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Mean Semi-Open Sets

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Abstract

This article introduces a new kind of mean sets that are also semi-open. Each such set contains a non-void proper semi-open set and also it is contained in another proper semi-open set. Within this work, we compare our new concepts against the corresponding concepts that were defined via open sets. We study some fundamental properties of mean semi-open sets and their complements and provide some new results. Moreover, we investigate the behavior of such sets when they are also minimal and maximal.

1. INTRODUCTION

Generalizing open sets is not a new research line in general topology. A vast number of studies and investigations were initiated to examine various kinds of open sets, ranging from semi-open , α-open and not ending with pre-open sets. Almost each such study investigated and explored various topological properties possessed by generalized open sets. For example, generalized interior and closure operators were defined depending on various generalized open sets. In [1], a study introduced a new kind of open set called minimal. Many significant results were explored. Then in [2], another study introduced and investigated maximal open sets. The complements of minimal and maximal open sets were investigated in [3]. Maximal and minimal clopen sets were introduced in [4]. Then many authors re-introduced minimal and maximal sets in terms of various generalized open sets such as [5] and [6]. Minimal soft sets have been re-investigated in [7]. Recently, minimal and maximal anti-open sets and their complements have been introduced in [8]. Mean open set was introduced in [9]. This kind of set appeared to possess many interesting properties that can be related to other kinds of sets, like minimal and maximal open sets. The study in [10] characterized mean open sets in connected T_1-spaces. Furthermore, the concept of mean sets has also been introduced on fuzzy open sets in [11].

In this work, we introduce mean sets in terms of semi-open sets. The duality of such sets is also introduced and explored in some detail.

This article consists of the following sections: Section 2 includes some basic but fundamental concepts. Section 3 is devoted to introducing mean semi-open sets and their complements and exploring some of their basic properties. Section 4 includes further results of mean semi-open sets.

2. PRELIMINARIES

Throughout this article, we use letter X to represent a topological space. An open set P in X is minimal open if the only open sets in P the sets ϕ and P itself. The dual of this set is the maximal open set, defined in [2]. An open set P is maximal in X if the only open sets that contain P are X and P itself. Suppose that P is a minimal open set then the complement of P (that is, $X - P$) is a maximal closed. In dual sense, the set

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$X - P$ is a minimal closed set whenever P is maximal open set. See also [3] for further reading.

Mean open sets have been introduced and investigated in [9] and [12]. A set O that is open in X is called mean if $R \subset O \subset Q$ where R and Q are open in X with $R \neq \emptyset$ and $Q \neq X$. Within this article, by $P \subset Q$ we mean that P is a subset of Q with $P \neq Q$. A semi-open set S of X is a set yields $B \subset P \subset \bar{B}$ where B is an open set of X and $\bar{B}$ is the closure of B, [13].
Let $SO(X) = \{P \subseteq X : P \text{ is semi} - \text{open in } X\}$.
The set $A^{o^s} = \cup\{P \in SO(X) : P \subseteq A\}$ represents the semi-interior of A. Now, a set F is semi-closed in X if $X - F \in SO(X)$. Consider the collection $SC(X) = \{F \subset X : F \text{ is semi} - \text{closed in } X\}$.
The set $\overline{A}^s = \cap\{F \in SC(X) : A \subseteq F\}$ represents the semi-closure of A. Further information can be found in [3].

Theorem 2.1. [13] Consider the product space of a topological space X and a topological space Y. If $P \in SO(X)$ and $Q \in SO(Y)$, then $P \times Q \in SO(X \times Y)$.

Let $P, Q \in SO(X)$ with $P, Q \neq X$. Then P is called minimal if the only semi-open set of P is either ϕ or P itself. On the other hand, Q is called maximal if the only semi-open set that contains Q is Q itself or X.

An easy observation is that minimal open sets are indeed minimal semi-open sets.

