Koebe's Covering Theorem (see [1]) states for each $f \in S$ the image $w = f(z), z \in U$,, in the w-plane contains the disk$\{w : |w| < 1/4\}$. From this theorem, every function $f \in S$ has an inverse f^{-1}which holds

$$f^{-1}(f(z)) = z, (z \in U)$$

and,

$$f(f^{-1}(w)) = w \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

A function $f \in \mathbb{A}$ is called bi-univalent in U if both f and f^{-1} are univalent in U.The class of all bi-univalent functions in U is denoted byΣ. This class was introduced by Lewin [2] and showed that $|a_2| \leq 1.51$ for the function in the class Σ. Recently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu in [4] proved that $|a_2| = \frac{4}{3}$. Several researchers have examined these subclasses of bi-univalent regular function and found estimates of the initial coefficients for functions in different subclasses [5-10].

A function $f \in \mathbb{A}$ is called a λ–pseudo starlike function in U if the following inequality holds are true (see [11]):

$$\mathcal{R}\left[\frac{z(f'(z))^{\lambda}}{f(z)}\right] \geq 0, (z \in U), \lambda \geq 1.$$

For two regular functions f_1 and f_2, the function f_1 subordination to f_2 in the disk U, is written as follows:

$$f_1(z) \prec f_2(z), \quad (z \in U),$$

if there is a regular function k with $k(0) = 0$ and $|k(z)| < 1$, such that

$$f_1(z) = f_2(k(z)), (z \in U).$$

In particular, when f_2 is univalent in U,

$$f_1 \prec f_2 \quad (z \in U) \iff f_1(0) = f_2(0) \text{ and } f_1(U) = f_2(U).$$

Lately, the polynomial defined below has been included in the topic of the geometric function theory of complex

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analysis. The Horadam polynomials are defined by the recurrence relationship as follows [12].

$$\hbar_n(r) = pr\hbar_{n-1}(r) + q\hbar_{n-2}(r) \quad r \in \mathbb{R}, n > 2, n \in \aleph, \qquad (2)$$

with

$$\hbar_1(r) = a,\ \hbar_2(r) = br \text{ and } \hbar_3(r) = pbr^2 + aq \text{ for some } p.q, a, b \in \mathbb{R}.$$

The characteristic equation of the iteration relation (2) is

$$t^2 - prt - q = 0.$$

The real roots of this equation are

$$\alpha_1 = \frac{pr+\sqrt{(pr)^2+4q}}{2}, \alpha_2 = \frac{pr-\sqrt{(pr)^2+4q}}{2}.$$

It should be noted that for specific values of parameters, the Horadam polynomial $\hbar_n(x)$ leads to different polynomials. To learn more about the details and special cases of the Horadam polynomials (see [13–14]). In [14], the Horadam polynomials $\hbar_n(r)$ are generated by:

$$g(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \quad (3)$$

The Horadam polynomials $\hbar_n(r)$ have recently been used in a similar context by Srivastava et al. [15]. Subsequently, many authors used a Horadam polynomial (see [16–19]).

The aim of the present work is to define two subclasses of the function classΣ using the Horadam polynomials $\hbar_n(r)$ and estimate the bounds of the coefficients a_2, a_3 and the Fekete–Szegö for functions of the subclasses presented throughout this work.

2. COEFFICIENT ESTIMATES FOR SUBCLASS $G_\Sigma(\lambda, p, q, r, g)$

We start with introducing the function subclass $G_\Sigma(\lambda, p, q, r, g)$ using the following definition..

Definition (2.1): Let $G(\lambda, \Sigma, p, q, r, g)$ be the class of the function $f \in \Sigma$ given by (1) under the next subordination

$$\left[\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - \frac{\lambda z^2 f''(z)+zf'(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\right] \prec g(r, z) - a, (z \in U),$$

$$\left[\frac{wg'(w)}{g(w)} + \frac{wg''(w)}{g'(w)} - \frac{\lambda w^2 g''(w)+wg'(w)}{\lambda w g'(w)+(1-\lambda)g(w)}\right] \prec g(r, w) - a, (w \in U),$$

and for $g = f^{-1}, 0 \leq \lambda \leq 1$.

