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$g_*^* - \mathcal{J}$-Closed Sets and Their Properties in in Ideal Topological Space

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Abstract

There are many research papers that deal with different types of generalized closed sets. Levine [4] introduced generalized closed (briefly, g-closed) sets and studied their basic properties and Veera Kumar [5] introduced g^*-closed sets in topological spaces. The purpose of this present paper is to define a new class of generalized idea closed sets called $g_*^* - I$-closed sets by using g^*-open set .In this paper, we introduce the $g_*^* - \mathcal{J}$-closed sets, characterizations and properties of $g_*^* - \mathcal{J}$-closed sets and its complement and other related sets. We prove that the class of $g_*^* - \mathcal{J}\_$ closed sets lies between the class of $\mathcal{J}g$-closed sets and the class of $g*$-closed sets. Also, we find some relations between $g_*^* - \mathcal{J}$-closed sets and already existing closed sets. $g_i^* - \mathcal{J}$-open neighborhood is introduced and their properties are investigated.

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1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [1] and Vaidyanatha swamy [2]. In 1990, Jankovic and Hamlett [3] once again investigated applications of topological ideals. The concept of generalized closed sets plays a significant role in topology. There are many research papers that deal with different types of generalized closed sets. Levine [4] introduced generalized closed (briefly, g-closed) sets and studied their basic properties and Veera Kumar [5] introduced g^*-closed sets in topological spaces. The purpose of this present paper is to define a new class of generalized idea closed sets called $g_*^* - I$-closed sets by using g^*-open set (which is a complement of g^*-closed set) and also we obtain the basic properties of called $g_*^* - I$-closed set in ideal topological spaces.

2. Preliminaries

An ideal I on a non-empty set X is a collection of subsets of X which satisfies the following properties [1], [2].
(i) $A \in I, B \in I \Rightarrow A \cup B \in I$
(ii) $A \in I, B \subset A \Rightarrow B \in I$

A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). Let Y be a subset of $X. I_Y = \{I \cap Y / I \in I\}$ is an ideal on Y and by $(Y, \tau/Y, I_Y)$ we denote the ideal topological subspace. Let $P(X)$ be the power set of X, then a set operator $(\ )^*: P(X) \to P(X)$ called the local function [1] of A with respect to τ and I is defined as follows:

For $A \subset X, A^*(I, \tau) = \{x \in X / U \cap A \notin I$ for every open set U containing $x\}$.

We write A^* instead of $A^*(I, \tau)$ in case there is no confusion. A Kuratowski closure operator $cl^*(\ )$ for a topology $\tau^*(I, \tau)$, called the τ^* - topology is defined by $cl^*(A) = A \cup A^*$ [6]

A subset A of a space (X, τ) is said to be semi-open [7] if $A \subset cl(\text{int}\,(A))$. A set operator $()^{*S}: P(X) \to P(X)$ called a semi-local function and $cl^{*s}()$ [7] of A with respect to τ and I are defined as follows:

For $A \subset X$, $A^{*S}(I, \tau) = \{x \in X / U \cap A \notin I$ for every semi open set U containing $x\}$. and $Cl^{*S}(A) = A \cup A^{*S}$.

Note: A^{*S} defined in [7] and A_* defined in [8] are the same. For a subset A of X, cl (A)(resp scl (A)) denotes the closure (resp semi closure) of A in (X, τ). Similarly $cl^*(A)$ and int $^*(A)$ denote the closure of A and interior of A in (X, τ^*).

A subset A of X is called * closed (resp.* S - closed) if $A^* \subseteq A$ (resp $A^{*S} \subseteq$ A) [3]. A is called * - dense in itself (resp .*S-dense) [3]. If $A \subset A^*$ (resp $\subset A^{*S}$) A is called * - perfect (resp .*s - perfect). If $A = A^*$ (resp $A = A^{*S}$) [3]. A subset$A$ of a topological space (X, τ) is said to be generalized closed (briefly g-closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ) [3]. The complement of g-closed set is said to be $g - open$.

Definition 2.1. A subset A of a topological space (X, τ) is said to be g^*-closed set if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ) [5].

Definition 2.2. A subset A of a space (X, τ, I) is said to be
(i) $g\mathcal{I}$ – closed [9] if $A^{*S} \subseteq U$ wherever $A \subseteq U$ and U is open in X.
(ii) $\mathcal{I}g -$ closed [10] if $A^* \subseteq U$ wherever $A \subseteq U$ and U is open in X.

