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# Nonparametric Estimation Method for the Distribution Function Using Various Types of Ranked Set Sampling

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**Abstract**

The purpose of this research is to estimate the cumulative distribution function $CDF$ using the local polynomial regression $LPR$ and compare it to parameter estimation using the method of moments and the maximum likelihood method to calculate both the mean square error and the bias using the ranked sets sample $RSS$ and the median ranked sets sample $MRSS$. As well as $RSS$ frequently produces more exact estimates than simple random sampling $SRS$ for the same sample size. By ranking samples based on some easily measurable characteristic, the variability within each set is decreased, resulting in more accurate estimations. We investigated three different degrees of local polynomial regression: the first, second, and third. The simulation analysis demonstrated that the second degree outperforms the other degrees. Also, when $LPR$ is used to analyze $RSS$ data, it takes advantage of the reduced variability within each ranked set, resulting in more precise and reliable regression function estimates. Following that, we investigated several degrees of bandwidth (0.1, 0.2, … and 0.9) and discovered that the bandwidth of degree 0.8 is superior to the other degrees based on a simulation study. Finally, we analyzed the relative efficiency of each of the three approaches:$LPR$, $MOM$, and $MLE$, and we discovered that $LPR$ is more efficient than the other methods for estimating the $CDF$ in different kernels (normal (gaussian), epanechinkov). The numerical results provide that the suggested estimator $CDF$ based on $LPR$ is more efficient than other methods, as predicted by the simulation analysis.

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## 1. Introduction

Cumulative Distribution Function CDF is a strong tool for understanding and evaluating random variables, as well as forecasting future occurrences; it is a foundational idea in probability theory and statistics. Also, establish that the occurrence is likely to take place until a specific point. When we utilize it, we encounter several obstacles. One of the challenges is studying nonparametric analysis in-depth and identifying several population features, such as odds, survival analysis, hazards, etc. Estimating the CDF using ranked sets sample RSS is more efficient than basic random sample SRS because RSS frequently yields more precise estimates with the same sample size, as well studying variables and measuring them is not easy; sometimes it is too expensive or time-consuming, but ranking the variables is easy or has a negligible cost. The first to introduce the rank-set sample was McIntyre (1952) for estimating the paster of yields in Australia and expressed expectations about how to develop the estimator that would be more effective for the paster of yields. Halls and Dell (1966) conducted a field trial evaluating its applicability to the estimation of forage yields in a pine-hardwood forest, the terminology ranked set sampling was, coined by Halls and Dell. Takahasi and Wakimoto (1968) proved the first theoretical result is that, when ranking is perfect, the ranked set

sample mean is an unbiased estimator of the population mean, and the variance of the ranked set sample mean is always smaller than the variance of the mean of a simple random sample of the same size. Research has continued in ranked sets since (1997). Muttlak (1997) suggested studying median ranked sets sampling to estimate the population mean instead of ranked sets sampling, and it is a strategy to minimize the error in ranking. Gulati (2004) studied the empirical distribution function of Stokes and Sagar with smooth estimators and properties using simulation to compare the smooth and empirical estimators. Frey (2012) derived the constraint to estimate the cumulative distribution function with the mean of the population to create a Woodruff-type confidence interval for the population quantile. Al-Saleh and Ahmad (2019) suggested a new technique of ranked sets sampling, which was called Moving extreme ranked sets sampling, to estimate the cumulative distribution function and then compared the proposed estimator with the corresponding estimator based on both. Zamanzade (2020) established two estimators in moving extreme ranked sets sampling with simulation and also showed that the proposed estimators provide a substantial improvement over their competitors and prove that the estimators are utilized to estimate the stress-strength probability. Abdallah and Al-Omari (2022) considered the problem of estimating the cumulative distribution function and the odds measure under moving extreme ranked set sampling.

The paper is structured as follows: Section 2 describes ranked sets sampling and median ranked sets sampling. Section 3 describes local polynomial regression. Section 4 Estimation of cumulative distribution function using the Maximum Likelihood Method, Method of Moments, and local polynomial regression. Section 5 Simulation study and conclusions.

## 2. Description RSS & MRSS

### 2-1. Methodology of ranked sets sample (RSS)

McIntyre (1952) was the first one who suggested the ranked sets sampling as a strategy to estimate the paster of yields. In the $RSS$ technique, taking samples is much cheaper than measurement of the variable. We will describe how to select ranked sets sampling in the following steps:

1. Draw randomly $m^2$ sample units from the population of interest. Divided the $m^2$ units into $m$ groups each one of the groups has a size of $m$.
2. Based on the judgment rank the unit sets without actual measurement by eyes or by a bit price method for the variable of interest.
3. From the first set, select the smallest order observation, discard the other units, and then from the second set, select the second smallest order observation, and discard the other units. The process continues until get the $m^{th}$ maximum order observation.
4. From steps 1-3 we can get the ranked sets sample RSS of one group
5. To obtain $RSS$ with size $k$, we can repeat steps (1-3) $r$ times, where $k = mr$.

Let $\{Y_{(i;i)j}, i = 1,2, \dots, m \; and \; j = 1,2, .., r\}$ be the $RSS$ element set, $Y_{(i;i)j}$ be the judgment order statistics of the $i^{th}$ sample of size $m$, and the $j^{th}$ cycle of the $r$ repeated. You should notice that we utilize square brackets [.], when the ranked sets sample is imperfect ranking it means there is an error in ranking, if there is no error in ranking it means that the ranked sets sample is perfect ranking we use the round brackets (.), it's very important to note that for each $i$ $\{Y_{(i;i)1}, Y_{(i;i)2}, \dots, Y_{(i;i)r}\}$ are independent and identically distributed $(iid)$. And for each $j$ $\{Y_{(1;1)j}, Y_{(2;2)j}, \dots, Y_{(m;m)j}\}$ are just independent.

### 2-2. Methodology of median ranked sets sample ($MRSS$)

Muttlak (1997) suggested a new strategy of ranked sets sampling, which is called median ranked sets sampling $MRSS$, to minimize errors in the process of ranking units within groups and to increase the efficiency of the estimator in the presence of errors in ranking, also to increase the efficiency over $RSS$ with perfect ranking. The following summarizes the $MRSS$ procedure for drawing a sample of size $k$. We will describe how to select median ranked sets sampling $MRSS$ in the following steps:

1) Drow randomly $m^2$ sample units from the population of interest.
2) Divided the $m^2$ units into $m$ groups each one of the groups has a size of $m$.
3) If the sample size of $m$ group is odd, the odd will be measured by rule $\frac{m+1}{2}$ it is equal to the units in the medial of the groups, if the sample size of the $m$ group is even, the even will be measured the first half group units with rule $\frac{m}{2}$ and the second half with rule $\frac{m+2}{2}$.
4) Steps 1-3 can be replayed $r$ times, if necessary to get $MRSS$ of size $k = mr$.

If the $m$ groups are odd, the median ranked sets sample odd, symbolized $MRSS_o$, where the units of the $MRSS_o$ for variable $Y$, is described as follows:

$\left\{Y_{i\left(\frac{m+1}{2}\right)j}, i = 1,2,..,m, j = 1,2,..,r\right\}$ Be the judgment order statistics of the $i^{th}$ sample of size $m$ and the $j^{th}$cycle of the $r$ repeated.

If the $m$ groups are even, the median ranked sets sample even, symbolized $MRSS_e$, Also $\left\{\frac{m}{2}\right\}$ is the units of the $MRSS_e$ for variable $Y$, is described as follows:

Where $\left\{Y_{i\left(\frac{m}{2}\right)j}; Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, i = 1,2,..,\frac{m}{2}, j = 1,2,..,r\right\}$ be the judgment order statistics of the $i^{th}$ sample of size $\left\{\frac{m}{2}\right\}$ and the $j^{th}$cycle of the $r$ repeated.

## 3. Local polynomial regression (LPR)

Local polynomial regression is a nonparametric technique used to generalize kernel regression, also used to model functions and smoothing one of the statistics plots, which is called the scatter plot. One of the most important uses is to find the relationship between the dependent variable and the independent variable, $LPR$ is better than other types of regression for having a good performance near the boundary. For each point of $z_o$ a $WLS$ which is low order weighted least square regression is fit at each point of $z$. By the Fan and Gijbels (1996), $(Z_i, Y_i)$ are defined according to the fixed model in equation (1).

$$Y_i = m(Z_i) + \sigma(Z_i)\varepsilon_i \qquad i = 1, \dots, n \tag{1}$$

Where $Z_i = \frac{i}{n}$, $\sigma(Z_i)$ is the variance of $Y_i$ at point $Z_i$, $\varepsilon_i$ is a residual error with normal distribution with mean zero and variance $\sigma^2$. For estimating $m(Z_i)$ we use a Tylor series.

$$m(z_i) \approx m(z) + m^{(1)}(z_i)(z_i - z) + \cdots + \frac{m^{(P)}(z_i)(z_i - z)^{(P)}}{P!} \tag{2}$$

We need the point in the area of z because it gives us a higher weight than the other point remaining, we can estimate the unknown parameters in equation (2) by using $WLS$ weighted least square, depending on the following formula:

$$z = \begin{bmatrix} 1 & (z_1 - z) & \cdots & (z_1 - z)^P \\ 1 & (z_2 - z) & \cdots & (z_2 - z)^P \\ \vdots & \vdots & & \\ 1 & (z_n - z) & \cdots & (z_n - z)^P \end{bmatrix}_{n*m} ; \; Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n*1} \quad \hat{\beta} = (Z^T W Z)^{-1} Z^T W Y$$

$$W = \begin{bmatrix} \frac{1}{h}k\left(\frac{(Z_1 - Z)}{h}\right) & 0 & \dots & 0 \\ 0 & \frac{1}{h}k\left(\frac{(Z_2 - Z)}{h}\right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{h}k\left(\frac{(Z_n - Z)}{h}\right) \end{bmatrix}_{n*m}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_o \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_n \end{bmatrix}_{n*1}, e_1 = \{1\ 0\ 0\ \cdots\cdots 0\} \text{ and } w = diag\left\{\frac{1}{h}k\left(\frac{(z_i - z)}{h}\right)\right\}\ i = 1,2,\dots,m\ ; j = 1,2,\dots,r$$

$k(.)$ represents kernel function, $h$ represents bandwidth. $W$ represents the diagonal elements matrix of weight, where $e_1 = \{1\ 0\ 0\ \cdots\cdots 0\}$ is the $(P+1)\ times\ 1$ with 1 in the first entry and 0 elsewhere.

## 4. Estimation of cumulative distribution function (CDF)

### 4.1. Estimation of (CDF) using The Maximum likelihood Method (MLE):

### 4.1.1. Based on ranked sets sampling (RSS).

Al-Saleh and Ahmad (2019) suggested and proved $CDF$ using $MLE$ based on $\left\{Y_{(i;i)j}, i = 1,2,\dots,m\ and\ j = 1,2,..,r\right\}$ the represent RSS that we selected from the population with $Pdf\ f(.)$ and $CDF\ F(.)$, and then we will use the maximum likelihood estimation $MLE$ for estimating $CDF\ F(Y_{RSS})$ depending on the $RSS$. They assumed that the variable $Y_i = \sum_{j=1}^{r} Y_{(i;i)j}\ ; i = 1,2,\dots,m$ is distributed according to a binomial distribution with mass parameter $m$ and success probability $F(Y_{RSS})$. Therefore, the estimator of the probability distribution function is defined according to the following relationship:

$$\hat{F}_{MLE}(Y_{RSS}) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i,i)j}}{mr} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i,i)j}}{k} = \bar{Y}_{RSS} \tag{3}$$

$\bar{Y}_{RSS}$ is the mean obtained by estimating the $CDF$ based on $MLE$ by using $RSS$.

### 4.1.2. Based on odd median ranked sets sampling ($MRSS_o$).

Let $\left\{Y_{i\left(\frac{m+1}{2}\right)j}, i = 1, \dots, m,\ j = 1, \dots, r\right\}$, be median ranked sets sample odd $MRSS_o$ of size $(k = mr)$, that we selected from the population with $Pdf\ f(.)$ and $CDF\ F(.)$, and then we will use the maximum likelihood estimation $MLE$ for estimating $CDF$ $F(Y_{MRSS_o})$ depending on the $MRSS_o$, note that for each $i$, $\left\{\mathrm{I}\left(Y_{i\left(\frac{m+1}{2}\right)1} \le Y_{MRSS_o}\right), \mathrm{I}\left(Y_{i\left(\frac{m+1}{2}\right)2} \le Y_{MRSS_o}\right), \dots, \mathrm{I}\left(Y_{i\left(\frac{m+1}{2}\right)r} \le Y_{MRSS_o}\right)\right\}$ are independent and identically distributed $(iid)$ each unit distributed Bernoulli distribution with probability of success $F\left(Y_{MRSS_o}\right)$, and $I(.)$ represent indicator.