Remark(2.2): Setting $\lambda = 0$ in the class $G(\lambda, \Sigma, p, q, r, g)$, we obtain $X(\Sigma, p, q, r, g)$.

Remark(2.3): Setting $\lambda = 1$ in the class $G(\lambda, \Sigma, p, q, r, g)$, we obtain $T(\Sigma, p, q, r, g)$.

At first, we state and prove the next result.

Theorem (2.4): Let $f \epsilon G_\Sigma(\lambda, p, q, r, g)$. Then

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|(br)^2\,(1-4\lambda+(\lambda+1)^2)-(bpr^2+qa)(2-\lambda)^2|}}, \quad (4)$$

$$|a_3| \leq \frac{|br|^2}{(2-\lambda)^2} + \frac{|br|}{2(3-2\lambda)}, \qquad (5)$$

$$|a_3 - \varepsilon a_2^2| \leq \begin{cases} \dfrac{|br|}{2(3-2\lambda)}, & if \\ |\varepsilon - 1| \leq \dfrac{|(br)^2\,(1-4\lambda+(\lambda+1)^2)-(bpr^2+qa)(2-\lambda)^2|}{2(3-2\lambda)(br)^2}, \\ \dfrac{|(br)^3\,(1-\varepsilon))|}{|(br)^2\,(1-4\lambda+(\lambda+1)^2)-(bpr^2+qa)(2-\lambda)^2|}, & if \\ |\varepsilon - 1| \geq \dfrac{|(br)^2\,(1-4\lambda+(\lambda+1)^2)-(bpr^2+qa)(2-\lambda)^2|}{2(3-2\lambda)(br)^2} \end{cases}$$

Proof: Since $f \epsilon G_\Sigma(\lambda, p, q, r, g)$, there are holomorphic functions π, v belong to $\mathbb{A}$ and $\pi, v: U \to U$ given

$\pi(z) = \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \cdots \quad (z \in U)$,

$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots, (w \in U)$,

such that $\pi(0) = v(0) = 0$ and $|\pi(z)| < 1, |v(w)| < 1, (z, w \in U)$ and we can write

$$\left[\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - \frac{\lambda z^2 f''(z)+zf'(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\right] = g(r, \pi(z)) - a,$$

$$\left[\frac{wg'(w)}{g(w)} + \frac{wg''(w)}{g'(w)} - \frac{\lambda w^2 g''(w)+wg'(w)}{\lambda w g'(w)+(1-\lambda)g(w)}\right] = g(r, v(w)) - a. \quad (6)$$

Or, in equivalent way,

$$\left[\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - \frac{\lambda z^2 f''(z)+zf'(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\right] = \hbar_2(r)\pi_1 z + [\{\hbar_2(r)\pi_2 + \hbar_3(r)\pi_1^2\} +]z^2 + \cdots, \quad (7)$$

and

$$\left[\frac{wg'(w)}{g(w)} + \frac{wg''(w)}{g'(w)} - \frac{\lambda w^2 g''(w)+wg'(w)}{\lambda w g'(w)+(1-\lambda)g(w)}\right] = \hbar_2(r)v_1 w + [\{\hbar_2(r)v_2 + \hbar_3(r)v_1^2\}]w^2 + \cdots. \quad (8)$$

From (7) and (8), it follows that

$$(2-\lambda)a_2 = \hbar_2(r)\pi_1, \qquad (9)$$

$$(6-4\lambda)a_3 - (5-(\lambda+1)^2)a_2^2 = \{\hbar_2(r)\pi_2 + \hbar_3(r)\pi_1^2\}, (10)$$

$$-(2-\lambda)a_2 = \hbar_2(r)v_1, \qquad (11)$$

$$(7-8\lambda+(\lambda+1)^2)a_2^2 - (6-4\lambda)a_3 = \{\hbar_2(r)v_2 + \hbar_3(r)v_1^2\}. \ (12)$$