Definition 2.3. A space (X, τ, I) is said to be a T_I-space if every I-generalized closed subset of X is τ^*-closed [10] [14].

Definition 2.4. A subset A of an ideal topological space (X, τ, I) is said to be I- compact if for every τ-open cover $\{\omega_\alpha : \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $(X - \cup \{\omega_\alpha : \alpha \in \Delta_0\}) \in I$ [11], [12].

Lemma 2.5. [13] Let $(X, \tau, \mathcal{I})$ be an ideal space and $W \subseteq X$. If $W \subseteq W^*$, then $W^* = Cl(W^*) = Cl(W) = Cl^*(W)$.

Theorem 2.6. Let $(X, \tau, \mathcal{I})$ be an ideal space. If W is an $\mathcal{I}g$-closed subset of X, then W is $\mathcal{I}$-compact [14], Theorem 2.17].

Note: In general the intersection of g-closed sets need not be g-closed.

Definition 2.7. [7] A topological space (X, τ) is said to be a g-multiplicative space if the arbitrary intersection of g-closed sets in X is g-closed.

Remark 2.8. [7]
1. In g-multiplicative spaces, $gCl(W)$ which is the intersection of all g-closed sets in X containing W is also g-closed.
2. Any indiscrete topological space (X,τ) is g-multiplicative.
3. If $X=\{x,y,z\}$ and $\tau=\{X,\emptyset,\{x\}\}$ then $\{x,z\}$ and $\{x,y\}$ are g-closed but $\{x\}$ is not g-closed and hence (X,τ) is not g-multiplicative.

Theorem 2.9. [10] (Theorem 3.20). Let $(X,\tau,\mathcal{J})$ be an ideal space and $W\subset Y\subset X$ where Y is α-open in X. Then $W^*(\mathcal{J}_Y,\tau_Y)=W^*(\mathcal{J},\tau)\cap Y$.

Lemma 2.11. [3] Let (X,τ) be a space, I and J be ideals on X, and let A and B be subsets of X. Then
(1) $A\subseteq B\Rightarrow A^*\subseteq B^*$.
(2) If $I\subseteq J$, then $A^*(I)\supseteq A^*(J)$.
(3) $A^*(I)=Cl(A^*)\subseteq Cl(A)$ (i.e, A^* is a closed subset of (A)).
(4) If $A\subseteq A^*$, then $A^*=Cl(A^*)=Cl(A)=Cl^*(A)$.
(5) $(A^*)^*\subseteq A^*$.
(6) $(A\cup B)^*=A^*\cup B^*$.
(7) If $U\in\tau$, then $U\cap A^*=U_x(U_x\cap A)^*\subseteq(U\cap A)^*$.
(8) If $A\in I$, then $A^*=\emptyset$.

Lemma 2.12. [3] For any two sets A and B of an ideal topological space $(X,\tau,\mathcal{J}),Cl^*(A\cup B)=Cl^*(A)\cup Cl^*(B)$.

3. Methodology

Definition 3.1: A subset W of an ideal space $(X,\tau,\mathcal{J})$ is said to be

1. $g_*^*-\mathcal{J}$-closed, if $Cl^*(W)\subset U$ whenever $W\subset U$ and U is g^*-open in X.
2. $g_*^*-\mathcal{J}$-open, if its complement is $g_*^*-\mathcal{J}$-closed set.

The collection of all $g_*^*-\mathcal{J}$-closed sets (resp $g_*^*-\mathcal{J}$-open sets) is denoted by ($g_*^*C(X)$ (resp $g_*^*O(X)$).

Remark 3.2: In any ideal topological space $(X,\tau,\mathcal{J})$,

1. Every $g_*^*-\mathcal{J}$-closed set is $\mathcal{J}g$-closed set.
2. Every $\mathcal{J}g$-closed set is $g\mathcal{J}$-closed set.
3. Every $g_*^*-\mathcal{J}$-closed set is $g\mathcal{J}$-closed set.
4. The converse of part (3) is not true in general, see the following example.