Let $\left\{Y_{i_{MRSS_o}} = \sum_{j=1}^{r} I\left(Y_{i\left(\frac{m+1}{2}\right)j} \le Y_{MRSS_o}\right), i = 1, \dots, m\right\}$, then variable $Y_{i_{MRSS_o}}$ distributed binomial with mass parameter $m$ and success probability $F\left(Y_{MRSS_o}\right)$.

The likelihood function is determined according to equation (4):

$$g\left(Y_{i_{MRSS_o}} \middle| m, F\left(Y_{MRSS_o}\right)\right) = \prod_{j=1}^{r}\binom{m}{Y_{i_{MRSS_o}}}\left(F\left(Y_{MRSS_o}\right)\right)^{Y_{i_{MRSS_o}}}\left(1 - F\left(Y_{MRSS_o}\right)\right)^{m - Y_{i_{MRSS_o}}} \tag{4}$$

Therefore, the estimator of the probability distribution function is defined according to the following relationship:

$$\hat{F}_{MLE}\left(Y_{MRSS_o}\right) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i\left(\frac{m+1}{2}\right)j}}{mr} \Rightarrow \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i\left(\frac{m+1}{2}\right)j}}{k} = \bar{Y}_{MRSS_o} \tag{5}$$

$\bar{Y}_{MRSS_o}$ is the mean obtained by estimating the $CDF$ based on $MLE$ by using $MRSS_o$.

### 4.1.3. Based on even median ranked sets sampling ($MRSS_e$).

The $MRSS_e$ element set is $\left\{Y_{i\left(\frac{m}{2}\right)j}, Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, i = 1,2, \dots, \frac{m}{2}\ and\ j = 1,2,..,r\right\}$ of size $(k = mr)$, that we selected from the population with $Pdf\ f(.)$ and $CDF\ F(.)$, and then we will use the maximum likelihood estimation $MLE$ for estimating $CDF$ $F(Y_{MRSS_e})$ depending on the $MRSS_e$, depending on the withdrawal method of median ranked sets, sample even note that for each $i$

$\left\{I\left(Y_{i\left(\frac{m}{2}\right)1}, Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)1} \le Y_{MRSS_e}\right),\ \dots\ , I\left(Y_{i\left(\frac{m}{2}\right)r}, Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)r} \le Y_{MRSS_e}\right)\right\}$ are independent and identically distributed $(iid)$ each unit distributed Bernoulli distribution with a probability of success $F\left(Y_{MRSS_e}\right)$, and $I(.)$ represent indicator.

Let $\left\{Y_{i_{MRSS_e}} = \sum_{j=1}^{r} I\left(Y_{i\left(\frac{m}{2}\right)1}, Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)1} \le Y_{MRSS_e}\right), i = 1, \dots, \frac{m}{2}\right\}$, then variable $Y_{i_{MRSS_e}}$ distributed binomial with mass parameter $m$ and success probability $F\left(Y_{MRSS_e}\right)$.

The likelihood function is determined according to equation (6):

$$w\left(Y_{i_{MRSS_e}} \middle| m, F\left(Y_{MRSS_e}\right)\right) = \prod_{j=1}^{r}\binom{\frac{m}{2}}{Y_{i_{MRSS_e}}}\left(F\left(Y_{MRSS_e}\right)\right)^{Y_{i_{MRSS_e}}}\left(1 - F\left(Y_{MRSS_e}\right)\right)^{\frac{m}{2} - Y_{i_{MRSS_e}}} \times \binom{m}{Y_{\frac{m}{2}+i_{MRSS_e}}}\left(F\left(Y_{MRSS_e}\right)\right)^{Y_{\frac{m}{2}+i_{MRSS_e}}}\left(1 - F\left(Y_{MRSS_e}\right)\right)^{m - Y_{\frac{m}{2}+i_{MRSS_e}}} \tag{6}$$

Therefore, the estimator of the probability distribution function is defined according to the following relationship:

$$\hat{F}_{MLE}\left(Y_{MRSS_e}\right) = \frac{\sum_{j=1}^{r}\left\{\sum_{i=1}^{\frac{m}{2}} Y_{i\left(\frac{m}{2}\right)j} + \sum_{i=\frac{m}{2}+1}^{m} Y_{i\left(\frac{m+2}{2}\right)j}\right\}}{k} = \bar{Y}_{MRSS_e} \tag{7}$$

$\bar{Y}_{MRSS_e}$ is the mean obtained by estimating the $CDF$ based on $MLE$ by using $MRSS_e$.

## 4.2. Estimation of (CDF) using the Method of Moments

### 4.2.1 Based on RSS:

Stokes and Sager (1988) suggested a new estimator, which is called the method of moments, for estimating the distribution function $F\ F(Y_{RSS})$, by using the ranked sets sample, the equation (8) shows us the suggested estimator.

$$\hat{F}_{MOM}(Y_{RSS}) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} I\left(Y_{(i,i)j} \le Y_{RSS}\right)}{k} \tag{8}$$

They proved that $CDF$ ranked set sample $\hat{F}_{RSS}(Y_{RSS})$ is an unbiased estimator for $F(Y_{RSS})$ and also has better efficiency than $CDF$ of simple random sample $\hat{F}_{SRS}(Y_{SRS})$.

### 4.2.2. Based on $\mathbf{MRSS_o}$:

Depending on the description of $MRSS_o$ in the subsection 4.1.2, the distribution $\left\{Y_{i\left(\frac{m+1}{2}\right)j}, i = 1,2,\dots,m\ and\ j = 1,2,..,r\right\}$, for $r$ independent binomial distribution with each mass parameter $m$ and success probability $P_{MRSS_o}$, can be obtained in equation (9)

$$g\left(Y_{i_{MRSS_o}}\middle|m, P_{MRSS_o}\right) = r\ \binom{m}{Y_{i_{MRSS_o}}}\left(P_{MRSS_o}\right)^{Y_{i_{MRSS_o}}}\left(1-P_{MRSS_o}\right)^{m-Y_{i_{MRSS_o}}} \tag{9}$$

We take the expectation for equation (9) for the purpose of calculating the distribution parameter estimate $P_{MRSS_o}$, which represents the $CDF\ F\left(Y_{MRSS_o}\right)$ and probability of success, based on the method of moments, through equation (10) we will get the mean:

$$E\left(Y_{i_{MRSS_o}}\right) = kP_{MRSS_o} = \sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i\left(\frac{m+1}{2}\right)j} \tag{10}$$

$$\hat{F}_{MOM}\ (Y_{MRSS_o}) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i\left(\frac{m+1}{2}\right)j}}{k} = \bar{Y}_{MRSS_o} \tag{11}$$

$\bar{Y}_{MRSS_o}$ is the mean obtained by estimating the $CDF$ based on $MOM$ by using $MRSS_o$

### 4.2.3. Based on $\mathbf{MRSS_e}$:

Depending on the description of $MRSS_e$ in the subsection 4.1.3, the distribution $\left\{Y_{i\left(\frac{m}{2}\right)j}, Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, i = 1,2,\dots,\frac{m}{2}; and\ j = 1,2,..,r\right\}$ for $r$ independent binomial distribution with mass parameter $m$ and success probability $P_{MRSS_e}$, which can be obtained in equation (12)

$$w\left(Y_{i_{MRSS_e}}\middle|k, P_{MRSS_e}\right) = r\left\{\binom{\frac{m}{2}}{Y_{i_{MRSS_e}}}\left(P_{MRSS_e}\right)^{Y_{i_{MRSS_e}}}\left(1-P_{MRSS_e}\right)^{\frac{m}{2}-Y_{i_{MRSS_e}}} + \binom{m}{Y_{\frac{m}{2}+i_{MRSS_e}}}\left(P_{MRSS_e}\right)^{Y_{\frac{m}{2}+i_{MRSS_e}}}\left(1-P_{MRSS_e}\right)^{m-Y_{\frac{m}{2}+i_{MRSS_e}}}\right\} \tag{12}$$

To calculate the distribution parameter estimate $P_{MRSS_e}$, Where $P_{MRSS_e} = F\left(Y_{MRSS_e}\right)$ which is the probability of success and represents the $CDF$ of the function, based on the method of moments, the expectation of the $P_{MRSS_e} = F\left(Y_{MRSS_e}\right)$ is defined according to the following:

$$E\left(Y_{i_{MRSS_e}}\right) = kP_{MRSS_e} = \sum_{j=1}^{r}\left\{\sum_{i=1}^{\frac{m}{2}} Y_{i\left(\frac{m}{2}\right)j} + \sum_{i=\frac{m}{2}+1}^{m} Y_{i\left(\frac{m+2}{2}\right)j}\right\} \tag{13}$$

$$\hat{F}_{MOM}(Y_{MRSS_e}) = \frac{\sum_{j=1}^{r}\left\{\sum_{i=1}^{\frac{m}{2}} Y_{i\left(\frac{m}{2}\right)j} + \sum_{i=\frac{m}{2}+1}^{m} Y_{i\left(\frac{m+2}{2}\right)j}\right\}}{k} = \bar{Y}_{MRSS_e} \tag{14}$$

$\bar{Y}_{MRSS_e}$ is the mean obtained by estimating the $CDF$ based on $MOM$ by using $MRSS_e$

## 4.3. Estimation of (CDF) using local polynomial regression (LPR)

### 4.3.1. LPR based on RSS:

Estimation of cumulative distribution function $CDF$ using local polynomial regression $LPR$ based on ranked sets sampling $RSS$, $F_{LPR}(Y_{RSS})$ can be linearized by using a Tylor series, depending on section (3) equation (2) that we defined $LPR$ with $P$ degree. Consider a fixed regression model:

$$F(Y_{RSS}) = F\left(Y_{(i,i)j}\right) + \sigma\left(Y_{(i,i)j}\right)\varepsilon_{ij}\,; i = 1,2,\dots m\,;\, j = 1,2,\dots r \tag{15}$$

$\sigma(Y_{(i,i)j})$ is the variance of $Y_{RSS}$ at point $Y_{(i,i)j}$, $\varepsilon_{ij}$ is a residual error with normal distribution with mean zero and variance $\sigma^2$.

$$F_{LPR}(Y_{RSS}) \approx F(Y) + F^{(1)}(Y)(Y_{RSS} - Y) + \frac{F^{(2)}(Y)}{2!}(Y_{RSS} - Y)^2 + \cdots + \frac{F^{(p)}(Y)}{P!}(Y_{RSS} - Y)^P \tag{16}$$

Where $\approx$ represent the approximate equality, $F^{(i)}(Y) = \left.\frac{\partial^{(i)}F_{LPR}(Y_{RSS})}{\partial {Y_{RSS}}^{(i)}}\right|_{Y_{RSS}=Y}$ $i = 1,2,\dots,P$

Where $P$ is the degree of the local polynomial range and $Y$ is an observation from a data neighborhood around $Y_{RSS}$. If the $CDF$ of the ranked sets sample $F_{LPR}(Y_{RSS})$ is unknown, the equation (16) is as follows:

$$F_{LPR}\left(Y_{(i,i)j}\right) \approx F(Y_{RSS}) + F^{(1)}\left(Y_{(i,i)j}\right)\left(Y_{(i,i)j} - Y_{RSS}\right) + \frac{F^{(2)}\left(Y_{(i,i)j}\right)}{2!}\left(Y_{(i,i)j} - Y_{RSS}\right)^2 + \cdots + \frac{F^{(P)}\left(Y_{(i,i)j}\right)}{P!}\left(Y_{(i,i)j} - Y_{RSS}\right)^P \tag{17}$$

$i = 1,2,\dots m\,;\, j = 1,2,\dots r$

By estimating the ranked sets sample units $F_{LPR}\left(Y_{(i,i)j}\right)$, the equation (17) will be as follows:

$$\hat{F}_{LPR}\left(Y_{(i,i)j}\right) \approx \beta_{o_{RSS}} + \beta_{1_{RSS}}\left(Y_{(i,i)j} - Y_{RSS}\right) + \beta_{2_{RSS}}\left(Y_{(i,i)j} - Y_{RSS}\right)^2 + \cdots + \beta_{P_{RSS}}\left(Y_{(i,i)j} - Y_{RSS}\right)^P \tag{18}$$

Where $\hat{\beta}_{i_{RSS}} = \frac{F^{(i)}\left(Y_{(i,i)j}\right)}{i\,!}$ $i = 0,1,2,\dots,P$

Using ordinary least square to estimate the unknown parameter $\beta_{RSS}$. Fan and Gijbles (1996) mentioned that the point in the area of $Y_{RSS}$ is giving us a higher weight than the other point. Rather, we can estimate the unknown parameters in equation (18) by using weighted least square depending on the following formula:

$$\hat{\beta}_{RSS} = \begin{bmatrix} \hat{\beta}_{o_{RSS}} \\ \hat{\beta}_{1_{RSS}} \\ \vdots \\ \hat{\beta}_{P_{RSS}} \end{bmatrix}_{P*1} \;;\; \hat{\beta}_{RSS} = \left({Y^*_{RSS}}^T W_{RSS} Y^*_{RSS}\right)^{-1} {Y^*_{RSS}}^T W_{RSS} \hat{F}_{RSS} \tag{19}$$