From (9) and (11), it follows that

$$\pi_1 = -v_1 \quad (13)$$

$$2(2-\lambda)^2 a_2^2 = \hbar_2^2(r)\,(\pi_1^2 + v_1^2). \ (14)$$

Adding (10) and (12), we obtain that

$$2[1-4\lambda+(\lambda+1)^2]a_2^2 = \hbar_2(r)(\pi_2+v_2) + \hbar_3(r)(\pi_1^2+v_1^2) \qquad (15)$$

Using (14) in (15), we conclude that

$$2\left[(1-4\lambda+(\lambda+1)^2) - \frac{\hbar_3(r)(2-\lambda)^2}{\hbar_2^2(r)}\right]a_2^2 = \hbar_2(r)(\pi_2+v_2), \qquad (16)$$

$$a_2^2 = \frac{\hbar_2^3(r)(\pi_2+v_2)}{2[\hbar_2^2(r)\,(1-4\lambda+(\lambda+1)^2) - \hbar_3(r)(2-\lambda)^2]} \ (17)$$

From (2) and (17), we have the required inequality (4).

Subtracting (12) from (10), we have

$$a_3 = a_2^2 + \frac{\hbar_2(r)(\pi_2 - v_2)}{4(3-2\lambda)}. \qquad (18)$$

In view of (13) and (14), equation (18) becomes

$$a_3 = \frac{\hbar_2{}^2(r)\left(\pi_1^2+v_1^2\right)}{2(2-\lambda)^2} + \frac{\hbar_2(r)(\pi_2-v_2)}{4(3-2\lambda)}.$$

By using $|\pi_i| < 1$, $|v_i| < 1$ and (2), we have the required inequality (5).

From (18), for $\varepsilon \in \mathbb{R}$, we write

$$a_3 - \varepsilon a_2{}^2 = \frac{\hbar_2(r)(\pi_2-v_2)}{4(3-2\lambda)} + (1-\varepsilon)a_2{}^2 \quad (19)$$

By substituting (17) in (19), we obtain that

$$a_3 - \varepsilon a_2{}^2 = \frac{\hbar_2(r)(\pi_2-v_2)}{4(3-2\lambda)} + \frac{(1-\varepsilon)\,\hbar_2{}^3(r)(\pi_2+v_2)}{2[\hbar_2{}^2(r)\,(1-4\lambda+(\lambda+1)^2) - \hbar_3(r)(2-\lambda)^2]} = \hbar_2(r)\left[\left(\Upsilon(\varepsilon,r) + \frac{1}{4(3-2\lambda)}\right)\pi_2 + \left(\Upsilon(\varepsilon,r) - \frac{1}{4(3-2\lambda)}\right)v_2\right],$$

where

$$\Upsilon(\varepsilon,r) = \frac{(1-\varepsilon)\,\hbar_2{}^2(r)}{2\,[\hbar_2{}^2(r)\,(1-4\lambda+(\lambda+1)^2) - \hbar_3(r)(2-\lambda)^2]}.$$

Thus, we deduce that

$$\left|a_3 - \varepsilon a_2{}^2\right| \le \begin{cases} \frac{\hbar_2(r)}{2(3-2\lambda)}, & 0 \le \Upsilon(\varepsilon,r) \le \frac{1}{4(3-2\lambda)}, \\ 2|\hbar_2(r)|\Upsilon(\varepsilon,r), & |\Upsilon(\varepsilon,r)| \ge \frac{1}{4(3-2\lambda)}. \end{cases}$$

From here with consideration of (2), it clearly shows that the proof of Theorem 2.4 is completed.

Corollary (2.5): If the function $f(z)$ in $X(\Sigma, p, q, r, g)$ given by (1), then

$$|a_2| \le \frac{|br|\sqrt{|br|}}{\sqrt{|2(br)^2 - 4(bpr^2+qa)|}},$$

$$|a_3| \le \frac{|br|^2}{4} + \frac{|br|}{6},$$

$$\left|a_3 - \varepsilon a_2{}^2\right| \le \begin{cases} \frac{|br|}{6}, & \text{if } |\varepsilon - 1| \le \frac{|2(br)^2 - 4(bpr^2+qa)|}{6(br)^2}, \\ \frac{|(br)^3\,(1-\varepsilon)|}{|2(br)^2 - 4(bpr^2+qa)|}, & \text{if } |\varepsilon - 1| \ge \frac{|2(br)^2 - 4(bpr^2+qa)|}{6(br)^2}. \end{cases}$$