Example 3.3. Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$, and $\mathcal{J}=\{\emptyset,\{c\}\}$. Put $A=\{b\}$ and the only open sets containing A are $\{a,b\}$ and X, then $A_*=\{b\}\subseteq\{a,b\}$, whenever $\{a,b\}$ is open and $\{b\}\subseteq\{a,b\}$. So A is $g\mathcal{J}$-closed set. But, since $A^*=\{b\}^*=\{b,c\}$, so $Cl^*(\{b\})=\{b,c\}\nsubseteq\{a,b\}$, whenever $\{b\}\subseteq\{a,b\}$ and $\{a,b\}$ is also g^*-open.

Theorem 3.4. Every $*$-closed set is $g_*^*-\mathcal{J}$-closed but not conversely.
Proof. Let W be a $*$-closed, then $W^*\subseteq W$. Let $W\subseteq U$ where U is g^*-open. Hence $Cl^*(W)\subseteq U$ whenever $W\subseteq U$ and U is g^*-open. Therefore W is $g_*^*-\mathcal{J}$-closed.

Example 3.5. Let $X=\{x,y,z\}$ with a topology $\tau=\{\emptyset,X,\{x\},\{y,z\}\}$ and an ideal $\mathcal{J}=\{\emptyset,\{z\}\}$. Then $g_*^*-\mathcal{J}$-closed sets are the power set of X and $*$-closed sets are $\emptyset,X,\{x\},\{z\},\{x,z\},\{y,z\}$. It is clear that $\{y\}$ is $g_*^*-\mathcal{J}$-closed set but it is not $*$-closed.

Theorem 3.6. If $(X,\tau,\mathcal{J})$ is an ideal topological space and $W \subset X$. Then the following are equivalent.

1. W is $g_*^* - \mathcal{J}$-closed,
2. For all $x \in Cl^*(W), g^*Cl(\{x\}) \cap W \neq \emptyset$,
3. $Cl^*(W) - W$ contains no nonempty g^*-closed set,
4. $W^* - W$ contains no nonempty g^*-closed set,

Proof. (1) ⇒ (2) Suppose $x \in Cl^*(W)$. If $g^*Cl(\{x\}) \cap W = \emptyset$, then $W \subseteq X - g^*Cl(\{x\})$. By Definition 3.1, $Cl^*(W) \subseteq X - g^*Cl(\{x\})$, which is a contradiction, since $x \in Cl^*(W)$.
(2) ⇒ (3) Suppose $F \subseteq Cl^*(W) - W, F$ is g^*-closed and $x \in F$. Since $F \subseteq X - W$ and F is g^*-closed, then $W \subseteq X - F$ and F is g^*-closed, $g^*Cl(\{x\}) \cap W = \emptyset$. Which is a contradiction. Since $x \in Cl^*(W)$ by (3), $g^*Cl(\{x\}) \cap W \neq \emptyset$. Therefore $Cl^*(W) - W$ contains no nonempty g^*-closed set.
(3) ⇒ (4) Since $l^*(W) - W = (W \cup W^*) - W = (W \cup W^*) \cap W^c = (W \cap W^c) \cup (W^* \cap W^c) = W^* \cap W^c = W^* - W$. Therefore $W^* - W$ contains no nonempty g^*-closed set.
(4) ⇒ (1) Let $W \subseteq U$ where U is a g^*-open set. Therefore $X - U \subseteq X - W$ and so $Cl^*(W) \cap (X - U) \subseteq Cl^*(W) \cap (X - W) = W^* - W$. Therefore $Cl^*(W) \cap (X - U) \subseteq W^* - W$.

Since $Cl^*(W)$ is always $*$-closed set, so $Cl^*(W)$ is g^*-closed set and so $Cl^*(W) \cap (X - U)$ is a g^*-closed set contained in $W^* - W$. Therefore $Cl^*(W) \cap (X - U) = \emptyset$ and hence $Cl^*(W) \subseteq U$. Therefore W is $g_*^* - \mathcal{J}$-closed.

Theorem 3.7. If $(X,\tau,\mathcal{J})$ is an ideal space, then W^* is always $g_*^* - \mathcal{J}$-closed for every subset W of X.

Proof. Let $W^* \subseteq U$ where U is g^*-open. Since $(W^*)^* \subseteq W^*$ so by Lemma 2.11, we have $Cl^*(W^*) \subseteq U$ whenever $W^* \subset U$ and U is g^*-open. Hence W^* is $g_*^* - \mathcal{J}$-closed.