Where

$$Y^*_{RSS} = \begin{bmatrix} 1 & \left(Y_{(1,1)1} - Y_{RSS}\right) & \dots & \left(Y_{(1,1)1} - Y_{RSS}\right)^P \\ \vdots & \vdots & & \vdots \\ 1 & \left(Y_{(m,m)1} - Y_{RSS}\right) & \dots & \left(Y_{(m,m)1} - Y_{RSS}\right)^P \\ \vdots & \vdots & & \vdots \\ 1 & \left(Y_{(m,m)r} - Y_{RSS}\right) & \dots & \left(Y_{(m,m)r} - Y_{RSS}\right)^P \end{bmatrix} ;\; \hat{F}_{RSS} = \begin{bmatrix} \hat{F}\left(Y_{(1,1)1}\right) \\ \vdots \\ \hat{F}\left(Y_{(m,m)1}\right) \\ \vdots \\ \hat{F}\left(Y_{(m,m)r}\right) \end{bmatrix}_{k*1}$$

$$\&\; W_{RSS} = diag\left\{\frac{1}{h_{RSS}} k_{RSS}\left(\frac{\left(Y_{(i,i)j} - Y_{RSS}\right)}{h_{RSS}}\right)\right\}, i = 1,\dots,m, j = 1,\dots,r \tag{20}$$

Since $k_{RSS}(.)$ represent kernel function, $h_{RSS}$ represent bandwidth that manages the size of the point in the zone of $Y_{RSS}$. $W_{RSS}$ represents diagonal elements matrix of weight, since the estimator of $\hat{\beta}_{o_{RSS}}$ is the $CDF$ of local polynomial regression in ranked sets sample, where $e_1$ is the vector has 1 in the first entry and somewhere else is zero.

$$\hat{F}_{LPR}(Y_{RSS}) = e_1 \times \hat{\beta}_{RSS} = e_1 \times \left({Y^*_{RSS}}^T W_{RSS} Y^*_{RSS}\right)^{-1} {Y^*_{RSS}}^T W_{RSS} \hat{F}_{RSS} = \hat{\beta}_{o_{RSS}}\,;\; e_1 = \{1\,0\,0\,\cdots\cdots 0\} \tag{21}$$

We have concluded in equation (21) that the estimator $\hat{\beta}_{o_{RSS}}$ is equal likely to the $\hat{F}_{LPR}(Y_{RSS})$ which is the estimator of the $CDF$ based on $LPR$ with degree $P$, this indicates that the estimator $\hat{\beta}_{o_{RSS}}$ is equal to the estimator $\hat{F}_{LPR_{(0)}}(Y_{RSS})$ which is the estimator of the $CDF$ based on $LPR$ with degree $P = 0$ in $RSS$, with the value of the vector $e_1^T = \{1\}$ in equation (22).

$$\hat{F}_{LPR_{(0)}}(Y_{RSS}) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} k_{RSS}\left(Y_{(i,i)j} - Y_{RSS}\right)\hat{F}_{RSS}}{\sum_{j=1}^{r}\sum_{i=1}^{m} k_{RSS}\left(Y_{(i,i)j} - Y_{RSS}\right)} \quad (22)$$

We have proved in equation (21) that the estimator $\hat{\beta}_{o_{RSS}}$ is equal likely to the $\hat{F}_{LPR}(Y_{RSS})$, and also the estimator $\hat{\beta}_{o_{RSS}}$ is equal to the $\hat{F}_{LPR_{(1)}}(Y_{RSS})$ which is the estimator of the $CDF$ based on $LPR$ with degree $P = 1$ in $RSS$, with the value of vector $e_1^T = \{1,0\}$, in equation (23).

$$\hat{F}_{LPR_{(1)}}(Y_{RSS}) = k_{RSS}^{-1}\frac{S_0\left(Y_{(i,i)j}, h_{RSS}\right) - S_1\left(Y_{(i,i)j}, h_{RSS}\right)\left(Y_{(i,i)j} - Y_{RSS}\right)}{S_0\left(Y_{(i,i)j}, h_{RSS}\right)S_2\left(Y_{(i,i)j}, h_{RSS}\right) - S_1\left(Y_{(i,i)j}, h_{RSS}\right)} \quad (23)$$

Where $S_L\left(Y_{(i,i)j}, h_{RSS}\right) = k_{RSS}^{-1}\sum_{j=1}^{r}\sum_{i=1}^{m}\left(Y_{(i,i)j}, Y_{RSS}\right)^L k_{RSS_h}\left(Y_{(i,i)j}, Y_{RSS}\right)$

The properties of $CDF$ will be studied based on local polynomial using ranked sets sampling depending on these conditions:

1- The function $F^{(2)}(Y)$ and $\sigma$ each continuous on $[0,1]$.

2- The kernel $k_{RSS}$ is symmetric about zero and is reinforced on $[-1,1]$.

3- The bandwidth $h_{RSS} = h_{RSS_k}$ it is a satisfactory sequence, $h_{RSS} \to 0 \; and \; kh_{RSS} \to \infty \; as \; k \to \infty$.

4- The point $Y_{(i,i)j}$ at which the estimate is made satisfactory $h_{RSS} < Y_{(i,i)j} < 1 - h_{RSS}$ for all $k \geq k_0$, where $k_0$ is fixed.

In this subsection, we will make a derivative for the bias and $MSE$ of the estimate $CDF$ based on local polynomial.

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{RSS})\right) = E\left\{e_1^T\left(Y_{RSS}^{*\,T}W_{RSS}Y_{RSS}^{*}\right)^{-1}Y_{RSS}^{*\,T}W_{RSS}\hat{F}_{RSS}\right\}$$

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{RSS})\right) = F_{RSS} + \frac{1}{2}h_{RSS}^{(2)}F_{RSS}^{(2)}\mu_2(k_{RSS}) + O\left(h_{RSS}^{(2)}k_{RSS}^{-1}\right) \quad (24)$$

$$Bias\left(\hat{F}_{LPR_{(1)}}(Y_{RSS})\right) = \frac{1}{2}h_{RSS}^{(2)}F_{RSS}^{(2)}\mu_2(k_{RSS}) + O\left(h_{RSS}^{(2)}k_{RSS}^{-1}\right) \quad (25)$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{RSS})\right) = e_1^T\left(Y_{RSS}^{*\,T}W_{RSS}Y_{RSS}^{*}\right)^{-1}Y_{RSS}^{*\,T}W_{RSS}V\left(F(Y_{RSS})\right)W_{RSS}Y_{RSS}^{*}\left(Y_{RSS}^{*\,T}W_{RSS}Y_{RSS}^{*}\right)^{-1}e_1 \quad (26)$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{RSS})\right) = (kh_{RSS})^{-1}R(k_{RSS})V\left(F(Y_{RSS})\right) + O\{(kh_{RSS})^{-1}\} \quad (27)$$

In the applied aspect, in Table 1.1, we use the R-Program to obtain values of the empirical $MSE$ and $Bias$ of $CDF$ based on $LPR$ by using $RSS$, with four level sets size ($m$ = 2,3,4,5) and number of cycles ($r$ = 3,5,7). We take degree $P = 2$ of $LPR$ with bandwidth 0.8 because they give a good result compared to other bandwidths and degrees and use kernel (normal, epanechnikov) with three values of probability ($P_{RSS}$ = 0.25,0.50,0.75).

**Table 1.1: The empirical mean square error and bias of CDF based on LPR by Using RSS**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| | **r = 3** | **m = 2, k = 6** | **m = 3, k = 9** | **m = 4, k = 12** | **m = 5, k = 15** |
| | $P_{RSS}$ **= 0.25** | Mse = 0.000107486<br>Bias = 0.0004636507 | Mse = 0.0001035949<br>Bias = 0.0004551812 | Mse = 0.0001014252<br>Bias = 0.0004503893 | Mse = 7.986923e-05<br>Bias = 0.0003996729 |
| | $P_{RSS}$ **= 0.50** | .946986e-05 Bias = 0.0003 | Mse = 5.383707e-05<br>Bias = 0.0003281374 | 4.66936e-05 Bias = 0.000309 | 4.70322e-05 Bias = 0.00030 |
| | $P_{RSS}$ **= 0.75** | Mse = 2.168546e-05<br>Bias = 0.0002082569 | Mse = 2.295053e-05<br>Bias = 0.0002142453 | Mse = 4.179461e-05<br>Bias = 0.000289118 | Mse = 2.598391e-05<br>Bias = 0.0002279645 |
| | **r = 5** | **m = 2, k = 10** | **m = 3, k = 15** | **m = 4, k = 20** | **m = 5, k = 25** |
| | $P_{RSS}$ **= 0.25** | Mse = 3.894282e-05<br>Bias = 0.00027908 | Mse = 2.321515e-05<br>Bias = 0.0002154769 | Mse = 3.138624e-05<br>Bias = 0.0002505444 | Mse = 2.607705e-05<br>Bias = 0.0002283727 |
| | $P_{RSS}$ **= 0.50** | .158395e-05 Bias = 0.0002 | Mse = 2.049009e-05<br>Bias = 0.0002024356 | 1.98626e-05 Bias = 0.000191 | .083776e-05 Bias = 0.00020 |
| | $P_{RSS}$ **= 0.75** | Mse = 1.459885e-05<br>Bias = 0.0001708733 | Mse = 1.734523e-05<br>Bias = 0.0001862537 | Mse = 1.692903e-05<br>Bias = 0.0001840056 | Mse = 1.845636e-05<br>Bias = 0.0001921268 |
| | **r = 7** | **m = 2, k = 14** | **m = 3, k = 21** | **m = 4, k = 28** | **m = 5, k = 35** |
| | $P_{RSS}$ **= 0.25** | Mse = 1.264575e-05<br>Bias = 0.000159033 | Mse = 1.352716e-05<br>Bias = 0.000164482 | Mse = 1.15039e-05<br>Bias = 0.0001516832 | Mse = 1.099064e-05<br>Bias = 0.0001482609 |
| | $P_{RSS}$ **= 0.50** | .211694e-05 Bias = 0.0001 | Mse = 1.238188e-05<br>Bias = 0.0001573651 | .398311e-05 Bias = 0.00016 | Mse = 9.6613e-06<br>bias = 0.0001390058 |
| | $P_{RSS}$ **= 0.75** | Mse = 7.667563e-06<br>Bias = 0.0001238351 | Mse = 9.508394e-06<br>Bias = 0.0001379014 | Mse = 1.167593e-05<br>Bias = 0.0001528131 | Mse = 6.628009e-06<br>Bias = 0.0001151348 |
| **Kernel / Epanechnikov** | | | | | |
| | **r = 3** | **m = 2, k = 6** | **m = 3, k = 9** | **m = 4, k = 12** | **m = 5, k = 15** |
| | $P_{RSS}$ **= 0.25** | Mse = 0.0001206227 | Mse = 8.56113e-05 | Mse = 6.935339e-05 | Mse = 8.86133e-05 |

| | | | | | |
|---|---|---|---|---|---|
| | | Bias = 0.0004911674 | Bias = 0.0004137905 | Bias = 0.0003724336 | Bias = 0.0004209829 |
| | $P_{RSS}$ = 0.50 | .661436e-05 Bias = 0.0003 | Mse = 5.640625e-05 Bias = 0.0003358757 | .636432e-05 Bias = 0.00033 | .704077e-05 Bias = 0.00033 |
| | $P_{RSS}$ = 0.75 | Mse = 1.99026e-05 Bias = 0.0001995124 | Mse = 2.287863e-05 Bias = 0.0002139095 | Mse = 3.196004e-05 Bias = 0.0002528242 | Mse = 4.15294e-05 Bias = 0.0002881992 |
| | **r = 5** | **m = 2, k = 10** | **m = 3, k = 15** | **m = 4, k = 20** | **m = 5, k = 25** |
| | $P_{RSS}$ = 0.25 | Mse = 3.369704e-05 Bias = 0.0002596037 | Mse = 2.682286e-05 Bias = 0.0002316154 | Mse = 1.952314e-05 Bias = 0.0001976013 | Mse = 2.263933e-05 Bias = 0.0002127878 |
| | $P_{RSS}$ = 0.50 | Mse = 2.034624e-05 Bias = 0.0002017238 | Mse = 1.930976e-05 Bias = 0.0001965185 | Mse = 1.933795e-05 Bias = 0.0001966619 | Mse = 2.062103e-05 Bias = 0.0002030814 |
| | $P_{RSS}$ = 0.75 | Mse = 7.79924e-06 Bias = 0.0001248939 | Mse = 1.103365e-05 Bias = 0.0001485506 | Mse = 1.969276e-05 Bias = 0.0001984578 | Mse = 1.909245e-05 Bias = 0.0001954096 |
| | **r = 7** | **m = 2, k = 14** | **m = 3, k = 21** | **m = 4, k = 28** | **m = 5, k = 35** |
| | $P_{RSS}$ = 0.25 | Mse = 1.560244e-05 Bias = 0.000176649 | Mse = 1.161882e-05 Bias = 0.000152439 | Mse = 1.186293e-05 Bias = 0.000154032 | Mse = 1.385473e-05 Bias = 0.0001664616 |
| | $P_{RSS}$ = 0.50 | .03374e-05 bias = 0.00014 | Mse = 1.377892e-05 Bias = 0.0001660056 | Mse = 1.074807e-05 Bias = 0.0001466156 | Mse = 1.121305e-05 Bias = 0.0001497535 |
| | $P_{RSS}$ = 0.75 | Mse = 8.002706e-06 Bias = 0.0001265125 | Mse = 6.681595e-06 Bias = 0.0001155993 | Mse = 6.970129e-06 Bias = 0.0001180689 | Mse = 1.073552e-05 Bias = 0.00014653 |

From the results of table 1.1, $figure$ 1, and $figure$ 2, we can observe the following:
1- For fixed values $P_{RSS}$ and $m$, when the number of cycles is increasing $r$ = 3,5 and 7, the values of Mse and Bias are decreasing. For more evidence, we can see $Figure$ 1 when the blue line gradually curves as the number of cycles increases.
2- For fixed values $m$ and sample size $k$, when the probability is increasing
$P_{RSS}$ = 0.25, 0.50, and 0.75, the blue line gradually curves; see $Figure$ 2, which means that values of Mse and Bias are decreasing.
3- Kernel epanechnikov gives us a better result than the gaussian kernel; for evidence, the result of empirical Mse and Bias in epanchnikov are less than gaussian.