Corollary (2.6): If the function $f(z)$ in $T(\Sigma, p, q, r, g)$ given by (1), then

$$|a_2| \le \frac{|br|\sqrt{|br|}}{\sqrt{|(br)^2 - bpr^2 + qa|}},$$

$$|a_3| \le |br|^2 + \frac{|br|}{2},$$

$$\left|a_3 - \varepsilon a_2{}^2\right| \le \begin{cases} \frac{|br|}{2}, & |\varepsilon - 1| \le \frac{|(br)^2 - (bpr^2+qa)|}{2(br)^2}, \\ \frac{|(br)^3\,(1-\varepsilon)|}{|(br)^2 - (bpr^2+qa)|}, & |\varepsilon - 1| \ge \frac{|(br)^2 - (bpr^2+qa)|}{2(br)^2}. \end{cases}$$

3. COEFFICIENT ESTIMATES FOR SUBCLASS $\mathcal{M}_\Sigma(\mu, \lambda, \rho, p, q, r, g)$

Definition (3.1): A function $f \in \Sigma$ of the form (1) is in $\mathcal{M}_\Sigma^q(\mu, \lambda, \rho, p, q, r, g)$ if it satisfies conditions

$$(1-\mu)\left[\rho\left(\frac{zf''(z)}{f'(z)} + 1\right) + (1-\rho)f'(z)\right] + \mu\left[\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right] \prec g(r,z) + 1 - a,\ (z \in \mathrm{U}),$$

$$(1-\mu)\left[\rho\left(\frac{w\mathrm{g}''(w)}{\mathrm{g}'(w)} + 1\right) + (1-\rho)\mathrm{g}'(w)\right] + \mu\left[\frac{w\left(\mathrm{g}'(w)\right)^\lambda}{\mathrm{g}(w)}\right] \prec g(r,w) + 1 - a,\ (w \in \mathrm{U}),$$

for $\mathrm{g} = f^{-1}, 0 \le \mu \le 1, 0 \le \rho \le 1$ and $\lambda \ge 1$.

Remark (3.2): For $\mu = 0$, the class $\mathcal{M}_\Sigma(\mu, \lambda, \rho, p, q, r, g)$ is shortened to the class $\mathcal{G}_\Sigma^*(\alpha, x)$ presented and investigated in [20]. In particular, in [20], for $\rho = 0$, we have

$$\mathcal{M}_\Sigma(0, \lambda, 0, p, q, r, g) \coloneqq \mathcal{H}_\Sigma(x),$$

also, for $\rho = 1$, we have

$$\mathcal{M}_\Sigma(0, \lambda, 1, p, q, r, g) \coloneqq \mathcal{K}_\Sigma(x).$$

Remark (3.3): For $\mu = a = 1, p = \mathrm{b} = 2, q = -1$ and $r \to t$, then $\mathcal{M}_\Sigma(\mu, \lambda, \rho, p, q, r, g)$ reduced to the class $\mathcal{L}\mathfrak{B}_\Sigma(\lambda, t)$ presented and investigated in [21].

Remark (3.4): For $\mu = 1$, we have the class $\mathcal{H}_\Sigma(\lambda, \rho, p, q, r, g)$.

Remark (3.5): For $\lambda = 1$, we have the class $\mathcal{N}_\Sigma(\mu, \rho, p, q, r, g)$.

Theorem (3.6): If $f \in \mathcal{M}_\Sigma(\mu, \lambda, \rho, p, q, r, g)$, then

$$|a_2| \le \frac{2|br|\sqrt{|br|}}{\sqrt{\left|\begin{matrix}(br)^2\,\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)+2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\} \\ -2(bpr^2+qa)\{2+\mu(2\lambda-3)\}^2\end{matrix}\right|}}, \quad (20)$$

$$|a_3| = \frac{br}{3(1-\mu)(\rho+1)+\mu(3\lambda-1)} + \frac{(br)^2}{\{2+\mu(2\lambda-3)\}^2}, \quad (21)$$