Theorem 3.8. Let $(X,\tau,\mathcal{J})$ be an ideal space. For every $W \in \mathcal{J}$, W is $g_*^* - \mathcal{J}$-closed.

Proof. Let $W \subseteq U$ where U is g^*-open set. Since $W^* = \emptyset$ for every $W \in \mathcal{J}$, then $Cl^*(W) = W \cup W^* = W \subseteq U$. Therefore, W is $g_*^* - \mathcal{J}$-closed.

Corollary 3.9. If $(X,\tau,\mathcal{J})$ is an ideal space and W is a $g_*^* - \mathcal{J}$-closed set, Then the following are equivalent:

1. W is a *-closed set,
2. $Cl^*(W) - W$ is a g^*-closed set,
3. $W^* - W$ is a g^*-closed set.

Proof. (1) ⇒ (2) If W is $*$-closed, then $W^* \subseteq W$ and so $Cl^*(W) - W = (W \cup W^*) - W = \emptyset$, so $Cl(\emptyset) = \emptyset \subseteq U$. Hence $Cl^*(W) - W$ is g^*-closed set.
(2) ⇒ (3) Since $Cl^*(W) - W = W^* - W$ and so $W^* - W$ is g^*-closed set.
(3) ⇒ (1) If $W^* - W$ is a g^*-closed set, since W is $g_*^* - \mathcal{J}$-closed set, by Theorem 3.6, $W^* - W = \emptyset$ and so W is $*$-closed.

Theorem 3.10. Let $(X,\tau,\mathcal{J})$ be an ideal space. Then every $g_*^* - \mathcal{J}$-closed, g^*-open set is *-closed set.

Proof. Since W is $g_*^* - \mathcal{J}$-closed and g^*-open. Then $Cl^*(W) \subseteq W$ whenever $W \subseteq W$ and W is g^*-open. Hence W is $*$-closed.

Corollary 3.11. If $(X,\tau,\mathcal{J})$ is a $T_{\mathcal{J}}$ ideal space and W is a $g_*^* - \mathcal{J}$-closed set, then W is $*$-closed set.

Proof. Since every $g_*^* - \mathcal{J}$-closed set is an $\mathcal{J}g$-closed set in an ideal space $(X,\tau,\mathcal{J})$ and X is $T_{\mathcal{J}}$ space, so every $\mathcal{J}g$-closed set is $*$-closed. So W is $*$-closed.

Theorem 3.12. If $(X,\tau,\mathcal{J})$ is an ideal space, Then every g^*-closed set is an $g^*_* - \mathcal{J}$-closed set but not conversely.

Proof. Let W be a g^*-closed set. If $W \subseteq U$, whenever U is g^*-open. Since every g^*-open is g-open and W is g^*-open, so $Cl(W) \subseteq U$. But, since $Cl^*(W) \subseteq Cl(W) \subseteq U$, whenever $W \subseteq U$ and U is g^*-open, so W is $g^*_i - \mathcal{J}$-closed.

Example 3.13. Let $X = \{x,y,z\}$ with a topology $\tau = \{\emptyset, X, \{x\}, \{x,z\}\}$ and an ideal $\mathcal{J} = \{\emptyset, \{x\}\}$. Then $g^*_* - \mathcal{J}$-closed sets are $\emptyset, X, \{x\}, \{y\}, \{x,y\}, \{y,z\}$ and g^*-closed sets are $\emptyset, X, \{y\}, \{y,z\}$. It is clear that $\{x\}$ is a $g^*_* - \mathcal{J}$-closed set but it is not g^*-closed in (X,τ).

Example 3.14. Let $X = \{x,y,z\}$ with a topology $\tau = \{\emptyset, X, \{x\}, \{x,z\}\}$ and an ideal $\mathcal{J} = \{\emptyset, \{y\}, \{z\}, \{y,z\}\}$. Clearly, the set $\{z\}$ is a $g^*_* - \mathcal{J}$-closed set but it is not g^*-closed in $(X,\tau,\mathcal{J})$.

Theorem 3.15. If $(X,\tau,\mathcal{J})$ is an ideal space, and W is a $*$-dense in itself, $g^*_* - \mathcal{J}$-closed subset of X, then W is g^*-closed.