### 4.3.2. LPR based on $MRSS_O$:

Estimation of cumulative distribution function $CDF$ using local polynomial regression $LPR$ based on median ranked sets sampling odd $MRSS_O$, we take $P$ degree of local polynomial regression, and using a Tylor series for linearizing $F_{LPR}(Y_{MRSS_O})$ as follows:

Consider a fixed regression model:

$$F\left(Y_{MRSS_O}\right) = F\left(Y_{i\left(\frac{m+1}{2}\right)j}\right) + \sigma\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)\varepsilon_{ij}\,;i\ =\ 1,2,\dots\,m\,;\ j\ =1,2,\dots\,r \tag{28}$$

$\sigma\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)$ is the variance of $Y_{MRSS_O}$ at point $Y_{i\left(\frac{m+1}{2}\right)j}$, $\varepsilon_{ij}$ is a residual error with normal distribution with mean zero and variance $\sigma^2$.

$$F_{LPR}\left(Y_{MRSS_O}\right) \approx F(Y) + F^{(1)}(Y)\left(Y_{MRSS_O} - Y\right) + \frac{F^{(2)}(Y)}{2!}\left(Y_{MRSS_O} - Y\right)^2 + \cdots + \frac{F^{(P)}(Y)}{P!}\left(Y_{MRSS_O} - Y\right)^P \tag{29}$$

Where $F^{(i)}(Y) = \left.\frac{\partial^{(i)}F\left(Y_{MRSS_O}\right)}{\partial {Y_{MRSS_O}}^{(i)}}\right|_{Y_{MRSS_O}=Y}$ $i = 1,2,\dots,P$ ; $Y$ is an observation from a data neighborhood around $Y_{MRSS_O}$, equation (29) is reformulated as follows:

$$F_{LPR}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right) \approx F\left(Y_{MRSS_O}\right) + F^{(1)}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right) + \frac{F^{(2)}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)}{2!}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)^2 + \cdots + \frac{F^{(P)}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)}{P!}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)^P \quad (30)$$

$$i = 1,2,\ldots m\,;\; j = 1,2,\ldots r$$

By estimating the ranked sets, sample units $F_{LPR}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)$ in equation (30) will be as follows:

$$\hat{F}_{LPR}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right) \approx \beta_{o_{MRSS_O}} + \beta_{1_{MRSS_O}}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right) + \beta_{2_{MRSS_O}}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)^2 + \cdots + \beta_{P_{MRSS_O}}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)^P \quad (31)$$

Where $\beta_{i_{MRSS_O}} = \frac{F^{(i)}\left(Y_{i\left(\frac{m+1}{2}\right)j}\right)}{i\,!}\; i = 0,1,2,\ldots,P$

$$\hat{\beta}_{MRSS_O} = \begin{bmatrix} \hat{\beta}_{o_{MRSS_O}} \\ \hat{\beta}_{1_{MRSS_O}} \\ \vdots \\ \hat{\beta}_{p_{MRSS_O}} \end{bmatrix}_{p*1} \;;\; \hat{\beta}_{MRSS_O} = \left(Y^*_{MRSS_O}{}^T W_{MRSS_O} Y^*_{MRSS_O}\right)^{-1} Y^*_{MRSS_O}{}^T W_{MRSS_O} \hat{F}_{MRSS_O} \quad (32)$$

$$Y^*_{MRSS_O} = \begin{bmatrix} 1 & \left(Y_{1\left(\frac{m+1}{2}\right)1} - Y_{MRSS_O}\right) & \cdots & \left(Y_{1\left(\frac{m+1}{2}\right)1} - Y_{MRSS_O}\right)^P \\ \vdots & \vdots & & \vdots \\ 1 & \left(Y_{m\left(\frac{m+1}{2}\right)1} - Y_{MRSS_O}\right) & \cdots & \left(Y_{m\left(\frac{m+1}{2}\right)1} - Y_{MRSS_O}\right)^P \\ \vdots & \vdots & & \vdots \\ 1 & \left(Y_{m\left(\frac{m+1}{2}\right)r} - Y_{MRSS_O}\right) & \cdots & \left(Y_{m\left(\frac{m+1}{2}\right)r} - Y_{MRSS_O}\right)^P \end{bmatrix}_{P*P} \quad \hat{F}_{MRSS_O} = \begin{bmatrix} \hat{F}\left(y_{1\left(\frac{m+1}{2}\right)1}\right) \\ \vdots \\ \hat{F}\left(y_{m\left(\frac{m+1}{2}\right)1}\right) \\ \vdots \\ \hat{F}\left(y_{m\left(\frac{m+1}{2}\right)r}\right) \end{bmatrix}_{P*1}$$

$$W_{MRSS_O} = diag\left\{\frac{1}{h_{MRSS_O}} k_{MRSS_O}\left(\frac{\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)}{h_{MRSS_O}}\right)\right\}, i = 1,\ldots,m, j = 1,\ldots,r \quad (33)$$

$k_{MRSS_O}(.)$ represents the kernel function of $MRSS_O$, and $h_{MRSS_O}$ represents the bandwidth that manages the size of the point in the zone of $Y_{MRSS_O}$. $W_{MRSS_O}$ represents the diagonal element matrix of weight since the estimator of $\hat{\beta}_{o_{MRSS_O}}$ is the $CDF$ of local polynomial regression in median ranked sets sample odd, where $e_1$ is the vector with 1 in the first entry and elsewhere is zero.

$$\hat{F}_{LPR}\left(Y_{MRSS_O}\right) = e_1 \times \hat{\beta}_{MRSS_O} = e_1 \times \left(Y^*_{MRSS_O}{}^T W_{MRSS_O} Y^*_{MRSS_O}\right)^{-1} Y^*_{MRSS_O}{}^T W_{MRSS_O} \hat{F}_{MRSS_O} = \hat{\beta}_{o_{MRSS_O}} \quad (34)$$

Estimation of the $CDF$ based on $LPR$ with degree $P = 0$, with the value of vector $e_1^T = \{1\}$, indicates that the estimator $\hat{\beta}_{o_{MRSS_O}}$ is equally likely to the $\hat{F}_{LPR}\left(Y_{MRSS_O}\right)$, which is equal to the estimator $\hat{F}_{LPR_{(0)}}\left(Y_{MRSS_O}\right)$ in equation (35).

$$\hat{F}_{LPR_{(0)}}\left(Y_{MRSS_O}\right) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} k_{MRSS_O}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)\hat{F}_{MRSS_O}}{\sum_{j=1}^{r}\sum_{i=1}^{m} k_{MRSS_O}\left(Y_{i\left(\frac{m+1}{2}\right)j} - Y_{MRSS_O}\right)} \quad (35)$$

We have proved in equation (34) that the estimator $\hat{\beta}_{o_{MRSS_O}}$ is equally likely to the estimator $\hat{F}_{LPR}(Y_{MRSS_O})$, which is equal to the estimator $\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})$, which is the estimator of the $CDF$ based on $LPR$ with degree $P = 1$, with the value of the vector being $e_1^T = \{1,0\}$, in equation (36).

$$\hat{F}_{LPR_{(1)}}(Y_{MRSS_O}) = k_{MRSS_O}^{-1} \frac{S_0\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h_{MRSS_O}\right) - S_1\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h_{MRSS_O}\right)\left(Y_{i\left(\frac{m+1}{2}\right)_j} - Y_{MRSS_O}\right)}{S_0\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h_{MRSS_O}\right) S_2\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h_{MRSS_O}\right) - S_1\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h_{MRSS_O}\right)} \tag{36}$$

Where $S_L\left(Y_{i\left(\frac{m+1}{2}\right)_j}, h\right) = k_{MRSS_O}^{-1} \sum_{j=1}^{r} \sum_{i=1}^{m} \left(Y_{i\left(\frac{m+1}{2}\right)_j}, Y_{MRSS_O}\right)^L k_{MRSS_{O_h}}\left(Y_{i\left(\frac{m+1}{2}\right)_j}, Y_{MRSS_O}\right)$

The properties of $CDF$ will be studied based on local polynomial using median ranked sets sampling odd, depending on these conditions:

1- The function $F^{(2)}(Y)$ and $\sigma$ each continuous on [0,1].

2- The kernel $k_{MRSS_O}$ is symmetric about zero and is reinforced on $[-1,1]$.

3- The bandwidth $h_{MRSS_O} = h_{MRSS_{O_k}}$ it is a satisfactory sequence, $h_{MRSS_O} \to 0 \; and \; kh_{MRSS_O} \to \infty \; as \; k \to \infty$.

4- The point $Y_{i\left(\frac{m+1}{2}\right)_j}$ at which the estimate is made satisfactory $h_{MRSS_O} < Y_{i\left(\frac{m+1}{2}\right)_j} < 1 - h_{MRSS_O}$ for all $k \geq k_0$, where $k_0$ is fixed.

In this subsection, we will make a derivative for the Bias and $MSE$ of the estimate $CDF$ based on local polynomial.

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})\right) = E\left\{e_1^T \left(Y_{MRSS_O}^{*\,T} W_{MRSS_O} Y_{MRSS_O}^*\right)^{-1} Y_{MRSS_O}^{*\,T} W_{MRSS_O} \hat{F}_{MRSS_O}\right\}$$

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})\right) = F_{MRSS_O} + \frac{1}{2} h_{MRSS_O}^{(2)} F_{MRSS_O}^{(2)} \mu_2(k_{MRSS_O}) + O\left(h_{MRSS_O}^{(2)} k_{MRSS_O}^{-1}\right) \tag{37}$$

$$Bias\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})\right) = \frac{1}{2} h_{MRSS_O}^{(2)} F_{MRSS_O}^{(2)} \mu_2(k_{MRSS_O}) + O\left(h_{MRSS_O}^{(2)} k_{MRSS_O}^{-1}\right) \tag{38}$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})\right) = e_1^T \left(Y_{MRSS_O}^{*\,T} W_{MRSS_O} Y_{MRSS_O}^*\right)^{-1} Y_{MRSS_O}^{*\,T} W_{MRSS_O} V\left(F(Y_{MRSS_O})\right) \tag{39}$$
$$W_{MRSS_O} Y_{MRSS_O}^{*\,T} \left(Y_{MRSS_O}^{*\,T} W_{MRSS_O} Y_{MRSS_O}^*\right)^{-1} e_1^T$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_O})\right) = \left(k_{MRSS_O} h_{MRSS_O}\right)^{-1} R(k_{MRSS_O}) V\left(F(Y_{MRSS_O})\right) + O\left\{\left(k_{MRSS_O} h_{MRSS_O}\right)^{-1}\right\} \tag{40}$$

In the Table 1.2, we use the R-Program to achieve the result of the empirical mean square error and bias of $CDF$ based on $LPR$ by using $MRSS\ odd$ with four levels of set size ($m$ = 3,5,7,9), with three values of probability ($P_{MRSS_O}$ = 0.25,0.50,0.75), and with the same number of cycles $r$, degree of kernel, type of kernel, and bandwidth as in the previous table 1.1.