$$|a_3 - \varepsilon a_2{}^2| \le \begin{cases} \frac{|br|}{3(1-\mu)(\rho+1)+\mu(3\lambda-1)} & \text{if } |\varepsilon - 1| \le \frac{|\boldsymbol{\theta_1} - \boldsymbol{\theta_2}|}{2(br)^2[3(1-\mu)(\rho+1)+\mu(3\lambda-1)]}, \\ \frac{|1-\varepsilon\ |\ |br|^3}{|\boldsymbol{\theta_1} - \boldsymbol{\theta_2}|} & \text{if } |\varepsilon - 1| \ge \frac{|\boldsymbol{\theta_1} - \boldsymbol{\theta_2}|}{2(br)^2[3(1-\mu)(\rho+1)+\mu(3\lambda-1)]} \end{cases}$$

Where $\boldsymbol{\theta_1} = (br)^2\{\mu(2\lambda^2 - 4\lambda + 1) - 4\rho(1-\mu) + 2(1-\mu)(\rho+3) + \mu(2\lambda^2 + 2\lambda - 1)\}$

$\boldsymbol{\theta_2} = 2(pbr^2 + qa)\{2 + \mu(2\lambda - 3)\}^2$

Proof: Since $f \in \mathcal{M}_\Sigma^q(\mu, \lambda, \rho, p, q, r, g)$, there are regular functions π, v belong to $\mathbb{A}$ and $\pi, v: \mathrm{U} \to \mathrm{U}$ given

$$\pi(z) = \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \cdots, \quad (z \in \mathrm{U}),$$

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 \ldots, \quad (w \in \mathrm{U}),$$

Such that $\pi(0)=\upsilon(0)=0$ and $|\pi(z)|<1, |\upsilon(w)|<1, (z,w\in U)$ satisfying

$$(1-\mu)\left[\rho\left(\frac{zf''(z)}{f'(z)}+1\right)+(1-\rho)f'(z)\right]+\mu\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right]=g\left(r,\pi(z)\right)+1-a,$$

$$(1-\mu)\left[\rho\left(\frac{wg''(w)}{g'(w)}+1\right)+(1-\rho)g'(w)\right]+\mu\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\right]=g\left(r,\upsilon(w)\right)+1-a,$$

$$(1-\mu)\left[\rho\left(\frac{zf''(z)}{f'(z)}+1\right)+(1-\rho)f'(z)\right]+\mu\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right]=1+h_2(r)\pi_1 z+[\{h_2(r)\pi_2+h_3(r)\pi_1{}^2\}+]z^2+\cdots, \quad (22)$$

$$(1-\mu)\left[\rho\left(\frac{wg''(w)}{g'(w)}+1\right)+(1-\rho)g'(w)\right]+\mu\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\right]=1+h_2(r)\upsilon_1 w+[\{h_2(r)\upsilon_2+h_3(r)\upsilon_1{}^2\}+]w^2+\cdots. \quad (23)$$

From (22) and (23), it follows that

$$\{2+\mu(2\lambda-3)\}a_2=h_2(r)\pi_1, \quad (24)$$

$$\{3(1-\mu)(\rho+1)+\mu(3\lambda-1)\}a_3+\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)\}a_2{}^2=h_2(r)\pi_2+h_3(r)\pi_1{}^2, \quad (25)$$

$$-\{2+\mu(2\lambda-3)\}a_2=h_2(r)\upsilon_1, \quad (26)$$

$$\{2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\}a_2{}^2-\{3(1-\mu)(\rho+1)+\mu(3\lambda-1)\}a_3=h_2(r)\upsilon_2+h_3(r)\upsilon_1{}^2. \quad (27)$$

The equations (24) and (26), lead to

$$\pi_1=-\upsilon_1. \quad (28)$$

Squaring and adding (24), (26), we have

$$2\{2+\mu(2\lambda-3)\}^2 a_2{}^2=h_2^2(r)(\pi_1{}^2+\upsilon_1{}^2). \quad (29)$$

Add (25) to (27), we conclude that

$$\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)+2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\}a_2{}^2=h_2(r)(\pi_2+\upsilon_2)+h_3(r)(\pi_1{}^2+\upsilon_1{}^2). \quad (30)$$

By substituting (29) in (30), we obtain that

$$\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)+2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\}a_2{}^2=h_2(r)(\pi_2+\upsilon_2)+h_3(r)\frac{2\{2+\mu(2\lambda-3)\}^2}{h_2^2(r)}a_2{}^2,$$

$$a_2{}^2=\frac{h_2^3(r)(\pi_2+\upsilon_2)}{\theta_3-\theta_4} \quad (31)$$

From (2) and (31), we have the required inequality (20).