Proof. Suppose W is a $*$-dense in itself, $g^*_* - \mathcal{J}$-closed subset of X. Let $W \subseteq U$ where U is g-open. Then, $Cl^*(W) \subseteq U$ whenever $W \subseteq U$ and U is g-open. Since W is $*$-dense in itself, so every g^*-open is g-open and W is $*$-dense in itself, by Lemma 2.5, $Cl(W) = Cl^*(W)$. Therefore $Cl(W) \subseteq U$ whenever $W \subseteq U$ and U is g-open. Hence W is g^*-closed.

Corollary 3.16. If $(X,\tau,\mathcal{J})$ is an ideal space where $\mathcal{J} = \{\emptyset\}$, then W is $g^*_* - \mathcal{J}$-closed if and only if W is g^*-closed

Proof. From the fact that for $\mathcal{J} = \{\emptyset\}, W^* = Cl(W) \supseteq W$. Therefore W is $*$-dense in itself. Since W is $g^*_* - \mathcal{J}$-closed, by Theorem 3.15, W is g^*-closed.

Conversely, by Theorem 3.12, every g^*-closed set is a $g^*_* - \mathcal{J}$-closed set.

Theorem 3.17. Let $(X,\tau,\mathcal{J})$ be an ideal space and $W \subseteq X$. Then W is $g^*_* - \mathcal{J}$-closed if and only if $W = F - N$ where F is $*$-closed and N contains no nonempty g^*-closed set.

Proof. If W is $g^*_* - \mathcal{J}$-closed, then by Theorem 3.6 (4), $N = W^* - W$ contains no nonempty g^*-closed set. If $F = Cl^*(W)$, then F is $*$-closed such that $F - N = (W \cup W^*) - (W^* - W) = (W \cup W^*) \cap (W^* \cap W^c)^c = (W \cup W^*) \cap ((W^*)^c \cup W) = (W \cup W^*) \cap (W \cup (W^*)^c) = W \cup (W^* \cap (W^*)^c) = W$.

Conversely, suppose $W = F - N$ where F is $*$-closed and N contains no nonempty g^*-closed set. Let U be a g^*-open set such that $W \subseteq U$. Then $F - N \subseteq U$ which implies that $F \cap (X - U) \subseteq N$. Now $W \subseteq F$ and $F^* \subseteq F$ then $W^* \subseteq F^*$ and so $(W^* \cup W) \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. By hypothesis, since $(W^* \cup W) \cap (X - U)$ is g^*-closed, $(W^* \cup W) \cap (X - U) = \emptyset$ and so $Cl^*(W) \subseteq U$. Hence W is $g^*_* - \mathcal{J}$-closed.

Theorem 3.18. Let $(X,\tau,\mathcal{J})$ be an ideal space and $W \subseteq X$. If $W \subseteq B \subseteq W^*$, then $W^* = B^*$ and B is $*$-dense in itself.

Proof. Since $W \subseteq B$, then $W^* \subseteq B^*$ and since $B \subseteq W^*$, then $B^* \subseteq (W^*)^* \subseteq W^*$. Therefore $W^* = B^*$ and $B \subseteq W^* \subseteq B^*$. Hence proved.

Theorem 3.19. Let $(X,\tau,\mathcal{J})$ be an ideal space and $W \subseteq X$. If W and B are subsets of X such that $W \subseteq B \subseteq Cl^*(W)$ and W is $g^*_* - \mathcal{J}$-closed, then B is $g^*_* - \mathcal{J}$-closed.

Proof. Let $B \subseteq U$ and U is g^*-open. Since $W \subseteq B$ and W is $g^*_* - \mathcal{J}$-closed, so $Cl^*(W) \subseteq U$. But, since $B \subseteq Cl^*(W)$, implies that $Cl^*(B) \subseteq Cl^*(Cl^*(W)) = Cl^*(W) \subseteq U$. Therefor $Cl^*(W) \subseteq U$, whenever $B \subseteq U$. And U is g^*-open. Thus B is $g^*_* - \mathcal{J}$-closed.

Corollary 3.20. Let $(X,\tau,\mathcal{J})$ be an ideal space. If W and B are subsets of X such that $W \subseteq B \subseteq W^*$ and W is $g^*_* - \mathcal{J}$-closed, then W and B are g^*-closed sets.