Lemma 2.2. [14] Let $P \in SO(X)$ and $Z \subset X$ be an open set. Then $P \cap Z \in SO(X)$.

Proposition 2.3. Let $P \in SO(X)$ and $Z \subset X$ be an open set.
1. Whenever P is minimal, then either $P \subset Z$ or $P \cap Z = \emptyset$.
2. Whenever P and Z are minimal, then either $P = Z$ or $P \cap Z = \emptyset$.

Proof. Suppose that $P \cap Z \neq \phi$. Since P is minimal semi-open and $P \cap Z$ is semi-open, Lemma 2.2. So, $P \cap Z = S$. Therefore, $P \subset Z$.
2. If $P \cap Z \neq \emptyset$, then by (1) we have $P \subset Z$. But the set Z is minimal with $P \cap Z \neq \emptyset$, then $P \cap Z = Z$. Thus, $Z \subset P$. Consequently, $P = Z$.

Theorem 2.4. [5] Let $P, Q \in SO(X)$. If P is maximal, then either $P \cup Q = X$ or $Q \subset P$. If both P and Q are maximal with $P \neq Q$. Then $P \cup Q = X$.

A semi-continuous is a map $h : X \to Y$, where X and Y are two topological spaces such that $h^{-1}(P)$ is semi-open in X for any open set P in Y.

3. MEAN SEMI-OPEN SETS

We present in this section mean semi-open sets and present few properties they have.

Definition 3.1. A semi-open set S of X is called mean semi-open if there exist $K, L \in SO(X)$ with $K \neq \emptyset$ and $L \neq X$ such that $K \subset S \subset L$.

Example 3.2. In the usual topological space $X = R$, let consider the set $S = [3,6)$ in X. Clearly S is a mean semi-open set since $K \subset S \subset L$ where $K = [4,5)$ and $L = [2,7)$ and $K, L \in SO(X)$.

Example 3.3. Consider the Euclidean space on $\mathbb{R}^2$. Let $S = \{(x,y) \in \mathbb{R}^2 : 0.2 \leq x < 0.8,\ 0.2 \leq y < 0.8\}$. It is clear that S is semi-open.
Now, let $L = \{(x,y) \in \mathbb{R}^2 : 0 \leq x < 1,\ 0 \leq y < 1\}$ and $K = \{(x,y) \in \mathbb{R}^2 : 0.4 \leq x < 0.6,\ 0.4 \leq y < 0.6\}$. Clearly, $K \subset S \subset L$. Thus, S is mean semi-open.

One can easily notice that if an open set R is mean, then R is also a mean semi-open. However, in general, the converse may not be true. Furthermore, the union of mean semi-open sets need not be mean semi-open sets. The same may occur for the intersection case; see Example 3.4.

Example 3.4. Let $X = \{1,2,3,4,5\}$ be a set and $\tau = \{\phi, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}, X\}$ be a topology on X. If we take the sets $S_1 = \{3,4,5\}$ and $S_2 = \{2,3,4\}$. It is clear that both S_1 and S_2 are mean semi-open, but both the sets $S_1 \cap S_2 = \{3,4\}$ and $S_1 \cup S_2 = \{2,3,4,5\}$ are not mean.

Definition 3.5. A mean semi-closed set C of X is a semi-closed set such that $D \subset F \subset H$ for some semi-closed sets D and H with $D \neq \emptyset$ and $H \neq X$

Example 3.6. Consider the topological space in Example 3.4. Notice that
$SC(X) = \{\emptyset, \{2,3,4,5\}, \{1,2,5\}, \{2,5\}, \{1,2\}, \{1,5\}, \{2\}, \{1\}, X\}$. It is clear that $\{2,5\}$ is a mean semi-closed in X since $\{2\} \subset \{2,5\} \subset \{1,2,5\}$.

Theorem 3.7. Let $P \in SO(X)$. Then P is mean if and only if $X - P$ is mean semi-closed.