**Table 1.2: The empirical mean square error and bias of CDF based on LPR by Using MRSS ODD**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| | **r = 3** | **m = 3, k = 9** | **m = 5, k = 15** | **m = 7, k = 21** | **m = 9, k = 27** |
| | $\boldsymbol{P_{MRSS_O}}$ **= 0.25** | Mse = 8.512535e-05 Bias = 0.0004126145 | Mse = 7.691912e-05 Bias = 0.0003922222 | Mse = 8.174425e-05 Bias = 0.0004043371 | Mse = 7.924077e-05 Bias = 0.0003980974 |
| | $\boldsymbol{P_{MRSS_O}}$ **= 0.50** | 865291e-05 Bias = 0.0003 | Mse = 5.529911e-05 Bias = 0.0003325631 | .341845e-05 Bias = 0.00035 | .342218e-05 Bias = 0.00035 |
| | $\boldsymbol{P_{MRSS_O}}$ **= 0.75** | Mse = 2.326374e-05 Bias = 0.0002157023 | Mse = 3.806345e-05 Bias = 0.000275911 | Mse = 4.086824e-05 Bias = 0.0002858959 | Mse = 3.77983e-05 Bias = 0.0002749484 |
| | **r = 5** | **m = 3, k = 15** | **m = 5, k = 25** | **m = 7, k = 35** | **m = 9, k = 45** |
| | $\boldsymbol{P_{MRSS_O}}$ **= 0.25** | Mse = 2.637953e-05 Bias = 0.0002296934 | Mse = 2.60296e-05 Bias = 0.0002281649 | Mse = 2.278614e-05 Bias = 0.0002134767 | Mse = 2.114632e-05 Bias = 0.0002056517 |
| | $\boldsymbol{P_{MRSS_O}}$ **= 0.50** | 524377e-05 Bias = 0.0001 | Mse = 1.809422e-05 Bias = 0.0001902326 | 1.96922e-05 Bias = 0.00019 | 1.632715e-05 Bias = 0.0001 |

| | | | | | |
|---|---|---|---|---|---|
| | $P_{MRSS_O}$ = 0.75 | Mse = 1.107483e-05<br>Bias = 0.0001488276 | Mse = 1.713709e-05<br>Bias = 0.0001851329 | Mse = 1.3506e-05<br>Bias = 0.0001643533 | Mse = 1.482615e-05<br>Bias = 0.0001721984 |
| | **r = 7** | **m = 3, k = 21** | **m = 5, k = 35** | **m = 7, k = 49** | **m = 9, k = 63** |
| | $P_{MRSS_O}$ = 0.25 | Mse = 1.56476e-05<br>Bias = 0.0001769045 | Mse = 9.443144e-06<br>Bias = 0.0001374274 | Mse = 1.042319e-05<br>Bias = 0.0001443828 | Mse = 1.071913e-05<br>Bias = 0.0001464181 |
| | $P_{MRSS_O}$ = 0.50 | 674636e-06 Bias = 0.0001 | Mse = 9.210139e-06<br>Bias = 0.0001357213 | .552546e-06 Bias = 0.00013 | 7.716423e-06 Bias = 0.0001 |
| | $P_{MRSS_O}$ = 0.75 | Mse = 9.022209e-06<br>Bias = 0.0001343295 | Mse = 6.555129e-06<br>Bias = 0.0001145 | Mse = 9.568994e-06<br>Bias = 0.0001383401 | Mse = 8.611258e-06<br>Bias = 0.0001312346 |
| **Kernel / Epanechnikov** | | | | | |
| | **r = 3** | **m = 3, k = 9** | **m = 5, k = 15** | **m = 7, k = 21** | **m = 9, k = 27** |
| | $P_{MRSS_O}$ = 0.25 | Mse = 0.0001048237<br>Bias = 0.0004578727 | Mse = 7.279169e-05<br>Bias = 0.0003815539 | Mse = 6.217946e-05<br>Bias = 0.0003526456 | Mse = 6.378918e-05<br>Bias = 0.0003571811 |
| | $P_{MRSS_O}$ = 0.50 | Mse = 5.373519e-05<br>Bias = 0.0003278267 | Mse = 5.708709e-05<br>Bias = 0.0003378967 | Mse = 7.544543e-05<br>Bias = 0.0003884467 | Mse = 5.500072e-05<br>Bias = 0.0003316647 |
| | $P_{MRSS_O}$ = 0.75 | Mse = 3.294451e-05<br>Bias = 0.0002566886 | Mse = 3.944943e-05<br>Bias = 0.0002808894 | Mse = 3.64872e-05<br>Bias = 0.0002701377 | Mse = 3.562654e-05<br>Bias = 0.0002669327 |
| | **r = 5** | **m = 3, k = 15** | **m = 5, k = 25** | **m = 7, k = 35** | **m = 9, k = 45** |
| | $P_{MRSS_O}$= 0.25 | Mse = 2.706822e-05<br>Bias = 0.0002326724 | Mse = 2.142626e-05<br>Bias = 0.0002070085 | Mse = 2.363208e-05<br>Bias = 0.0002174032 | Mse = 1.916575e-05<br>Bias = 0.0001957843 |
| | $P_{MRSS_O}$ = 0.50 | Mse = 2.02618e-05<br>Bias = 0.0002013047 | Mse = 1.732828e-05<br>Bias = 0.0001861627 | Mse = 2.406087e-05<br>Bias = 0.0002193667 | Mse = 1.673458e-05<br>Bias = 0.0001829458 |
| | $P_{MRSS_O}$ = 0.75 | Mse = 1.15668e-05<br>Bias = 0.0001520973 | Mse = 1.023598e-05<br>Bias = 0.0001430803 | Mse = 1.64224e-05<br>Bias = 0.0001812313 | Mse = 1.666829e-05<br>Bias = 0.0001825831 |
| | **r = 7** | **m = 3, k = 21** | **m = 5, k = 35** | **m = 7, k = 49** | **m = 9, k = 63** |
| | $P_{MRSS_O}$ = 0.25 | Mse = 1.297796e-05<br>Bias = 0.0001611084 | Mse = 1.355451e-05<br>Bias = 0.0001646481 | Mse = 1.365714e-05<br>Bias = 0.0001652703 | Mse = 1.144232e-05<br>Bias = 0.0001512767) |
| | $P_{MRSS_O}$ = 0.50 | .17379e-05 bias = 0.00015 | Mse = 1.000622e-05<br>Bias = 0.0001414653 | Mse = 1.265157e-05<br>Bias = 0.0001590696 | Mse = 1.054311e-05<br>Bias = 0.000145211) |
| | $P_{MRSS_O}$ = 0.75 | Mse = 9.345159e-06<br>Bias = 0.0001367125 | Mse = 8.477178e-06<br>Bias = 0.0001302089 | Mse = 8.385006e-06<br>Bias = 0.0001294991 | Mse = 1.065344e-05<br>Bias = 0.0001459688) |

The following steps describe table 1.2, $Figure$ 3, and $Figure$ 4:

1- When the number of cycles is increasing $r$ = 3,5 and 7, for fixed values $P_{MRSS_O}$ and $m$, the values of Mse and Bias are decreasing, and also $Figure$ 3 proves that the blue line is curving gradually when we increase the cycles $r$.

2- In $Figure$ 4, we notice that the blue line continues to curve as the probability increases ( $P_{MRSS_O}$ = 0.25, 0.50, 0.75), and this indicates a decrease in the values of Mse and bias for fixed values $m$ and sample size $k$.

3- In table 1.2, we notice that the kernel epanechnikov is better than the kernel gaussian. Evidence of this is that the experimental values in Mse and Bias are lower than what they are in the kernel gaussian.

### 4.3.3. LPR based on $MRSS_e$:

Estimation of cumulative distribution function $CDF$ using local polynomial regression $LPR$ based on median ranked sets sampling even $MRSS_e$, we can linearize the $CDF$, $F(Y_{MRSS_e})$, by using a Tylor series and with $P$ degree of local polynomial regression.

The regression fixed model based on median ranked sets sample even:

$$F(Y_{MRSS_e}) = F\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right) + \sigma\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)\varepsilon_{ij} \quad (39)$$

$$i = 1,2,\dots\frac{m}{2}\ ;\ j = 1,2,\dots r$$

$$F_{LPR}(Y_{MRSS_e}) \approx F(Y) + F^{(1)}(Y)(Y_{MRSS_e} - Y) + \frac{F^{(2)}(Y)}{2!}(Y_{MRSS_e} - Y)^2 + \cdots + \frac{F^{(P)}(Y)}{P!}(Y_{MRSS_e} - Y)^P \quad (40)$$

where $F^{(i)}(Y) = \left.\frac{\partial^{(i)}F(Y_{MRSS_e})}{\partial {Y_{MRSS_e}}^{(i)}}\right|_{Y_{MRSS_e}=Y}$ $i = 1,2,\dots,P$ ; $Y$ is an observation from the data neighborhood around $Y_{MRSS_e}$; if the $CDF$ of ranked sets sample $F_{LPR}(Y_{MRSS_e})$ is unknown, the equation (40) will be as follows:

$$F_{LPR}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)\approx \tag{41}$$
$$F(Y_e)+F^{(1)}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)+$$
$$+\frac{F^{(2)}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)}{2!}\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)^2+\cdots$$
$$+\frac{F^{(P)}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)}{P!}\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)^P$$

$i=1,2,\dots,\frac{m}{2}\ ;\ j=1,2,\dots,r,\quad$ where $\approx\left\{\frac{m}{2}+\frac{m}{2}=m\right\}$

By estimating the ranked sets sample units $F_{LPR}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)$, the equation (41) will be:

$$\hat{F}_{LPR}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)\approx\beta_{o_{MRSS_e}}+\beta_{1_{MRSS_e}}\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)+ \tag{42}$$
$$\beta_{2_{MRSS_e}}\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)^2+\cdots+\beta_{P_{MRSS_e}}\left(\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}\right)^P$$

Where $\beta_{i_{MRSS_e}}=\frac{F^{(i)}\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)}{i\,!}\ i=0,1,2,\dots,P$

$$\hat{\beta}_{MRSS_e}=\begin{bmatrix}\hat{\beta}_{o_{MRSS_e}}\\ \hat{\beta}_{1_{MRSS_e}}\\ \vdots\\ \hat{\beta}_{p_{MRSS_e}}\end{bmatrix}_{p*1}\ ;\ \hat{\beta}_{MRSS_e}=\left(Y^*_{MRSS_e}{}^T W_{MRSS_e}Y^*_{MRSS_e}\right)^{-1}Y^*_{MRSS_e}{}^T W_{MRSS_e}\hat{F}_{MRSS_e} \tag{43}$$

$$Y^*_{MRSS_e}=\begin{bmatrix}1 & \left(\left(Y_{1\left(\frac{m}{2}\right)1}+Y_{\frac{m}{2}+1\left(\frac{m+2}{2}\right)1}\right)-Y_{MRSS_e}\right) & \dots & \left(\left(Y_{1\left(\frac{m}{2}\right)1}+Y_{\frac{m}{2}+1\left(\frac{m+2}{2}\right)1}\right)-Y_{MRSS_e}\right)^P\\ \vdots & \vdots & & \vdots\\ 1 & \left(\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)1}+Y_{m\left(\frac{m+2}{2}\right)1}\right)-Y_{MRSS_e}\right) & \dots & \left(\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)1}+Y_{m\left(\frac{m+2}{2}\right)1}\right)-Y_{MRSS_e}\right)^P\\ \vdots & \vdots & & \vdots\\ 1 & \left(\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)r}+Y_{m\left(\frac{m+2}{2}\right)r}\right)-Y_{MRSS_e}\right) & \dots & \left(\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)r}+Y_{m\left(\frac{m+2}{2}\right)r}\right)-Y_{MRSS_e}\right)^P\end{bmatrix}_{P*P}$$

$$\hat{F}_{MRSS_e}=\begin{bmatrix}\hat{F}\left(Y_{1\left(\frac{m}{2}\right)1}+Y_{\frac{m}{2}+1\left(\frac{m+2}{2}\right)1}\right)\\ \vdots\\ \hat{F}\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)1}+Y_{m\left(\frac{m+2}{2}\right)1}\right)\\ \vdots\\ \hat{F}\left(Y_{\frac{m}{2}\left(\frac{m}{2}\right)r}+Y_{m\left(\frac{m+2}{2}\right)r}\right)\end{bmatrix}_{k*1}$$

$$W_{MRSS_e}=diag\left\{\frac{1}{h_{MRSS_e}}k_{MRSS_e}\left(\frac{\left(Y_{i\left(\frac{m}{2}\right)j}+Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)-Y_{MRSS_e}}{h_{MRSS_e}}\right)\right\},i=1,\dots,\frac{m}{2},j=1,\dots,r \tag{44}$$