Proof. Let W and B be subsets of X such that $W \subseteq B \subseteq W^*$ which implies that $W \subseteq B \subseteq W^* \subseteq Cl^*(W)$ and W is $g_*^* - \mathcal{J}$-closed. By Theorem 3.19, B is $g_*^* - \mathcal{J}$-closed. Since $W \subseteq B \subseteq W^*$, then $W^* = B^*$ and so W and B are $*$-dense in itself. By Theorem 3.15, W and B are g^*-closed.

Theorem 3.21. Let $(X, \tau, \mathcal{J})$ be an ideal space and $W \subseteq X$. Then W is $g_*^* - \mathcal{J}$-open if and only if $F \subseteq \text{int }^*(W)$ whenever F is g^*-closed and $F \subseteq W$.

Proof. Suppose W is $g_*^* - \mathcal{J}$-open. If F is g^*-closed and $F \subseteq W$, then $X - W \subseteq X - F$ and so $Cl^*(X - W) \subseteq X - F$. Therefore $F \subseteq X - Cl^*(X - W) = \text{int }^*(W)$. Hence $F \subseteq \text{int}^* (W)$.

Conversely, suppose the condition holds. Let U be a g^*-open set such that $X - W \subseteq U$. Then by hypothesis $X - U \subseteq W$ and so $X - U \subseteq \text{int}^* (W)$. Therefore $Cl^*(X - W) \subseteq U$. Thus, $X - W$ is $g_*^* - \mathcal{J}$-closed. Hence W is $g_*^* - \mathcal{J}$-open.

Corollary 3.22. Let $(X, \tau, \mathcal{J})$ be an ideal space and $W \subseteq X$. If W is a $g_*^* - \mathcal{J}$-open, then $F \subseteq \text{int }^*(W)$ whenever F is closed and $F \subseteq W$.

Proof. Since every closed set is g^*-closed set, so by Theorem 3.21 we get the result.

The following theorem gives a property of $g_*^* - \mathcal{J}$-closed.

Theorem 3.23. Let $(X, \tau, \mathcal{J})$ be an ideal space and $W \subseteq X$. If W is $g_*^* - \mathcal{J}$-open and $\text{int }^*(W) \subseteq B \subseteq W$, then B is $g_*^* - \mathcal{J}$-open.

Proof. Since W is $g_*^* - \mathcal{J}$-open, then $X - W$ is $g_*^* - \mathcal{J}$-closed. By Theorem 3.6 (4), $Cl^*(X - W) - (X - W)$ contains no nonempty g^*-closed set. Since $\text{int }(W) \subseteq int^*(B)$ which implies that $Cl^*(X - B) \subseteq Cl^*(X - W)$ and so $Cl^*(X - B) - (X - B) \subseteq Cl^*(X - W) - (X - W)$ by Theorem 3.6 we get, $X - B$ is $g_*^* - \mathcal{J}$-closed. Thus, B is $g_*^* - \mathcal{J}$ - open.

The following theorem gives a characterization of $g_*^* - \mathcal{J}$-closed sets in terms of $g_*^* - \mathcal{J}$-open sets.

Theorem 3.24. If $(X, \tau, \mathcal{J})$ be an ideal topological space and $W \subseteq X$. Then the following are equivalent:

1. W is $g_*^* - \mathcal{J}$-closed,
2. $W \cup (X - W^*)$ is $g_*^* - \mathcal{J}$-closed,
3. $W^* - W$ is $g_*^* - \mathcal{J}$-open.