In order to find a_3, we subtract (27) from (25) to obtain

$$[6(1-\mu)(\rho+1)+2\mu(3\lambda-1)](a_2{}^2-a_3)=(\pi_2-\upsilon_2)h_2(r)+(\pi_1{}^2-\upsilon_1{}^2)h_3(r). \quad (32)$$

In view of (28) and (32), we have

$$a_3=\frac{(\pi_2-\upsilon_2)h_2(r)}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}+a_2{}^2. \quad (33)$$

In view of (28) and (29), equation (33) becomes

$$a_3=\frac{(\pi_2-\upsilon_2)h_2(r)}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}+\frac{h_2^2(r)(\pi_1{}^2+\upsilon_1{}^2)}{2\{2+\mu(2\lambda-3)\}^2}.$$

Now, by using equation (2) and applying $|\pi_i|\le 1, |\upsilon_i|\le 1$, we deduce that

$$|a_3|=\frac{|br|}{3(1-\mu)(\rho+1)+\mu(3\lambda-1)}+\frac{(br)^2}{\{2+\mu(2\lambda-3)\}^2}.$$

For $\varepsilon\in\mathbb{R}$, using (33), we conclude that

$$a_3-\varepsilon a_2{}^2=\frac{(\pi_2-\upsilon_2)h_2(r)}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}+(1-\varepsilon)a_2{}^2. \quad (34)$$

By substituting (31) in (34), we have

$$a_3-\varepsilon a_2{}^2=\frac{(\pi_2-\upsilon_2)h_2(r)}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}+\frac{(1-\varepsilon)h_2^3(r)(\pi_2+\upsilon_2)}{\theta_3-\theta_4}.$$

Where $\theta_3=h_2^2(r)\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)+2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\}$

$$\theta_4=h_2^2(r)\{\mu(2\lambda^2-4\lambda+1)-4\rho(1-\mu)+2(1-\mu)(\rho+3)+\mu(2\lambda^2+2\lambda-1)\}$$

Simplifying the above equation, we obtain that

$$a_3-\varepsilon a_2{}^2=h_2(r)\left\{\left[\frac{1}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}+\vartheta(\varepsilon,r)\right]\pi_2+\left[\vartheta(\varepsilon,r)-\frac{1}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}\upsilon_2\right]\right\},$$

where

$$\vartheta(\varepsilon,r)=\frac{(1-\varepsilon)\,h_2^2(r)}{\theta_3-2\,h_3(r)\{2+\mu(2\lambda-3)\}^2}.$$

Thus, we conclude that

$$|a_3-\varepsilon a_2{}^2|\le\begin{cases}\dfrac{h_2(r)}{3(1-\mu)(\rho+1)+\mu(3\lambda-1)}, & if\ 0\le|\vartheta(\varepsilon,r)|\le\dfrac{1}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)},\\ 2|h_2(r)||\vartheta(\varepsilon,r)|, & if\ |\vartheta(\varepsilon,r)|\ge\dfrac{1}{6(1-\mu)(\rho+1)+2\mu(3\lambda-1)}\end{cases}$$

By using (2), it shows clearly the proof of Theorem 3.6 is completed.