$k_{MRSS_e}(.)$ represents the kernel function of $MRSS_e$, and $h_{MRSS_e}$ represents the bandwidth that manages the size of the point in the zone of $Y_{MRSS_e}$. $W_{MRSS_e}$ represents the diagonal element matrix of weight since the estimator of $\hat{\beta}_{o_{MRSS_e}}$ is the $CDF$ of local polynomial regression in median ranked sets sample even, where $e_1$ is the vector with 1 in the first entry and elsewhere is zero.

$$\hat{F}\left(Y_{MRSS_e}\right)=e_1\times\hat{\beta}_{MRSS_e}=e_1\times\left(Y_{MRSS_e}{}^T W_{MRSS_e}Y_{MRSS_e}\right)^{-1}Y_{MRSS_e}{}^T W_{MRSS_e}\hat{F}_{MRSS_e}=\hat{\beta}_{o_{MRSS_e}} \tag{45}$$

In equation (45), we get the estimator $\hat{\beta}_{o_{MRSS_e}}$, which is equally likely to the $\hat{F}_{LPR}(Y_{MRSS_e})$, which is equal to the $\hat{F}_{LPR_{(0)}}(Y_{MRSS_e})$ the estimator of the $CDF$ based on $LPR$ with degree $P = 0$, with the value of vector $e_1^T = \{1\}$, in equation (46).

$$\hat{F}_{LPR_{(0)}}(Y_{MRSS_e}) = \frac{\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} k_{MRSS_e}\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j} - Y_{MRSS_e}\right)\hat{F}_{MRSS_e}}{\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} k_{MRSS_e}\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j} - Y_{MRSS_e}\right)} \quad (46)$$

In the equation (47) we prove $\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})$, which is the estimator of the $CDF$ based on $LPR$ with degree $P = 1$, with the value of the vector being $e_1^T = \{1,0\}$, this indicates that the estimation of $\hat{\beta}_{o_{MRSS_e}}$ is equally likely to the $\hat{F}_{LPR}(Y_{MRSS_e})$, which is equal to the $\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})$ in equation (47).

$$\hat{F}_{LPR_{(1)}}(Y_{MRSS_e}) =$$
$$k_{MRSS_e}^{-1}\frac{S_0\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right) - S_1\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right)\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j} - Y_{MRSS_e}\right)}{S_0\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right)S_2\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right) - S_1\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right)} \quad (47)$$

Where $S_L\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, h_{MRSS_e}\right) =$

$$k_{MRSS_e}^{-1}\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}}\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, Y_{MRSS_e}\right)^L k_{MRSS_{e_h}}\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}, Y_{MRSS_e}\right) \quad (48)$$

The properties of $CDF$ will be studied based on local polynomial using median ranked sets sampling even depending on these conditions:

1- The function $F^{(2)}(Y)$ and $\sigma$ each continuous on $[0,1]$.

2- The kernel $k_{MRSS_e}$ is symmetric about zero and is reinforced on $[-1,1]$.

3- The bandwidth $h_{MRSS_e} = h_{MRSS_{e_k}}$ it is a satisfactory sequence, $h_{MRSS_e} \to 0 \; and \; kh_{MRSS_e} \to \infty \; as \; k \to \infty$.

4- The point $\left(Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j}\right)$ at which the estimate is made satisfactory.

$h_{MRSS_e} < Y_{i\left(\frac{m}{2}\right)j} + Y_{\frac{m}{2}+i\left(\frac{m+2}{2}\right)j} < 1 - h_{MRSS_e}$ for all $k \geq k_0$, where $k_0$ is fixed.

In this subsection, we will make a derivative for the bias and $MSE$ of the estimate $CDF$ based on local polynomial.

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})\right) = E\left\{e_1^T\left(Y_{MRSS_e}^{*\,T}W_{MRSS_e}Y_{MRSS_e}^{*}\right)^{-1}Y_{MRSS_e}^{*\,T}W_{MRSS_e}\hat{F}_{MRSS_e}\right\}$$

$$E\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})\right) = F_{MRSS_e} + \frac{1}{2}h_{MRSS_e}^{(2)}F_{MRSS_e}^{(2)}\mu_2(k_{MRSS_e}) + O\left(h_{MRSS_e}^{(2)}k_{MRSS_e}^{-1}\right) \quad (49)$$

$$Bias\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})\right) = \frac{1}{2}h_{MRSS_e}^{(2)}F_{MRSS_e}^{(2)}\mu_2(k_{MRSS_e}) + O\left(h_{MRSS_e}^{(2)}k_{MRSS_e}^{-1}\right) \quad (50)$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})\right) = e_1^T\left(Y_{MRSS_e}^{*\,T}W_{MRSS_e}Y_{MRSS_e}^{*}\right)^{-1}Y_{MRSS_e}^{*\,T}W_{MRSS_e}V\left(F(Y_{MRSS_e})\right) \quad (51)$$
$$W_{MRSS_e}Y_{MRSS_e}^{*\,T}\left(Y_{MRSS_e}^{*\,T}W_{MRSS_e}Y_{MRSS_e}^{*}\right)^{-1}e_1^T$$

$$MSE\left(\hat{F}_{LPR_{(1)}}(Y_{MRSS_e})\right) = \left(k_{MRSS_e}h_{MRSS_e}\right)^{-1}R(k_{MRSS_e})V\left(F(Y_{MRSS_e})\right) + O\left\{\left(k_{MRSS_e}h_{MRSS_e}\right)^{-1}\right\} \quad (52)$$

In Table 1.3, we will solve the empirical mean square error and bias of $CDF$ based on $LPR$ by using $MRSS\ even$ with four levels of set size ($m$ = 2,4,6,8), with three levels of probability ($P_{MRSS_e}$ = 0.25,0.50,0.75) with the same number of cycles $r$, degree of kernel, type of kernel, and bandwidth as in the previous tables.

**Table 1.3: The empirical mean square error and bias of CDF based on LPR by Using MRSS EVEN**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| | **r = 3** | **m = 2, k = 6** | **m = 4, k = 12** | **m = 6, k = 18** | **m = 8, k = 24** |
| | $\boldsymbol{P_{MRSS_e}}$ **= 0.25** | Mse = 0.0001086189<br>Bias = 0.0004660877 | Mse = 9.811098e-05<br>Bias = 0.0004429695 | Mse = 6.557998e-05<br>Bias = 0.0003621601 | Mse = 8.739888e-05<br>Bias = 0.0004180882 |
| | $\boldsymbol{P_{MRSS_e}}$ **= 0.50** | .418607e-05 Bias = 0.0003 | Mse = 5.512962e-05<br>Bias = 0.000332053 | .664477e-05 Bias = 0.0003 | 312043e-05 Bias = 0.0003 |
| | $\boldsymbol{P_{MRSS_e}}$ **= 0.75** | Mse = 1.983807e-05<br>Bias = 0.0001991887 | Mse = 2.367623e-05<br>Bias = 0.0002176062 | Mse = 3.517974e-05<br>Bias = 0.0002652536 | Mse = 3.618193e-05<br>Bias = 0.0002690053 |

| | | | | | |
|---|---|---|---|---|---|
| | **r = 5** | **m = 2, k = 10** | **m = 4, k = 20** | **m = 6, k = 30** | **m = 8, k = 40** |
| | $P_{MRSS_e}$ **= 0.25** | Mse = 3.73182e-05 Bias = 0.0002731966 | Mse = 2.330108e-05 Bias = 0.0002158753 | Mse = 2.754449e-05 Bias = 0.0002347104 | Mse = 2.789499e-05 Bias = 0.000236199 |
| | $P_{MRSS_e}$ **= 0.50** | .611155e-05 Bias = 0.0001 | Mse = 2.056189e-05 Bias = 0.00020279 | .56869e-05 Bias = 0.00017 | .545758e-05 Bias = 0.0001 |
| | $P_{MRSS_e}$ **= 0.75** | Mse = 1.09177e-05 Bias = 0.000147768 | Mse = 1.606776e-05 Bias = 0.0001792639 | Mse = 1.418493e-05 Bias = 0.0001684336 | Mse = 1.940943e-05 Bias = 0.000197025 |
| | **r = 7** | **m = 2, k = 14** | **m = 4, k = 28** | **m = 6, k = 42** | **m = 8, k = 56** |
| | $P_{MRSS_e}$ **= 0.25** | Mse = 1.541827e-05 Bias = 0.0001756034 | Mse = 1.022398e-05 Bias = 0.0001429964 | Mse = 1.08598e-05 Bias = 0.0001473757 | Mse = 1.15695e-05 Bias = 0.0001521151 |
| | $P_{MRSS_e}$ **= 0.50** | .078183e-05 Bias = 0.0001 | Mse = 1.026643e-05 Bias = 0.000143292 | .229568e-05 Bias = 0.0001 | .829758e-06 Bias = 0.0001 |
| | $P_{MRSS_e}$ **= 0.75** | Mse = 6.130213e-06 Bias = 0.0001107268 | Mse = 7.483774e-06 Bias = 0.0001223419 | Mse = 8.35094e-06 Bias = 0.0001292358 | Mse = 8.344404e-06 Bias = 0.0001291852 |
| **Kernel / Epanechnikov** | | | | | |
| | **r = 3** | **m = 2, k = 6** | **m = 4, k = 12** | **m = 6, k = 18** | **m = 8, k = 24** |
| | $P_{MRSS_e}$ **= 0.25** | Mse = 0.0001040585 Bias = 0.0004561985 | Mse = 0.0001081452 Bias = 0.0004650703 | Mse = 7.688896e-05 Bias = 0.0003921453 | Mse = 8.177441e-05 Bias = 0.0004044117 |
| | $P_{MRSS_e}$ **= 0.50** | Mse = 5.506826e-05 Bias = 0.0003318682 | Mse = 4.512977e-05 Bias = 0.0003004323 | Mse = 4.609768e-05 Bias = 0.0003036369 | Mse = 6.321508e-05 Bias = 0.0003555702 |
| | $P_{MRSS_e}$ **= 0.75** | Mse = 2.04245e-05 Bias = 0.0002021114 | Mse = 3.123355e-05 Bias = 0.0002499342 | Mse = 3.560996e-05 Bias = 0.0002668706 | Mse = 4.591781e-05 Bias = 0.0003030439 |
| | **r = 3** | **m = 2, k = 10** | **m = 4, k = 20** | **m = 6, k = 30** | **m = 8, k = 40** |
| | $P_{MRSS_e}$ **= 0.25** | Mse = 3.372501e-05 Bias = 0.0002597114 | Mse = 2.574094e-05 Bias = 0.0002268962 | Mse = 2.332808e-05 Bias = 0.0002160004 | Mse = 2.738825e-05 Bias = 0.0002340438 |
| | $P_{MRSS_e}$ **= 0.50** | Mse = 2.510962e-05 Bias = 0.0002240965 | Mse = 1.965092e-05 Bias = 0.0001982469 | Mse = 1.50904e-05 Bias = 0.0001737262 | Mse = 1.491986e-05 Bias = 0.0001727418 |
| | $P_{MRSS_e}$ **= 0.75** | Mse = 7.975208e-06 Bias = 0.000126295 | Mse 1.677073e-05 Bias = 0.0001831433 | Mse = 1.190957e-05 Bias = 0.0001543345 | Mse = 1.722819e-05 Bias = 0.0001856243 |
| | **r = 7** | **m = 2, k = 14** | **m = 4, k = 28** | **m = 6, k = 42** | **m = 8, k = 56** |
| | $P_{MRSS_e}$**= 0.25** | Mse = 1.558296e-05 Bias = 0.0001765387 | Mse = 1.510541e-05 Bias = 0.0001738126 | Mse = 9.30583e-06 Bias = 0.0001364246 | Mse = 1.232422e-05 Bias = 0.0001569982 |
| | $P_{MRSS_e}$ **= 0.50** | Mse = 1.284562e-05 Bias = 0.0001602849 | Mse = 9.537822e-06 Bias = 0.0001381146 | Mse = 9.276924e-06 Bias = 0.0001362125 | Mse = 1.013299e-05 Bias = 0.0001423586 |
| | $P_{MRSS_e}$ **= 0.75** | Mse = 7.110055e-06 Bias = 0.0001192481 | Mse = 8.564457e-06 Bias = 0.0001308775 | Mse = 8.637282e-06 Bias = 0.0001314327 | Mse = 8.434095e-06 Bias = 0.0001298776 |

Table 1.3, $Figure$ 5, and $Figure$ 6, were interpreted as follows:

1- The values of $P_{MRSS_e}$ and $m$ are immutability, Mse and Bias are decreasing, such as the blue line curving in the $figure$ 5, this happens when the repetitions are increasing, $r$ = 3,5,7.

2- As the probability $P_{MRSS_e}$ = 0.25,0.50, and 0.75 increases, the value of both Mse and Bias decreases if the sample size $k$ and $m$ are stable. $Figure$ 6 proves that through the curvature of the blue line.

3- On the basis of the result empirical Mse and Bias, we confirmed that the kernel epanechnikov is better than the kernel gaussian.