Proof. (1) ⇒ (2) Suppose W is $g_*^* - \mathcal{J}$-closed. If U is any g^*-open set such that $\cup (X - W^*) \subseteq U$, then $X - U \subseteq X - \big(W \cup (X - W^*)\big) = X \cap (W \cup (W^*)^c)^c = W^* \cap W^c = W^* - W$. Since W is $g_*^* - \mathcal{J}$-closed, by Theorem 3.6 (4), it follows that $X - U = \emptyset$ and so $X = U$. Therefore $W \cup (X - W^*) \subseteq U$ which implies that $W \cup (X - W^*) \subseteq X$ and so $Cl^*\big(W \cup (X - W^*)\big) \subseteq X = U$. Hence $W \cup (X - W^*)$ is $g_*^* - \mathcal{J}$-closed.
(2) ⇒ (1) Suppose $W \cup (X - W^*)$ is $g_*^* - \mathcal{J}$-closed. If F is any g^*-closed set such that $F \subseteq W^* - W$, then $F \subseteq W^*$ and $F \subseteq X \setminus W$ which implies that $X - W^* \subseteq X - F$ and $W \subseteq X - F$. Therefore $W \cup (X - W^*) \subseteq W \cup (X - F) = X - F$ and $X - F$ is g^*-open. Since $Cl^*\big(W \cup (X - W^*)\big) \subseteq X - F$ and since $\big(W \cup (X - W^*)\big)^* \subseteq Cl^*\big(W \cup (X - W^*)\big) \subseteq X - F$ which implies that $W^* \cup (X - W^*)^* \subseteq X - F$ and so $W^* \subseteq X - F$ which implies that $F \subseteq X - W^*$. Since $F \subseteq W^*$, it follows that $F = \emptyset$. Hence by Theorem 3.6 W is $g_*^* - \mathcal{J}$ closed.
(2) ⇒ (3) Since $-(W^* - W) = X \cap (W^* \cap W^c)^c = X \cap ((W^*)^c \cup W) = (X \cap\ (W^*)c) \cup (X \cap W) = W \cup (X - W^*)$. Therefore, $X - (W^* - W)$ is $g_*^* - \mathcal{J}$-closed. Hence, $W^* - W$ is $g_*^* - \mathcal{J} -$ open. The equivalence is clear.

Theorem 3.25. If $(X, \tau, \mathcal{J})$ is an ideal topological space. Then every subset of X is $g_*^* - \mathcal{J}$ closed if and only if every g^*-open set is $*$-closed.

Proof. Suppose every subset of X is $g_*^* - \mathcal{J}$-closed. If $U \subseteq X$ is g^*-open, then U is $g_*^* - \mathcal{J}$ closed and so $Cl^*(U) \subseteq U$, then $U^* \subseteq Cl^*(U) \subseteq U$. Hence U is $*$-closed.

Conversely, suppose that every g^*-open set is $*$-closed. If U is g^*-open set such that $\subseteq U \subseteq X$, then $W^* \cup W = Cl^*(W) \subseteq U^* \cup U = U$ and so W is $g_*^* - \mathcal{J}$-closed.

Corollary 3.26. Let $(X, \tau, \mathcal{J})$ be an ideal space. If W is a $g_*^* - \mathcal{J}$-closed subset of X, then W is $\mathcal{J}$-compact.

Proof. The proof follows from the fact that every $g_*^* - \mathcal{J}$-closed is $\mathcal{J}g$-closed.

Definition 3.27. Let N be a subset of $(X, \tau, \mathcal{J})$ and $x \in X$. The subset N of X is called a $g_*^* - \mathcal{J}$-open neighbourhood of x if there exists $g_*^* - \mathcal{J}$-open set U containing x such that $U \subset N$.

Theorem 3.28. For each $(X, \tau, \mathcal{J})$ either $\{x\}$ is g^*-closed or $\{x\}^c$ is $g_*^* - \mathcal{J}$-closed in X.
Proof. $\{x\}$ is not g^*-closed, then $\{x\}^c$ is not g^*-open.Therefore the only g^*-open set containing $\{x\}^c$ is X and $Cl^*(\{x\}^c) \subseteq X$ which proves that $\{x\}^c$ is $g_*^* - \mathcal{J}$-closed.

Theorem 3.29. If W and B are $g_*^* - \mathcal{J}$-closed sets in an ideal space $(X, \tau, \mathcal{J})$, then $W \cup B$ is also a $g_*^* - \mathcal{J}$-closed set.

Proof. Let U be a g^*-open subset of $(X, \tau, \mathcal{J})$ containing $W \cup B$. Then $W \subset U$ and $\subset U$. Since W and B are $g_*^* - \mathcal{J}$-closed, $Cl^*(W) \subset U$ and $Cl^*(B) \subset U$. By Lemma 2.12, $Cl^*(W \cup B) = Cl^*(W) \cup Cl^*(B) \subseteq U \cup U = U$. where $W \cup B \subset U$ and U is g^*-open which implies $W \cup B$ is $g_*^* - \mathcal{J}$-closed.

Theorem 3.30. Let $(X, \tau, \mathcal{J})$ be a g-multiplicative ideal space and let W be $g_*^* - \mathcal{J}$-closed. Then W is τ^*-closed $\Leftrightarrow W^* - W$ is closed.