Proof. Assume that P is a mean semi-open.
So $K \subset P \subset L$ where $K, L \in SO(X)$ with $K \neq \emptyset$ and $L \neq X$. The complement of $K \subset P \subset L$ gives
$X - L \subset X - P \subset X - L$. Clearly, $X - L$ is non-empty and distinct from $X - P$. Moreover, $X - K$ differs from $X - P$ and X. Therefore, $X - P$ is a mean semi-closed set.
Now, let $P \in SO(X)$ such that $X - P$ is mean semi-closed set.
Thus, $F \subset X - P \subset H$ where F and H are semi-closed sets differ from each other with $F \neq \emptyset$ and $H \neq X$.
Hence, $X - H \subset P \subset X - F$. But $\emptyset \neq X - H \neq P$ and $P \neq X - F \neq X$. Therefore, P is mean semi closed set.

Proposition 3.8. Assume that E be a minimal semi-open, L be a minimal open subsets of X with $E \neq$L. Let $R \in \{E, L\}$. Then either $R = \overline{R}^s$ or $X - \overline{R}^s$ is a mean semi-open set in X.

Proof. From Proposition 2.3, we have $E \cap L = \emptyset$. Hence $E \cap \overline{L}^s = \emptyset$. So $E \subset X - \overline{L}^s$. But both E and L are minimal semi-open, so $X - \overline{L}^s \neq \emptyset, X$.

If the set $X-\overline{L}^s$ is not mean semi-open set, then this set is either minimal semi open or maximal semi-open. Since, $E\subset X-\overline{L}^s$, hence $X-\overline{L}^s$ is not minimal.
Thus, $X-\overline{L}^s$ is maximal. Now, L is semi-open, by Theorem 2.4, we have either $L\subset X-\overline{L}^s$ or $L\cup\left(X-\overline{L}^s\right)=X$. The former case cannot occur, so we have $L\cup\left(X-\overline{L}^s\right)=X$. This implies that $L=\overline{L}^s$. If this case does not occur, then $\overline{L}^s$ is a mean semi-open.
We can proceed with a similar argument to the one above so we can get that either $E=\overline{E}^s$ or $X-\overline{E}^s$ is mean semi-open.

Proposition 3.9. Assume that P and Q be distinct maximal semi-closed subsets of X. Let $H\in\{P,Q\}$, then either $H=H^{s^o}$ or $X-H^{s^o}$ is a mean semi-closed set.

Proof. Since both P and Q are distinct maximal semi-closed sets, so their complement sets, $X-P$ and $X-Q$ are minimal semi-open sets with $X-P\neq X-Q$. Proposition 3.8 leads to either $G=\overline{G}^s$ or $X-\overline{G}^s$ is a mean semi-open set, where $G=X-P$ or $G=X-Q$.
In case that $G=X-P$, then clearly $P=P^{o^s}$, as $G=\overline{G}^s$, or since $X-\overline{G}^s$ is a mean semi-open set, we get that $X-\left(\overline{X-P}^s\right)=P^{o^s}\in SO(X)$ and it is a mean in X.
Hence, Theorem 3.7 implies that $X-P^{o^s}$ is a mean semi-closed set. Now, a similar argument can be applied for $X-Q$ to find that either $Q=Q^{o^s}$ or $X-Q^{o^s}$ is mean semi-closed by Theorem 3.7.

4. MORE RESEULTS on MEAN SEMI-OPEN SETS

Proposition 4.1. Suppose that $P,Q\in SO(X)$ are both maximal with $P\neq Q$. If $R\in SO(X)$ is a mean, then, $P\cap Q\neq\emptyset$.