## 5. SIMULATION STUDY AND CONCLUSION

### 5.1. MONTE CARLO COMPARISONS

In this part, a comparison is made between the $CDF$ estimator for the three methods (moment, maximum likelihood, and local polynomial regression) based on the relative efficiency $RE$, which is defined according to the following relationship:

$$RE_i(CDF) = \frac{MSE\left(\hat{F}_i(Y_k)\right)}{MSE\left(\hat{F}(Y_k)\right)} \qquad i = MOM, MLE\ ; k = RSS, MRSS \tag{53}$$

To generate $RSS$ and $MRSS$, we assume that the ranking process is done using imperfect ranking mode, with a binomial distribution when simulation $n$ = 500 and sample size $k = m * r$ in four levels of sets size $m$ = 2,3,4,5, in $RSS$. And in $MRSS$ $m$ = 3,5,7,9 for type $odd$, also $m$ = 2,4,6,8 for type $even$, cycles $r$ = 3,5,7. With three degrees of probability (0.25, 0.50, 0.75), we used two types of kernels (normal, epanechnikov). We use bandwidth 0.8 with degree 2 for $LPR$ because they give a good result compared to other bandwidths and degrees. In Tables 2.1, 2.2, and 2.3, we use R-programing to obtain the result of the relative efficiency $RE$.

**Table 2.1: The Relative Efficiency (RE) Of the Method Of Moment And MLE To LPR based on RSS**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| **r = 3** | | **m = 2, k = 6** | **m = 3, k = 9** | **m = 4, k = 12** | **m = 5, k = 15** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 1.135750702 | 1.16406889 | 1.17344999 | 1.471932307 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 1.137516514 | 1.162210688 | 1.174450728 | 1.474069551 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 8.206516713 | 8.963245957 | 10.20831334 | 10.00206242 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 8.208776681 | 8.952738327 | 10.19476759 | 10.00882799 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 5.063775451 | 47.27995388 | 25.63634402 | 40.74663898 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 5.063065298 | 47.27842886 | 25.64026797 | 40.71708223 |
| **r = 5** | | **m = 2, k = 10** | **m = 3, k = 15** | **m = 4, k = 20** | **m = 5, k = 25** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 3.081517979 | 5.066695671 | 3.676639827 | 4.322041795 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 3.089373086 | 5.06586001 | 3.67284517 | 4.334577723 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 2.225527765 | 22.94889871 | 23.22915932 | 21.62112914 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 2.226000338 | 22.96821049 | 23.25431716 | 21.65576338 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 7.401329557 | 61.04156589 | 61.22146396 | 55.01398976 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 7.400843217 | 60.98252949 | 61.2500539 | 54.99610974 |
| **r = 7** | | **m = 2, k = 14** | **m = 3, k = 21** | **m = 4, k = 28** | **m = 5, k = 35** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 9.350327185 | 8.481033713 | 9.664600701 | 9.828317550 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 9.314370441 | 8.501444501 | 9.677066038 | 9.871335973 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 3.896041410 | 37.03106475 | 31.81991703 | 44.73795452 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 3.899763472 | 37.01806995 | 31.86743149 | 44.7348804 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 1.386600932 | 108.5673353 | 85.88318018 | 146.7357995 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 1.385961876 | 108.5636544 | 85.8482365 | 146.9243931 |
| **Kernel / Epanechikov** | | | | | |
| **r = 3** | | **m = 2, k = 6** | **m = 3, k = 9** | **m = 4, k = 12** | **m = 5, k = 15** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 1.012059090 | 1.408594426 | 1.716100684 | 1.326686852 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 1.013632592 | 1.406345891 | 1.717564203 | 1.328613199 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 8.620434815 | 8.554989917 | 8.456819846 | 8.24706609 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 8.622808771 | 8.544960886 | 8.445598208 | 8.252644556 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 5.517384663 | 47.42853921 | 33.52502062 | 25.49415595 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 5.516610895 | 47.4270094 | 33.53015203 | 25.47566302 |
| **r = 5** | | **m = 2, k = 10** | **m = 3, k = 15** | **m = 4, k = 20** | **m = 5, k = 25** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 3.561232678 | 4.385218429 | 5.910724402 | 4.978331956 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 3.570310627 | 4.384495166 | 5.904623949 | 4.992771429 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 2.360911893 | 24.35167501 | 23.85938013 | 21.84837033 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 2.361413214 | 24.37216724 | 23.88522051 | 21.88336858 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 1.385402937 | 95.95917942 | 52.62949429 | 53.18112657 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 1.385311902 | 95.86637242 | 52.65407185 | 53.16384225 |
| **r = 7** | | **m = 2, k = 14** | **m = 3, k = 21** | **m = 4, k = 28** | **m = 5, k = 35** |
| **MOM & LPR** | $P_{RSS}$ **= 0.25** | 7.578423631 | 9.874006138 | 9.372102845 | 7.796579219 |
| **MLE & LPR** | $P_{RSS}$ **= 0.25** | 7.549280754 | 9.897769309 | 9.384190921 | 7.830704748 |
| **MOM & LPR** | $P_{RSS}$ **= 0.50** | 4.566728578 | 33.27649772 | 41.39732994 | 38.54676471 |
| **MLE & LPR** | $P_{RSS}$ **= 0.50** | 4.571091377 | 33.26482046 | 41.45914569 | 38.54411601 |
| **MOM & LPR** | $P_{RSS}$ **= 0.75** | 1.328531874 | 154.4991877 | 143.8662039 | 90.59330149 |
| **MLE & LPR** | $P_{RSS}$ **= 0.75** | 1.327919581 | 154.4939494 | 143.8076684 | 90.7097374 |

**Table 2.2: The Relative Efficiency (RE) Of the Method Of Moment And MLE To LPR Of MRSS ODD**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| **r = 3** | | **m = 3, k = 9** | **m = 5, k = 15** | **m = 7, k = 21** | **m = 9, k = 27** |
| **MOM & LPR** | $P_{MRSS_O}$ **= 0.25** | 1.415389188 | 1.527881494 | 1.402094949 | 1.406503243 |
| **MLE & LPR** | $P_{MRSS_O}$ **= 0.25** | 1.414098151 | 1.531513881 | 1.405189967 | 1.412945634 |
| **MOM & LPR** | $P_{MRSS_O}$ **= 0.50** | 8.226359101 | 8.515932716 | 7.231069192 | 7.052961598 |
| **MLE & LPR** | $P_{MRSS_O}$ **= 0.50** | 8.222205855 | 8.499512922 | 7.225271195 | 7.053201262 |
| **MOM & LPR** | $P_{MRSS_O}$ **= 0.75** | 4.664795944 | 27.80029136 | 25.26318726 | 26.65122506 |
| **MLE & LPR** | $P_{MRSS_O}$ **= 0.75** | 4.662367272 | 27.79661329 | 25.27047898 | 26.64191247 |
| **r = 5** | | **m = 3, k = 15** | **m = 5, k = 25** | **m = 7, k = 35** | **m = 9, k = 45** |
| **MOM & LPR** | $P_{MRSS_O}$ **= 0.25** | 4.458475947 | 4.336927959 | 4.756751253 | 4.901259415 |
| **MLE & LPR** | $P_{MRSS_O}$ **= 0.25** | 4.466569344 | 4.335760058 | 4.73894657 | 4.898157221 |

| | | | | | |
|---|---|---|---|---|---|
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.50** | 3.088215710 | 24.9128672 | 21.95669351 | 25.34578294 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.50** | 3.088062861 | 24.93722305 | 21.95971501 | 25.32617144 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.75** | 9.552986366 | 59.23718671 | 72.01075818 | 62.79158109 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.75** | 9.559189622 | 59.21191988 | 72.07851325 | 62.82913636 |
| **r = 7** | | **m = 3, k = 21** | **m = 5, k = 35** | **m = 7, k = 49** | **m = 9, k = 63** |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.25** | 7.326612388 | 11.49452979 | 9.720133663 | 8.919277964 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.25** | 1.322835909 | 11.41768038 | 9.745375456 | 8.955212783 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.50** | 5.287188996 | 47.05310094 | 47.58948973 | 49.52667836 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.50** | 5.287012619 | 46.91815183 | 47.52536847 | 49.5257712 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.75** | 1.145476679 | 148.3637927 | 95.61809737 | 99.81082903 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.75** | 1.144674214 | 148.3842194 | 95.57852163 | 99.64525508 |
| **Kernel / Epanechikov** | | | | | |
| **r = 3** | | **m = 3, k = 9** | **m = 5, k = 15** | **m = 7, k = 21** | **m = 9, k = 27** |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.25** | 1.149410868 | 1.614515338 | 1.843264641 | 1.747199133 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.25** | **1.148362441** | 1.618353688 | 1.847333509 | 1.755202058 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.50** | 8.979216413 | 8.249211862 | 6.078342982 | 8.132878988 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.50** | 8.974683071 | 8.233306339 | 6.073469261 | 8.133155348 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.75** | 3.294042012 | 26.82358148 | 28.29655331 | 28.27585839 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.75** | 3.292327007 | 26.82003263 | 28.30472056 | 28.26597812 |
| **r = 5** | | **m = 3, k = 15** | **m = 5, k = 25** | **m = 7, k = 35** | **m = 9, k = 45** |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.25** | 4.345040051 | 5.268698317 | 4.586477365 | 5.407750805 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.25** | 4.352927529 | 5.267279497 | 4.569310023 | 5.404328033 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.50** | 2.323389334 | 26.0140591 | 17.9700734 | 24.7286995 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.50** | 2.323274339 | 26.03949151 | 17.9725463 | 24.70956546 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.75** | 9.146669779 | 99.17496908 | 59.22260449 | 55.85200401 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.75** | 9.152609192 | 99.13266732 | 59.27832716 | 55.88540876 |
| **r = 7** | | **m = 3, k = 21** | **m = 5, k = 35** | **m = 7, k = 49** | **m = 9, k = 63** |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.25** | 8.833738122 | 8.007998814 | 7.418449251 | 8.355552021 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.25** | 8.842005986 | 7.954459438 | 7.437713899 | 8.389215649 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.50** | 3.907380366 | 43.30962142 | 32.17081358 | 36.2482038 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.50** | 3.907250019 | 43.18540868 | 32.12746718 | 36.24753986 |
| **MOM & LPR** | $P_{MRSS_0}$ **= 0.75** | 1.105891296 | 114.7249474 | 109.1196595 | 80.67786555 |
| **MLE & LPR** | $P_{MRSS_0}$ **= 0.75** | 1.105116564 | 114.7407427 | 109.0744956 | 80.54403085 |

**Table 2.3: The Relative Efficiency (RE) Of the Method Of Moment And MLE To LPR based on MRSS EVEN**

| **Kernel / Normal** | | | | | |
|---|---|---|---|---|---|
| **r = 3** | | **m = 2, k = 6** | **m = 4, k = 12** | **m = 6, k = 18** | **m = 8, k = 24** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 1.121467811 | 1.213447261 | 1.764840428 | 1.297525781 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 1.124377986 | 1.211776704 | 1.770773642 | 1.297350721 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 9.008062404 | 8.634465465 | 9.953317382 | 7.173374769 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 9.009959571 | 8.638287367 | 9.958276137 | 7.185803392 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 5.533058407 | 45.2334261 | 29.71497231 | 28.20200581 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 5.534016162 | 45.28242039 | 29.74194806 | 28.18686013 |
| **r = 5** | | **m = 2, k = 10** | **m = 4, k = 20** | **m = 6, k = 30** | **m = 8, k = 40** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 3.217770418 | 4.952221099 | 4.012744473 | 3.795039898 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 3.225021571 | 4.94776637 | 4.016705337 | 3.786597522 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 2.979935512 | 22.41561938 | 28.1837648 | 27.35911443 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 2.980518944 | 22.41636834 | 28.1842429 | 27.38291505 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 9.897176145 | 64.53177045 | 70.13118147 | 49.04326917 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 9.894483270 | 64.54179052 | 70.10334207 | 49.09162196 |
| **r = 7** | | **m = 2, k = 14** | **m = 4, k = 28** | **m = 6, k = 42** | **m = 8, k = 56** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 7.653310002 | 10.89775215 | 9.644330466 | 8.469671982 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 7.665166066 | 10.89591333 | 9.640288035 | 8.519949004 |

| | | | | | |
|---|---|---|---|---|---|
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 4.383895869 | 43.43432917 | 34.10624707 | 57.80866613 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 4.379929010 | 43.36149957 | 34.12770176 | 57.71607135 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.733270932 | 133.91933 | 112.9850412 | 106.2509198 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.734050677 | 133.9057005 | 113.0363767 | 106.5025735 |
| **Kernel / Epanechikov** | | | | | |
| **r = 3** | | **m = 2, k = 6** | **m = 4, k = 12** | **m = 6, k = 18** | **m = 8, k = 24** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 1.170616528 | 1.100857921 | 1.505264215 | 1.386770018 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 1.173654243 | 1.099342366 | 1.510324759 | 1.386582918 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 8.863753821 | 10.54768947 | 10.07144394 | 7.162634295 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 8.865620595 | 10.55235823 | 10.07646155 | 7.175044309 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 5.374192759 | 34.28867356 | 29.35597232 | 22.22237951 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 5.375123014 | 34.32581311 | 29.38262217 | 22.21044514 |
| **r = 5** | | **m = 2, k = 10** | **m = 4, k = 20** | **m = 6, k = 30** | **m = 8, k = 40** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 3.560603837 | 4.48282386 | 4.738023875 | 3.865256086 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 3.568627556 | 4.478791373 | 4.742700642 | 3.856657508 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 1.912071150 | 23.45475428 | 29.29782511 | 28.34515203 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 1.912445509 | 23.45553796 | 29.29832211 | 28.36981044 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.354878769 | 61.82682567 | 83.52995952 | 55.25257732 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.354510127 | 61.83642573 | 83.49680131 | 55.30705199 |
| **r = 7** | | **m = 2, k = 14** | **m = 4, k = 28** | **m = 6, k = 42** | **m = 8, k = 56** |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.25** | 7.572425265 | 7.376059306 | 11.25482628 | 7.950999739 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.25** | 7.584156027 | 7.374814719 | 11.2501088 | 7.998197858 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.50** | 3.679574828 | 46.75234031 | 45.20458505 | 38.96374121 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.50** | 3.676245288 | 46.67394715 | 45.2330212 | 38.9013312 |
| **MOM & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.494407568 | 117.0210791 | 109.2393765 | 105.1210118 |
| **MLE & LPR** | $P_{MRSS_e}$ **= 0.75** | 1.495079855 | 117.0091694 | 109.2890101 | 105.3699893 |

The next five steps are a description and explanation of relative efficiency $RE$ tables :

1- $LPR$ was more efficient than the method of moment and maximum likelihood at the same set size and cycles for estimating $CDF$.