Proof. Necessity: W is τ^*-closed $\Longrightarrow W^* \subset W \Longrightarrow W^* - W = \emptyset$ which is closed.
Sufficiency: Let $W^* - W$ be closed. Then it is g-closed By (4) of theorem 3.6, $W^* - W = \emptyset$ which implies $W^* \subset W$.

Theorem 3.31. Let $(X, \tau, \mathcal{J})$ be a g-multiplicative ideal space and $W \subset X$. If W is $g_*^* - \mathcal{J}$ closed then $W \cup (X - W^*)$ is $g_*^* - \mathcal{J}$-closed.

Proof. Let U be g^*-open and $W \cup (X - W^*) \subset U$
Then $X - U \subset X - [W \cup (X - W^*)] = W^* - W$. Since W is $g_*^* - \mathcal{J}$-closed, $W^* - W$ contains no non-empty g^*-closed set. Therefore $X - U = \emptyset$ which implies $X = U$. Thus X is the only g^*-open set containing $W \cup (X - W^*)$, then $Cl^*\big(W \cup (X - W^*)\big) \subseteq X$, which proves $W \cup (X - W^*)$ is $g_*^* - \mathcal{J}$-closed.

Theorem 3.32. Let W be a subset of a g-multiplicative ideal space $(X, \tau, \mathcal{J})$. If W is $g_*^* - \mathcal{J}$ closed then $W^* - W$ is $g_*^* - \mathcal{J}$-open

Proof. Since $X - (W^* - W) = W \cup (X - W^*)$, the proof follows from Theorem 3.30.

Theorem 3.33. Let $(X, \tau, \mathcal{J})$ be an ideal space and $W \subset Y \subset X$. If W is $g_*^* - \mathcal{J}$-closed in $(Y, \tau_Y, \mathcal{J}_Y), Y$ is α-open and τ^*-closed in X. Then W is $g_*^* - \mathcal{J}$-closed in X.

Proof. Let $W \subset U$ and U be g^*-open in X. Then $W^*(\mathcal{J}_Y, \tau/Y) = W^*(\mathcal{J}, \tau) \cap Y \subset U \cap Y$. Then $Y \subset U \cup (X - W^*(\mathcal{J}, \tau))$. Since Y is τ^*-closed, $Y^* \subset Y$. Therefore $W^* \subset Y^* \subset Y \subset U \cup (X - W^*(\tau, \mathcal{J}))$. This implies $W^* \subset U$ and hence $Cl^*(W) \subset U$. So W is $g_*^* - \mathcal{J}$-closed in X.

Theorem 3.34. Let $(X, \tau, \mathcal{J})$ be an ideal space. If every g^*-open set is τ^*-closed, then every subset of X is $g_*^* - \mathcal{J}$-closed.

Proof. Let $W \subset U$ and U be a g^*-open set in X. Then $Cl^*(W) \subset Cl^*(U) = U$ which proves W is $g_*^* - \mathcal{J}$-closed.

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$g_*^* - \mathcal{I}$ مجموعات مغلقة وخصائصها في الفضاء الطوبولوجي المثالي

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الخلاصة: في هذه الورقة ، ونحن نقدم ز \ _أست \ ^أست \ –ماثكال}أنا – {مجموعات مغلقة ، التوصيفات وخصائص ز \ _ أست \ ^أست \ –ماثكال}أنا – {مجموعات مغلقة ومكملة لها ومجموعات أخرى ذات الصلة .نثبت أن فئة ز \ _أست \ ^أست \ –ماثكال}أنا – _ {مجموعات مغلقة تقع بين فئة \ماثكال{أنا}ز–مجموعات مغلقة وفئة ز\أست–مجموعات مغلقة .أيضا ، نجد بعض العلاقات بين ز \ _أست \ ^أست \ – ماثكال}أنا – {مجموعات مغلقة ومجموعات مغلقة موجودة بالفعل .يتم تقديم الحي المفتوح ويتم التحقيق في ممتلكاتهم.

الكلمات المفتاحية: الفضاء الطوبولوجي المثالي ، ز \ ^هو–مجموعة مغلقة ، ز \ ^أستي–مجموعة مغلقة ، ز \ _أست \ ^أست–أنا–مجموعة مفتوحة