Corollary (3.7): If $f\epsilon\mathcal{H}_\Sigma(\lambda,\rho,p,q,r,g)$, then

$$|a_2|\le\frac{2|br|\sqrt{|br|}}{\sqrt{\left|2\lambda(br)^2(2\lambda-1)-2(bpr^2+qa)(2\lambda-1)^2\right|}},$$

$$|a_3|\le\frac{br}{(3\lambda-1)}+\frac{(br)^2}{(2\lambda-1)^2},$$

$$|a_3-\varepsilon a_2{}^2| \leq \begin{cases} \frac{|br|}{(3\lambda-1)} \text{ if} \\ |\varepsilon-1| \leq \frac{|2\lambda(br)^2\,(2\lambda-1)-2(pbr^2+qa)(2\lambda-1)^2|}{2(br)^2(3\lambda-1)}, \\ \frac{|1-\varepsilon\ |\ |br|^3}{|2\lambda(br)^2\,(2\lambda-1)-2(pbr^2+qa)(2\lambda-1)^2|}, if \\ |\varepsilon-1| \geq \frac{|2\lambda(br)^2\,(2\lambda-1)-2(pbr^2+qa)(2\lambda-1)^2|}{2(br)^2(3\lambda-1)}. \end{cases}$$

Corollary (3.8): If $f\epsilon\mathcal{N}_\Sigma(\mu,\rho,p,q,r,g)$, then

$$|a_2| \leq \frac{2|br|\sqrt{|br|}}{\sqrt{|(br)^2\,\{2\mu-4\rho(1-\mu)+2(1-\mu)(\rho+3)\}-2(bpr^2+qa)(2-\mu\,)^2|}},$$

$$|a_3| \leq \frac{br}{3(1-\mu)(\rho+1)+2\mu}+\frac{(br)^2}{(2-\mu)^2}$$

$$|a_3-\varepsilon a_2{}^2| \leq \begin{cases} \frac{|br|}{3(1-\mu)(\rho+1)+2\mu}, \quad if \\ |\varepsilon-1| \leq \frac{|\theta_5-2(pbr^2+qa)(2-\mu)^2|}{2(br)^2[3(1-\mu)(\rho+1)+2\mu]}, \\ \frac{|1-\varepsilon\ |\ |br|^3}{|\theta_5-2(pbr^2+qa)(2-\mu)^2|}, if \\ |\varepsilon-1| \geq \frac{|\theta_5-2(pbr^2+qa)\{2+\mu(2\lambda-3)\}^2|}{2(br)^2[3(1-\mu)(\rho+1)+2\mu]}, \end{cases}$$

where $\theta_5 = 2\lambda(br)^2\,(2\lambda-1)$.

4. CONCLUSIONS

In this work, new subclasses of bi-univalent functions defined by means of the Horadam polynomials $h_n(r)$ are studied. The central purpose of this study is that bounds for the initial coefficients are established. Furthermore, the researchers solve Fekete-Szego¨ functional problems for functions in$G_\Sigma(\lambda,p,q,r,g) and \mathcal{M}_\Sigma^q(\mu,\lambda,\rho,p,q,r,g)$ in the present work. In the form of corollaries, several exceptional and unique cases of the key theorems are illustrated.

5. REGULAR AND Bi-UNIVALENT FUNCTIONS AND APPLICATIONS

Recent RCS studies discuss the hidden body's response to electromagnetic. Cloaking of electromagnetic has gained interest in the scientific field, especially amongst scientists who are interested in materials-artificial composites which have properties of exotic electromagnetic. In the mathematical sense, in complex plane cloak of two dimensions and cloaked object can be considered as simple connected regions. There are equivalence between the above regions and conformal maps of the unit circle according to the theorem of Riemann Mapping. Suppose that cloaked object and cloak are respectively denoted by the functions $g(z)$ and $q(z)$, then we obtain $g(z) \prec q(z)$.

For the cloak, it is better to be a three dimensional since the cloak relies on the cloaked body and the rays which are reflected by the body could be cloaked because of not consisting all reflected rays.

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Arabic Abstract

في هذا البحث، نُقدّم ونبحث فئات فرعية جديدة من الدوال ثنائية التكافؤ المرتبطة بمتعددات حدود هورادام. بالإضافة إلى ذلك، نجد تقديرات لأول معاملين للدوال في هذه الفئات الفرعية. كما حصلنا على متباينات فيكيتي-سيغو لهذه الفئات من الدوال. كما نُقدّم روابط وثيقة الصلة لنتائجنا بتلك التي نُوقشت في الأبحاث السابقة.