2- In different kernels, $LPR$ remains more efficient.

3- By increasing the number of cycles, which is $r$, the efficiency of $LPR$ will increase.

4- Evidently, using $LPR$ significantly improves the behavior of the method of moment and $MLE$. It is apparent that all the proposed procedures based on $LPR$ can be the best choice.

5- The relative efficiency in the case of $MRSS\ odd, even$ has a higher efficiency result with the $LPR$ method compared to the $RSS$.

## 5.2 CONCLUSION

This article is concerned with estimating the cumulative distribution function $CDF$ based on the local polynomial regression $LPR$ depending on $RSS$ and $MRSS$. A new $CDF$ estimator depends on $LPR$ is derived. The resulting proposed estimator is used to introduce three ways of estimating $CDF$ (the method of moments, the maximum likelihood method, and $LPR$), based on $RSS$ and $MRSS$. The method of moments and the maximum likelihood method based on $RSS$ were suggested by Al-Saleh and Ahmad (2019). In this study we get the same result of $CDF$ estimate for the method of moments and the maximum likelihood method, $CDF$ estimator based on $LPR$ can have some advantages over their competitors for fixed or non-fixed samples; on the empirical side, we use kernel (normal, epanechikov) with bandwidth (0.1, 0.2, …, 0.9) and three levels of degree of kernel, We have concluded that kernel epanechikov is a little better than normal, with bandwidth 0.8 giving the best result. Compared to the others, bandwidth and degree 2 are also like that. We have concluded that $CDF$ of $LPR$ based on $MRSS$ is better than $CDF$ of $LPR$ based on $RSS$ because the data in $MRSS$ is stable and is less prone to ranking errors for fixed or non-fixed samples. Depending on the relative efficiency, we get that there is no big difference between both kernels, nonetheless we conclude that relative efficiency depends on whether the $MRSS$ is better than the $RSS$ and has more efficiency. We recommend using $LPR$ based on estimators.

**Fig. 1:** The mean square error and bias of CDF based on RSS with. $P = 0.25$.

$\boldsymbol{m = 2, r = 3, k = 6}$ $\boldsymbol{m = 2, r = 5, k = 10}$ $\boldsymbol{m = 2, r = 7\ k = 14}$

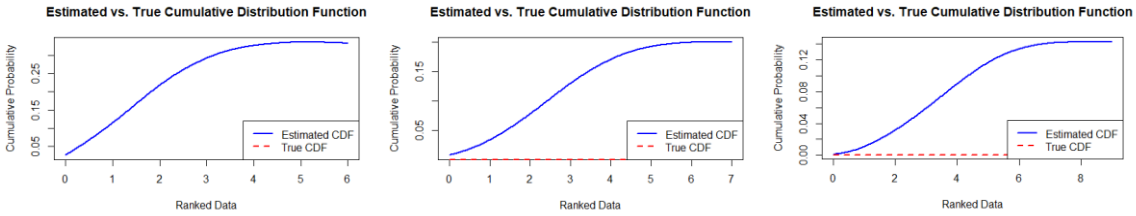



**Fig. 2:** The mean square error and bias of CDF based on RSS with. $m = 5, r = 3, k = 15$

$\boldsymbol{P = 0.25}$ $\boldsymbol{P = 0.5}$ $\boldsymbol{P = 0.75}$

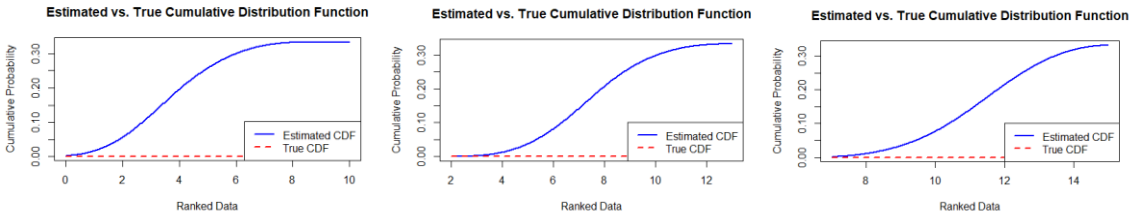



**Fig. 3:** The mean square error and bias of CDF based on MRSS ODD with. $m = 5, P = 0.5$.

$\boldsymbol{m = 5, r = 3, k = 15}$ $\boldsymbol{m = 5, r = 5, k = 25}$ $\boldsymbol{m = 5, r = 7,\ k = 35}$

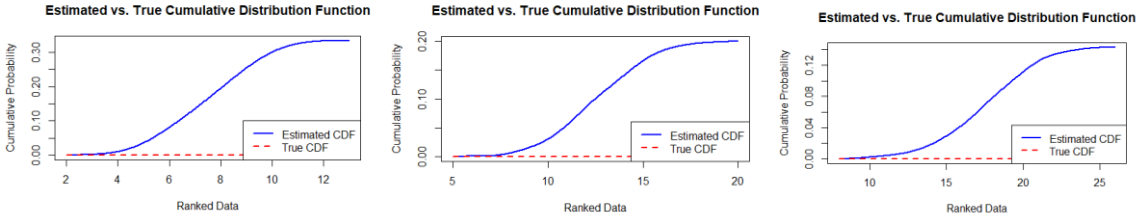



**Fig. 4:** The mean square error and bias of CDF based on MRSS ODD with. $m = 9, r = 5, k = 45$

$\boldsymbol{P = 0.25}$ $\boldsymbol{P = 0.5}$ $\boldsymbol{P = 0.75}$

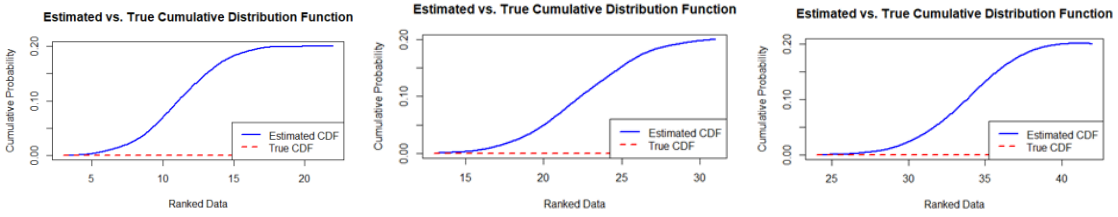



**Fig. 5:** The mean square error and bias of CDF based on MRSS EVEN with. $m = 6, P = 0.75$.

$\boldsymbol{m = 6, r = 3, k = 18}$ $\boldsymbol{m = 6, r = 5, k = 35}$ $\boldsymbol{m = 6, r = 7\ k = 42}$

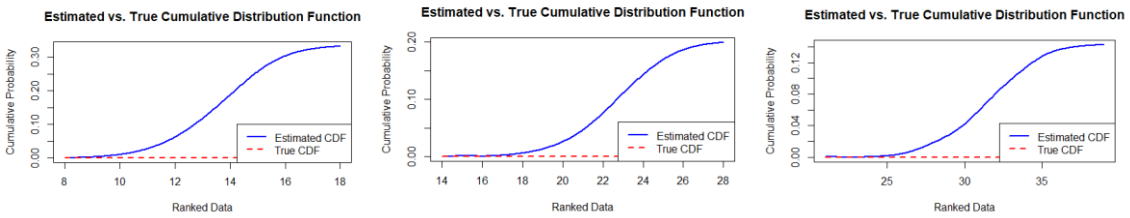
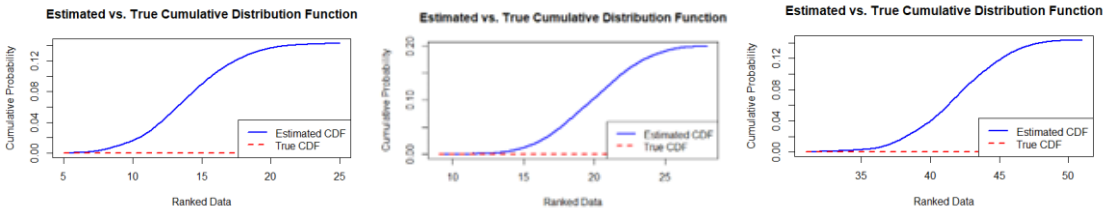



**Fig. 6:** The mean square error and bias of CDF based on MRSS EVEN with. $m$ = 8,r = 7,k = 56

$\boldsymbol{P = 0.25}$ $\boldsymbol{P = 0.5}$ $\boldsymbol{P = 0.75}$

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**طريقة التقدير اللامعلمية لدالة التوزيع باستخدام انواع مختلفة من مجموعات العينات المرتبة**

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**الخلاصة:** الغرض من هذا البحث هو تقدير دالة التوزيع التراكمي $CDF$ باستخدام الانحدار متعدد الحدود المحلي $LPR$ ومقارنته مع الطرق المعلمية وهي كل من طريقة العزوم وطريقة الامكان الأعظم لغرض حساب متسوط مربع الخطأ والتحيز بالأعتماد على منهجية مجموعة عينات المرتبة $RSS$ ومنهجية مجموعة عينات المرتبة الوسطى $MRSS$. بالإضافة إلى أن $RSS$ غالبًا تنتج تقديرات أكثر دقة من عينة عشوتئية بسيطة $SRS$ و لنفس حجم العينة, ويتم ذالك من خلال ترتيب العينات بناءً على بعض الخصائص التي يمكن قياسها بسهولة, يتم تقليل التباين داخل كل مجموعة, مما يؤدي إلى تقديرات أكثر دقة. لقد قمنا بدراسة ثلاث درجات مختلفة من الانحدار متعدد الحدود المحلي: الدرجة الأولى والثانية والثالثة, أظهر تحليل المحاكاة أن الدرجة الثانية تتفوق على الدرجات الأخرى, وعندما يتم استخدام $LPR$ لتحليل بيانات $RSS$, فإنه يستفيد من التباين المنخفض داخل كل مجموعة مرتبة, وهذا ما يؤدي إلى تقديرات أكثر دقة وموثوقية لدالة الانحدار , إضافة الى ذالك , قمنا بدراسة درجات مختلفة من النطاق الترددي $(0.1, 0.2,\ldots,0.9)$ وتبينا لنا من خلال دراسة المحاكاة أن النطاق الترددي للدرجة $0.8$ يتفوق على الدرجات الأخرى. بعد ذالك, قمنا بتحليل الكفاءة النسبية لكل من الأساليب الثلاثة: $LPR\ MOM$ و $MLE$ , و تبينا لنا أن طريقة $LPR$ هي أكثر كفاءة من الطرق الأخرى لتقدير $CDF$ في نواة مختلفة (gaussian) normal), epanechinkov), و من ثم أظهرت دراسة المحاكاة أن مقدر $CDF$ المقترح بناءً على $LPR$ أكثر كفاءة من الطرق المعلمية.

**الكلمات المفتاحية:** الانحدار متعدد الحدود المحلي، عينة مجموعة مرتبة، متوسط عينة مجموعة مرتبة، خطأ مربع دالة التوزيع التراكمي