Proof. By Theorem 2.4, $P\cup Q=X$. Since R is a mean set. So R can not be maximal or minimal semi-open.
So $R\notin\{P,Q\}$. On the other hand, $R\neq\emptyset,X$.
Theorem 2.4 leads to either $R\subset P$ or $R\cup P=X$ and $R\subset Q$ or $R\cup Q=X$.
Consequently, we have to check the following possible cases:
$R\subset P$ and $R\subset Q$.
Verification: Since $R\subset P,Q$ and never $R=P,Q$, so $R\subset(P\cap Q)$. Hence, $P\cap Q\neq\emptyset$.
$R\subset P$ and $R\cup Q=X$.
Verification: If $R\cap Q\neq\emptyset$, then $P\cap Q\neq\emptyset$. Assume now $R\cap Q=\emptyset$. But, $R\subset P$, so there exists an $x\in P-R$. But $R\cup Q=X$, so $x\in Q$. Therefore, $P\cap Q\neq\emptyset$.
$R\subset Q$ and $R\cup P=X$.
Verification: Similar proof to the previous one.
$R\cup P=X$ and $R\cup Q=X$.
Verification: Since $R\cup P=X$ and $R\cup Q=X$, so $R\cup(P\cap Q)=X$. Hence, if $P\cap Q=\emptyset$, then $R=X$. But $R\neq X$, therefore, $P\cap Q\neq\emptyset$.

Proposition 3.4. Consider the product space $X\times Y$ where X and Y be two spaces. Let S and T be a mean semi-open subsets of X and Y, respectively. Then $P\times Q$ is a mean semi-open set in $X\times Y$.

Proof. Consider the set $P\times Q$. By the hypothesis, we get that $\phi\neq K_X\subset P\subset L_X\neq X$ where $K_X,L_X\in SO(X)$. Similarly, $\phi\neq K_Y\subset Q\subset L_Y\neq Y$ where $K_Y,L_Y\in SO(Y)$. By Theorem 2.1, $\{K_X\times K_Y,\ L_X\times L_Y, P\times Q\}\subset SO(X\times Y)$. Since P and Q is a mean semi open set in X and Y, respectively. Consequently, $K_X\times K_Y\neq\phi$ and $L_X\times L_Y\neq X$. Therefore, $K_X\times K_Y\subset P\times Q\subset L_X\times L_Y$.

Proposition 4.3. Consider the following bijective map h from space X to space Y. If h is semi-continuous and P is a mean open set Y, then $h^{-1}(P)\in SO(X)$ and it is also mean.

Proof. Since P is a mean open subset of Y, then
$\phi\neq K\subset P\subset L\neq Y$ where $K,L\subset Y$ and both K and L are open in Y. Semi-continuity of h implies that $h^{-1}(P),h^{-1}(K),h^{-1}(L)\in SO(X)$.
But h, is bijective, so
$\phi\neq h^{-1}(K)\subset h^{-1}(P)\subset h^{-1}(L)\neq h^{-1}(Y)=X$.
Hence, $h^{-1}(P)$ iسs mean semi-open set in X.

5. CONCLUSION

We present the notion of mean semi-open sets. Such sets are defined via semi-open sets. investigated The properties of such sets along with their complements namely mean semi-closed sets, are investigated. In addition, the study offer some interesting results regarding being mean semi-open sets maximal and minimal. Besides, other similar results for mean semi-closed sets are presented.

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Arabic Abstract

في هذه المقالة تقدم نوعا جديدا من مجموعات المتوسط المعرفة بدلالة المجموعات شبه المفتوحة. تحتوي كل مجموعة على مجموعة شبه مفتوحة فعلية غير فارغة وفي نفس الوقت توجد نلك المجموعة في مجموعة شبه مفتوحة فعلية أخرى. في هذا العمل نقارن المفاهيم التي قدمناه مع المفاهيم المناضرة لها والمعرفة بدلالة المجموعات المفتوحة. ندرس بعض خصائص متوسط المجموعات شبه المفتوحة وكذلك مكملاتها مع تقديم بعض النتائج الجديدو. بالاضافة الى ذلك، فإننا ندرس سلوك هذه المجموعات عندما تكون أيضا مجموعات صغرى و عندما تكون مجموعات كبرى.